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BASIC MATHEMATICS

Basic Mathematics

FOR ENGINEERS

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PREFACE

This text presents the mathematics required for the intelligent pursuit of elementary engineering courses and serves as a preparation for a course in the calculus. It contains those topics from algebra, trigonometry, and analytic geometry which are needed to meet this dual objective.

The book is intended to be used with students who have had at least two years of high school mathematics. However, it is completely self-contained except that it relies upon the usual simple facts from arithmetic and geometry. Since the necessary facts from geometry are collected in the Appendix, it is possible to use the book successfully with intelligent students who have had very little mathematical training beyond ordinary arithmetic, or who have largely forgotten their high school mathematics. An attempt has been made to make the explanations and discussions clear and complete so that the book can also be used for home study, or in courses where greater reliance than usual is placed upon the student's reading. Many illustrative examples are worked out in full so that the student can see in operation the principles discussed in the text and has a clear procedure after which to pattern his own work.

Since the purpose of the book has been to relate mathematics as far as possible to its engineering applications, use has been made of engineering symbols and terminology, and numerous illustrations, graphs, examples, and exercises from engineering. Thus without sacrificing any of the mathematical rigor expected of a textbook at this level, we have attempted to utilize the practical sophistication of the modern student.

Certain special features of the book are designed to make it especially adapted to engineering curricula:

1. Principles of accuracy in numerical computations are discussed in the first chapter, and the student is reminded of them throughout the text.
2. The slide rule is discussed in the first chapter, and the student can use it to solve problems throughout the text.
3. Graphical methods are stressed.
4. Since the trigonometric functions are needed early in most physics and engineering courses, they are introduced early in the book.

5. A chapter on vector algebra is placed early in the text.
6. The graphs of the trigonometric functions are discussed fully.
7. The Doolittle method of solving simultaneous linear equations is treated.
8. Two introductory chapters on differential and integral calculus are supplied for use in courses where the fundamental ideas of the calculus are needed.
9. In order to motivate the student's work as much as possible, each chapter has a brief introduction which places the material of that chapter in its mathematical and engineering setting. Each chapter concludes with a Progress Report which summarizes the material treated in the chapter.
10. There is a great abundance of exercises for the student, many from engineering and the physical sciences.

Much of the material contained in this book was developed and used by the authors in the mathematics sections of the training courses in communication engineering offered to members of the Signal Corps Reserve by the Illinois Institute of Technology.

The authors owe a particular debt of gratitude to Dr. Jesse E. Hobson, Director of the Department of Electrical Engineering and Director of the Signal Corps Training Courses, without whose encouragement and assistance this book could not have been written. The authors also acknowledge their debt to the late Dr. Edward Helly. In addition to contributing some of the material for a number of the chapters, he improved the book in many places by his suggestions and criticisms. By his untimely death the authors lost a fine friend and an inspiring collaborator. Many members of the mathematics department of the Signal Corps training staff helped in the production of the manuscript. Their assistance, particularly that of Mr. Philip L. Browne, is gratefully acknowledged.

THE AUTHORS

Illinois Institute of Technology
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CHAPTER 1

NUMERICAL COMPUTATIONS

Scientists and engineers must have the ability to compute easily, quickly, and accurately. All modern technological developments which depend upon the theories devised by these men would have been impossible without great care and skill in computation.

The ability to compute depends upon a knowledge of the fundamental properties of numbers. In order to compute rapidly, one must have had much practice and should be able to use any devices to speed and simplify computation that may be available to him. Further, especially if the numbers with which he is concerned were obtained experimentally, he should be able to estimate the accuracy of his result.

In this chapter we shall study the fundamental properties of numbers and the operations with them; we shall learn to use the slide rule, a device for simplifying and speeding computations; and finally we shall give certain rules which can be used to determine the accuracy of our results.

1-1. The Integers. The number system is the result of long development. The necessity of counting objects forced man to create the simplest numbers, called the **integers** or **natural numbers**, which we denote by the symbols 1, 2, 3, 4, etc. Problems of everyday life suggested certain operations with these numbers, and finally the four operations of addition, subtraction, multiplication, and division were developed.

Various properties of these numbers and of the operations with them were discovered quite intuitively and were used almost instinctively. As a basis for more complicated problems and methods which are to be studied in this book, a knowledge of the following laws and properties is essential.

1. *The process of counting starts with the integer one and can be carried on indefinitely.*

2. *Of two given integers, one occurs earlier than the other in counting.* This fact is expressed as follows. The integer encountered first is **smaller** than the second, or the second integer is **greater** than the first. For brevity, we use the sign $<$ for **smaller than** and the sign $>$ for **greater than**. For example, $3 < 5$ and $5 > 3$. Note that the *narrowing*

end of the symbol is always nearer the smaller integer, and the *widening* end is always nearer the larger integer.

3. If a given integer is greater than a second integer, and the second greater than a third, then the first integer is greater than the third. For example, if $7 > 4$ and $4 > 2$, then $7 > 2$.

4. The result of an addition or a multiplication does not depend on the arrangement of the integers. This is called the **commutative law**.

Example 1.

$$3 + 5 + 7 = 7 + 5 + 3 = 5 + 7 + 3 = 15.$$

Example 2.

$$3 \times 7 = 7 \times 3 = 21.$$

5. In the addition or multiplication of more than two integers, the result is the same no matter how the integers are grouped and the computation performed. This is called the **associative law** or the **law of grouping**.

Example 3.

$$17 + 13 + 26 + 14 = 70;$$

$$17 + 13 = 30, 26 + 14 = 40;$$

$$30 + 40 = 70.$$

Example 4.

$$3 \times 7 \times 2 \times 5 = 210;$$

$$3 \times 7 = 21, 2 \times 5 = 10;$$

$$21 \times 10 = 210.$$

6. If the sum of two or more integers is to be multiplied by an integer, the result can be found by multiplying each of the first two or more integers by the latter and adding the products. This is called the **distributive law**.

Example 5. The sum of 13 and 18 is to be multiplied by 7.

(a) Add: $13 + 18 = 31.$

Multiply: $31 \times 7 = 217.$

(b) Multiply each: $13 \times 7 = 91, 18 \times 7 = 126.$

Add: $91 + 126 = 217.$

1-2. Negative Numbers. Addition and multiplication of two integers are always possible, and the result is again an integer. But subtraction and division cannot always be carried out with only positive integers. Thus, there is no number representing the difference $3 - 5$. Yet there are many occasions in life that require "taking more away than you have."

Assume, for example, that a car starts at a point *A* on an east and west highway and goes 150 miles east to a point *B* and returns 100 miles

to C . Knowing the distance CB to be 100 miles, we can find the distance from C to A by the subtraction

$$150 - 100 = 50.$$

Now assume that the car makes a return trip of 200 miles, starting at B . The final position D is obviously 50 miles west of A . But the subtraction $150 - 200$ is not possible with the positive integers.

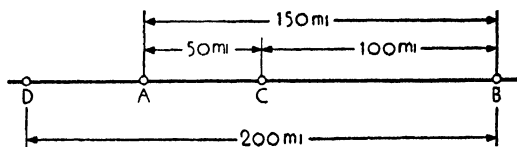


FIG. 1-1.

This example suggests a method of distinguishing the distances east of A from those west of A . One of these directions is usually considered **positive**, the other **negative**, and we say:

Point B indicates a positive distance of 150 miles from A .

Point C indicates a positive distance of 50 miles.

Point D indicates a negative distance of 50 miles.

We can express these statements in a simpler way by saying:

The distance AB is $+150$ miles.

The distance AC is $+50$ miles.

The distance AD is -50 miles.

So, instead of being satisfied with the statement that the subtraction $150 - 200$ is impossible, we define new numbers which we call **negative numbers** (denoted by a minus sign) and which make possible such operations. Thus for each positive integer there is a corresponding negative integer. We have then -1 , -2 , -3 , -4 , etc. Finally, to provide a result for the difference of equal integers, the number zero (denoted by 0) is introduced. With the positive and negative integers and zero the process of subtraction is always possible. The result of the subtraction above is then

$$150 - 200 = -50.$$

The signs $+$ and $-$ are thus used with a double meaning. First, the processes of addition and subtraction are indicated by these signs; and second, the signs are used to distinguish the positive and negative integers. In the first case they are regarded as *signs of operation*, and in the second case as *signs of quality*. When used as signs of operation they

are always expressed, but $+$ is usually omitted as a sign of quality. We shall regard the terms *integer* and *number* as meaning a quantity together with its sign of quality. Thus the signs in $20 - 5 + 7$ are signs of operation; but the signs of $+6$ and -8 are signs of quality. Here 6 is usually written instead of $+6$.

The use of positive and negative signs also indicates direction in many engineering problems. If a spring is fastened to a rigid support at one end, and the other is acted upon by several forces, some pushing and some pulling, we may designate those forces which tend to compress the spring as positive and those which tend to elongate the spring as negative. Similarly, the ammeter on the dashboard of an automobile has positive numbers to indicate that the battery is being charged, whereas negative numbers show discharge. In this case, the sign of the number indicates the direction of flow of electric current.

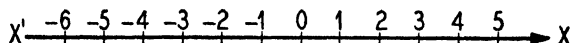


FIG. 1-2.

A very convenient representation of the integers is suggested by the car problem above. Draw a straight line and mark an arbitrary point 0 on it. After choosing a suitable unit, plot the positive integers as distances to the right, the negative integers as distances to the left. In this way a point of the line corresponds to each integer. We note that if one positive integer is greater than a second, the first lies to the right of the second on the line. This remark can be used to extend the ideas of *greater than* and *less than* to the negative integers and zero, as follows.

1. If one integer lies to the right of a second integer on the line, the first is greater than the second. Thus, $5 > 3$, $2 > 0$, $5 > -3$, $-5 > -7$, and $0 > -6$.

2. If one integer lies to the left of a second integer on the line, the first is less than the second. Thus $3 < 5$, $0 < 3$, $-2 < 1$, $-3 < 0$, and $-5 < -3$.

EXERCISES

Place the proper sign, $<$ or $>$, between the integers of the following pairs.

- | | | | |
|--------------|-------------|------------|------------|
| 1. 5, 3. | 2. 6, 8. | 3. -1, 0. | 4. -3, 5. |
| 5. -6, -8. | 6. -2, -1. | 7. 6, -6. | 8. 3, -2. |
| 9. 5, -3. | 10. +6, -8. | 11. 1, -3. | 12. 4, 2. |
| 13. -8, -10. | 14. 3, -3. | 15. -3, 4. | 16. 2, -5. |

1-3. Rational and Irrational Numbers: The System of Real Numbers. The product of any two positive integers is a positive integer. However,

the quotient of any two positive integers is not necessarily an integer. To provide a result for division of positive integers by positive integers, positive fractions must be introduced. By introducing also the corresponding negative fractions, we can provide a result for the quotient of any two integers, as we shall see in Sec. 1-6.

The integers and fractions form the rational numbers. Hence, a **rational number** is one which can be represented as the quotient of two integers.

There are, however, numbers which cannot be expressed as the quotient of two integers; these are called **irrational**. For instance, it is not possible to find a rational number which when multiplied by itself gives 2. In other words, the square root of 2 is not rational.

Rational and irrational numbers form the system of real numbers. In Chapter 12, the operation of taking square roots will lead to the introduction of other new elements, the imaginary numbers.

To summarize:

Positive integers	{	Subtraction leads to negative numbers.
		Division leads to rational fractions.
		Extraction of roots leads to irrational numbers.

The point on the **number line** of Fig. 1-2 corresponding to any number can now be located as follows. *Measure from 0 a distance equal to the given number times the unit of measurement, measuring to the right if the given number is positive, to the left if it is negative.* In this way there is a point on the line corresponding to every number.

Using this representation, we can now extend very simply the ideas of *greater than* and *less than* to all real numbers by restating the definitions given at the end of the preceding section, substituting *number* for *integer*.

1. *If one number lies to the right of a second number on the line, the first is greater than the second.*

2. *If one number lies to the left of a second number on the line, the first is less than the second.*

By these definitions, every positive number is greater than 0 and every negative number is less than 0.

The absolute value of a positive number is the number itself; the absolute value of a negative number is the positive number obtained by changing its sign of quality from - to +.

The absolute value is denoted by parallel vertical bars on either side of the number; the absolute values of +5 and -5 are denoted by $|+5|$ and $|-5|$ respectively. By the definition, $|+5| = |-5| = 5$.

1-4. The Basic Generalization of Algebra. In arithmetic, operations are performed with specific numbers. However, when it is desired to make general arguments about numbers, letters are used, such as a, b, C, D, x, y, z . These letters, in any particular application, represent numbers and may be replaced by numbers at the will of the reader. Thus by using letters we may designate, in effect, an entire group of numbers at once, rather than being limited to one at a time. This generalization of the idea of the number, in addition to furnishing an extremely economical means of expressing ideas about physical quantities, makes possible the achievement of many general results which apply to all or large groups of numbers. It is the basic characteristic of algebra and furnishes the mathematician and the engineer with a practical tool of great power. Many uses of this tool in practical applications will be seen in this book.

1-5. The Signs of Arithmetic and Algebra.

CONCEPT	SIGN	EXAMPLE
Addition	+	$5 + 2$
Subtraction	-	$3 - 2$
Multiplication	\cdot or \times	$3 \cdot 4$ or 3×4
Division	\div	$6 \div 2$ or $\frac{6}{2}$
Signs of		
{ Parentheses	()	$(2 + 3)$
{ Brackets	[]	$[2 - (3 - 6)]$
{ Braces	{ }	$\{1 - [3 + (3 - 2)]\}$
Equality	=	$3 = 3$
Not equal	\neq	$7 \neq 5$
Less than	<	$2 < 5$
Greater than	>	$2 > -3$
Equal to or less than	\leq	$a \leq b$
Equal to or greater than	\geq	$c \geq d$
Plus or minus	\pm	$x = \pm 2$

1-6. The Laws of Signs. The following rules govern the operations of addition, subtraction, multiplication, and division in the system of real numbers:

1. Addition and subtraction.

(a) *To add two numbers with like signs of quality, add their absolute values and attach their common sign of quality to the result.*

Example 1.

$$+5 + (+4) = | +5 | + | +4 | = + | 5 + 4 | = +9 = 9.$$

Example 2.

$$(-5) + (-4) = - (| -5 | + | -4 |) = -(5 + 4) = -9.$$

(b) *To add two numbers with unlike signs of quality, subtract the smaller absolute value from the larger and attach to the result the sign of quality of the number with the larger absolute value.*

Example 3.

$$6 + (-2) = + | 6 - 2 | = +4 = 4.$$

Example 4.

$$7 + (-9) = - | 9 - 7 | = -2.$$

(c) *To subtract one number from another, change the sign of quality of the number to be subtracted (the subtrahend) and proceed as in addition. By this rule addition and subtraction may be regarded as essentially the same process.*

Example 5.

$$12 - (+20) = - | 20 - 12 | = -8.$$

2. Multiplication and division.

(a) *The product of two numbers of like signs of quality is the product of their absolute values; the product of two numbers of unlike signs of quality is the negative of the product of their absolute values.*

Example 6.

$$12 \times 4 = | 12 | \times | 4 | = 48.$$

Example 7.

$$(12) \times (-4) = -(| 12 | \times | -4 |) = -(12 \times 4) = -48.$$

(b) *The quotient of two numbers of like signs of quality is the quotient of their absolute values; the quotient of two numbers of unlike signs of quality is the negative of the quotient of their absolute values.*

Example 8.

$$\frac{-6}{-2} = \frac{| -6 |}{| -2 |} = \frac{6}{2} = 3.$$

Example 9.

$$\frac{10}{-5} = - \frac{| 10 |}{| -5 |} = - \frac{10}{5} = -2.$$

The **reciprocal** of a number is defined as one divided by the number. The reciprocal of 6 is $\frac{1}{6}$; the reciprocal of -2 is $-\frac{1}{2}$. We will see in Sec. 1-10 that *division by a given number is the same as multiplication by the reciprocal of that number, so that multiplication and division may be regarded as essentially the same process.*

1-7. Operations with Zero. Where a is any number, the operations with zero are defined as follows.

$$a \pm 0 = a.$$

$$a \cdot 0 = 0.$$

$$\frac{0}{a} = 0, \text{ if } a \neq 0.$$

Note that division by zero is not included. To see why this is so we need only examine the definition of the quotient q of a and b , $q = \frac{a}{b}$ if $a = b \cdot q$.

Let $a = 5$ and $b = 0$. Then the quotient $\frac{5}{0}$ should yield the number q such that $5 = 0 \cdot q$. But there is no number q such that $0 \cdot q = 5$, since by definition 0 times any number is zero. Hence, neither 5 nor any other number can be divided by zero without contradiction. By the definition, the quotient $\frac{0}{0}$ must be a number q such that $0 = q \cdot 0$. Now any value of q satisfies this relation, and hence no unique value of q is determined. We therefore assign no particular value to $\frac{0}{0}$. Since it is desirable that the number system be free of contradiction and that the process of division lead to a definite result, *we exclude the process of division by zero from arithmetic.*

1-8. Fundamental Laws of Algebra. It is important to note that *the general properties of positive integers and of the operations with them*, as described in Sec. 1-1, *hold for all real numbers*, with the exception of Statement 1, which obviously applies only to positive integers.

For completeness, we shall restate these properties here more briefly with letters. That the first statement below is equivalent to Statements 1 and 2 of Sec. 1-3 may be easily verified.

1. $a > b$ if $a - b > 0$,

$a < b$ if $a - b < 0$.

2. If $a > b$ and $b > c$, then $a > c$.

3. Commutative law:

$$a + b = b + a, a \times b = b \times a.$$

4. Associative law:

$$(a + b) + c = a + (b + c), (a \times b) \times c = a \times (b \times c).$$

5. Distributive law:

$$(a + b) \times c = a \times c + b \times c.$$

EXERCISES

1. Add:

- (a) 2 and -11 ,
- (b) -7 and -12 ,
- (c) 8 and 1,
- (d) -1 and 86,
- (e) 26 and -8 .

3. Add:

- (a) 27, -20 , and 7,
- (b) 11, -2 , and 99,
- (c) 16, 27, and -20 ,
- (d) 11, 22, and 0,
- (e) 5, -12 , -5 , and -14 .

5. Subtract:

- (a) 2 from -11 ,
- (b) -11 from 2,
- (c) -7 from -12 ,
- (d) 7 from -12 ,
- (e) 12 from -7 .

7. Subtract:

- (a) -4 from -3 ,
- (b) 16 from -19 ,
- (c) 19 from -16 ,
- (d) 20 from 0,
- (e) 21 from -21 .

9. Multiply:

- (a) 2 by -11 ,
- (b) -7 by 12,
- (c) 8 by -12 ,
- (d) -1 by -12 ,
- (e) 26 by -3 .

11. Multiply:

- (a) 2 by 11 by -6 ,
- (b) -11 by 2 by 6,
- (c) -11 by 3 by 0,
- (d) 5 by -12 by -5 ,
- (e) -11 by -7 by -3 .

13. Divide:

- (a) 0 by -12 ,
- (b) 12 by -2 ,
- (c) -8 by -8 ,
- (d) 86 by -43 ,
- (e) -26 by 13.

2. Add:

- (a) 11 and -42 ,
- (b) 7 and -12 ,
- (c) -3 and 86,
- (d) -20 and -8 ,
- (e) 4 and -3 .

4. Add:

- (a) 14, -18 , 42, and -85 ,
- (b) ~~82~~, 56, -14 , and -28 ,
- (c) -72 , 43, 19, and -108 ,
- (d) 15, -18 , 0, and -47 ,
- (e) 18, 16, -43 , and -56 .

6. Subtract:

- (a) 8 from 1,
- (b) -8 from 1,
- (c) 20 from -8 ,
- (d) -20 from -8 ,
- (e) 4 from -3 .

8. Subtract:

- (a) -18 from 86,
- (b) -22 from -76 ,
- (c) -37 from 26,
- (d) 42 from 0,
- (e) -85 from 106.

10. Multiply:

- (a) -8 by 8,
- (b) -4 by 0,
- (c) 5 by -22 ,
- (d) -9 by 7,
- (e) -5 by -22 .

12. Multiply:

- (a) 3 by -8 by 4,
- (b) 4 by 6 by -10 ,
- (c) -8 by -6 by 37,
- (d) -3 by -6 by 0,
- (e) -4 by -8 by -2 .

14. Divide:

- (a) -7 by -7 ,
- (b) -21 by -3 ,
- (c) -36 by 8,
- (d) 105 by -21 ,
- (e) -45 by -9 .

15. Divide:

- (a) -8 by -16 ,
- (b) -7 by 0 ,
- (c) 81 by -18 ,
- (d) 108 by -12 ,
- (e) 12 by -108 .

16. Divide:

- (a) 87 by 0 ,
- (b) -144 by -16 ,
- (c) -72 by 24 ,
- (d) -42 by 84 ,
- (e) 36 by -12 .

17. Multiply:

- (a) 2 by the sum of 3 and 4 ,
- (b) -3 by the sum of 5 and -2 ,
- (c) -5 by the sum of 3 and -1 ,
- (d) 3 by the sum of -5 and 3 ,
- (e) -5 by the sum of -6 and 4 .

18. Multiply:

- (a) 3 by the sum of 5 and -8 ,
- (b) 2 by the sum of 15 and -10 ,
- (c) -5 by the sum of -8 and -2 ,
- (d) -6 by the sum of -3 and 5 ,
- (e) -7 by the sum of -8 and 3 .

1-9. Signs of Grouping. A table of the signs of grouping was given in Sec. 1-5. When a given number multiplies the quantity within signs of grouping, by the distributive law each number forming the sum or difference in the signs of grouping is to be multiplied by the given number. In such cases, the sign of multiplication appearing in front of the bracket is often omitted. Other properties of operations with signs of grouping will be evident from the following illustrative examples.

Example 1. $8 \times (5 - 6) = 8(5 - 6) = 40 - 48 = -8$. Also $8 \cdot (5 - 6) = 8(-1) = -8$.

Example 2. $(-8) \times (-5) \cdot (+6) = -8(-5)(+6) = -(-40)(6) = -(-240) = 240$.

Example 3. $-1(8 - 6 + 4) = -8 + 6 - 4 = -2 - 4 = -6$, or $-1(8 - 6 + 4) = -1(6) = -6$.

Example 4. $+ [8 + (-6)] = +8 + (-6) = 8 - 6 = 2$, or $+ [8 - 6] = 8 - 6 = 2$.

Example 5. $-(8 - 6) = -8 - (-6) = -8 + 6 = -2$.

Example 6. $5 - 3 + 6 = 5 + (-3 + 6)$, or $5 - 3 + 6 = 5 - (+3 - 6)$.

As already indicated in the preceding examples, we can state the following rules concerning the removal and insertion of signs of grouping preceded by a $+$ or $-$ sign. A **term** is a number of an expression which is set off from the other numbers by $+$ or $-$ signs.

Rule 1. Signs of grouping preceded by a plus sign may be removed or inserted by rewriting each enclosed term or group of terms with its original sign.

Rule 2. Signs of grouping preceded by a minus sign may be removed or inserted if the sign of each of the enclosed terms or groups of terms is changed.

Example 7. $8 - \{16 + [(4 - 3) + 5] - 2\} = 8 - \{16 + [4 - 3 + 5] - 2\} = 8 - \{16 + 4 - 3 + 5 - 2\} = 8 - 16 - 4 + 3 - 5 + 2 = 13 - 25 = -12$. Also $8 - \{16 + [(4 - 3) + 5] - 2\} = 8 - \{16 + [1 + 5] - 2\} = 8 - \{16 + 6 - 2\} = 8 - \{22 - 2\} = 8 - \{20\} = 8 - 20 = -12$.

EXERCISES

Remove the signs of grouping and find the values of the following expressions:

- | | |
|--|---------------------------------|
| 1. $2 + (3 - 9)$. | 2. $(2 + 3) - 9$. |
| 3. $2 - (3 - 9)$. | 4. $2 - (3 + 9)$. |
| 5. $(2 - 3) + 9$. | 6. $3 - [4 + 2(5 - 3)]$. |
| 7. $3 + [4 - 2(5 - 3)]$. | 8. $3 + [4 - 2(5 + 3)]$. |
| 9. $3 - [4 - 2(5 - 3)]$. | 10. $3 - 2[5 - 3(3 - 5)]$. |
| 11. $(7 - 5) + 3 - (6 - 7) + 10$. | 12. $-[7 - 5(3 - 6) - 7(10)]$. |
| 13. $-3[(4 + 3)2 - (6 + 7) + 2]$. | |
| 14. $5 - 3\{[7 - 2(11 - 13) + 7(3 + 2)] + 17\}$. | |
| 15. $5 + 3\{7 - [2(11 - 13) - 7(3 - 2) + 17]\}$. | |
| 16. $\{18 + [69 - 42 + (67 - 14)] + 31\} - 6$. | |
| 17. $\{52 - [18 - (14 - 8 + 7)] + 16\}$. | |
| 18. $\{26 + [14 - 8(2 + 3)]\} - \{16 + 8(6 + 4)\}$. | |
| 19. $\{46 - [18 + 3(4 - 8)]\} + 3\{8 - 6[4 + (8 - 6)]\}$. | |
| 20. $\{13 - 2[8(3 - 2) - 7]\} - 37(18 - 16)$. | |

1-10. Fractions. The term fraction was used in Sec. 1-3 to refer to rational numbers which were not integers. More generally the term refers to the quotient of any two numbers. The **dividend**, or number to be **divided**, is the **numerator**; the **divisor**, or the number by which the numerator is divided, is the **denominator**.

1. *The sum of two fractions with a common denominator is the sum of their numerators divided by their common denominator. Symbolically:*

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

2. *The product of two fractions is the product of their numerators divided by the product of their denominators. Symbolically:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

A negative number may be regarded as the product of -1 times the absolute value of the number. For example, $-6 = (-1) \cdot |6|$, $-3 = (-1) \cdot |3|$, $-a = (-1) \cdot |a|$, if $a > 0$. Using this fact,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b},$$

and

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$

The last formula gives the rule for subtraction of fractions.

Since $\frac{c}{c} = 1$, we have:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}.$$

Hence, the numerator and denominator of a fraction can each be multiplied (or divided) by the same number without changing the value of the fraction.

A fraction is said to be *inverted* by the interchange of its numerator and denominator. For example, inverting $\frac{3}{5}$ we get $\frac{5}{3}$; inverting $\frac{a}{b}$ we get $\frac{b}{a}$.

Since

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot b \cdot d}{\frac{c}{d} \cdot b \cdot d} = \frac{a \cdot d}{b \cdot c},$$

we have

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}.$$

Or, the quotient of two fractions is equal to the fraction of the dividend multiplied by the inverted fraction of the divisor. Note in the above example

that $\frac{a}{b}$ is the dividend and $\frac{c}{d}$ the divisor.

Since the reciprocal of a number is one divided by that number, it follows that the *reciprocal of a fraction is equal to that fraction inverted*.

An integer is said to be a **multiple** of a second integer if the quotient of the first by the second is an integer. In this case the second integer is said to **divide evenly** into the first integer.

A **prime integer** is one which can be divided evenly only by itself, its negative, 1, and -1 . Thus 3, 5, -11 are primes, whereas -4 , 0, 12 are not. Every integer can be resolved into a product of positive prime integers greater than one (called **factors**) with the sign of the integer itself prefixed.

If we resolve the numerator and denominator of a fraction each into the products of their prime factors greater than one, it is possible to eliminate factors common to numerator and denominator by dividing both by these factors. When all such common factors have been eliminated from numerator and denominator, the fraction is in its **lowest terms**.

Example 1.

$$\frac{156}{72} = \frac{2 \cdot 2 \cdot 3 \cdot 13}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \frac{13}{2 \cdot 3} = \frac{13}{6}.$$

Example 2.

$$\frac{-96}{108} = -\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3} = -\frac{2 \cdot 2 \cdot 2}{3 \cdot 3} = -\frac{8}{9}.$$

Example 3.

$$\frac{5}{18} \cdot \frac{6}{10} = \frac{5 \cdot 6}{18 \cdot 10} = \frac{3 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 3 \cdot 2 \cdot 5} = \frac{1}{5}.$$

Example 4.

$$\frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{8} \cdot \frac{8}{5} = \frac{3 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 5} = \frac{9}{20}.$$

Example 5.

$$\frac{5}{8} + \frac{6}{8} - \frac{3}{8} = \frac{5+6-3}{8} = \frac{8}{8} = 1.$$

The least common multiple (designated as **L.C.M.**) of a group of integers is the smallest positive integer which is a multiple of each of the given integers. Thus the least common multiple is the smallest positive integer into which each of the given integers will divide evenly.

The greatest common factor (designated as **G.C.F.**) of a group of integers is the largest positive integer which divides evenly into each integer of the set.

We observe that, if the quotient of two integers is an integer, every positive prime factor of the denominator greater than 1 occurs at least as many times as a prime factor of the numerator. Using this observation, we arrive at a simple method for finding the least common multiple and the greatest common divisor of a set of integers.

1. Factor each integer of the set into the product of positive primes greater than 1.

2. To find the least common multiple, select each prime factor the greatest number of times it occurs in any one number of the set. The product of these factors is the least common multiple.

3. To find the greatest common factor, select each prime factor the greatest number of times it occurs as a factor of every number of the set. The product of these numbers is the greatest common factor.

Example 6. Find the greatest common factor, and least common multiple of 56, -32, and 96.

Factoring each expression we obtain

$$\begin{aligned} 56 &= 2 \cdot 2 \cdot 2 \cdot 7, \\ -32 &= -2 \cdot 2 \cdot 2 \cdot 2 \cdot 2, \\ 96 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3. \end{aligned}$$

By the rule above, the G.C.F. = $2 \cdot 2 \cdot 2 = 8$, and the L.C.M. = $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 672$. As a check on the G.C.F., $56/8 = 7$, $-32/8 = -4$, and $96/8$

= 12. Since 7, -4, 12 have no common factors other than +1 or -1, 8 is the correct G.C.F. As a check on the L.C.M., $672/56 = 12$, $672/(-32) = -21$, and $672/96 = 7$; since 12, -21, and 7 have no common factors, 672 is the correct L.C.M.

When fractions with different denominators are to be added, a common denominator can be obtained by multiplying numerator and denominator of the fractions by certain integers. The most convenient denominator is the least common multiple of the given denominators, termed the **least common denominator**. Once the common denominator has been obtained, the previous rule of addition applies.

Example 7. Add $\frac{5}{8} - \frac{7}{12} + \frac{13}{36}$. We may write this sum as

$$\frac{5}{2 \cdot 2 \cdot 2} - \frac{7}{2 \cdot 2 \cdot 3} + \frac{13}{2 \cdot 2 \cdot 3 \cdot 3}.$$

Thus the L.C.M. of the denominators is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$ so that the sum becomes

$$\begin{aligned} \frac{5(3 \cdot 3)}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} + \frac{-7(2 \cdot 3)}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} + \frac{13(2)}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} &= \frac{45}{72} + \frac{-42}{72} + \frac{26}{72} \\ &= \frac{45 - 42 + 26}{72} = \frac{29}{72}. \end{aligned}$$

Example 8. Simplify:

$$\frac{\frac{5}{8} - \frac{5}{8} + \frac{1}{12}}{1 - \frac{2}{3} - \frac{1}{15}}.$$

The expression can be simplified as follows.

$$\begin{aligned} \frac{\frac{5}{2 \cdot 2 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{1}{2 \cdot 2 \cdot 3}}{1 - \frac{2}{3} - \frac{1}{3 \cdot 5}} &= \frac{\frac{15}{24} - \frac{20}{24} + \frac{2}{24}}{\frac{15}{15} - \frac{10}{15} - \frac{1}{15}} \\ &= \frac{\frac{15 - 20 + 2}{24}}{\frac{15 - 10 - 1}{15}} = \frac{\frac{-3}{24}}{\frac{4}{15}} \\ &= -\frac{3}{24} \cdot \frac{15}{4} = -\frac{15}{32}. \end{aligned}$$

EXERCISES

Write the following fractions in lowest terms.

1. (a) $\frac{1}{3} \frac{8}{8}$,

2. (a) $\frac{-84}{108}$,

(b) $\frac{1}{12}$,

(b) $\frac{-144}{256}$,

(c) $\frac{357}{144}$.

(c) $\frac{81}{144}$.

3. (a) $\frac{-128}{1028}$,

(b) $\frac{800}{1158}$,

(c) $\frac{111}{36}$.

4. (a) $\frac{87}{58}$,

(b) $\frac{-180}{720}$,

(c) $\frac{18}{66}$.

Find the L.C.M. of:

5. 6, 18, and 22.

7. 4, 5, -6, and -112.

9. 84, 24, -11, and 44.

11. 49, -98, 196, and -294.

6. 3, 5, 20, and 32.

8. 12, 14, -22, and 33.

10. 18, -24, 36, and -144.

12. 30, 72, -84, and -108.

Find the G.C.F. of:

13. 26 and 39.

15. -1000, 250, and 500.

17. 26, -52, and -91.

19. 44, -121, and 77.

14. 24, 48, and -30.

16. 81, -18, and 27.

18. 36, 81, and 108.

20. 42, 21, and -84.

Combine the following fractions, stating your result as a fraction in lowest terms.

21. $\frac{1}{3} + \frac{1}{8} - \frac{1}{12}$.

23. $\frac{4}{15} - \frac{7}{20} + \frac{19}{30}$.

25. $\frac{1}{3} + \frac{1}{4} - \frac{3}{5} + \frac{5}{6}$.

27. $3 + \frac{1}{18} - \frac{5}{12}$.

29. $\frac{5}{8} \cdot \frac{19}{25}$.

31. $-\frac{3}{8} \cdot \frac{1}{5} \cdot \frac{20}{9}$.

22. $\frac{1}{8} + \frac{3}{12} - \frac{5}{16}$.

24. $\frac{9}{22} - \frac{5}{8} + \frac{1}{11}$.

26. $-\frac{1}{8} + \frac{5}{12} - \frac{1}{32}$.

28. $\frac{1}{14} - \frac{5}{21} + \frac{3}{49}$.

30. $\frac{3}{7} \cdot (-\frac{21}{9}) \cdot (-\frac{13}{8})$.

32. $(-\frac{5}{12}) \cdot (-\frac{3}{5}) \cdot \frac{4}{3}$.

33. $\frac{\frac{21}{36}}{\frac{7}{13}}$.

35. $\frac{\frac{-96}{49}}{\frac{-21}{144}}$.

37. $\frac{3 + 7}{\frac{5}{8}}$.

39. $\frac{\frac{3}{7} + \frac{4}{5}}{\frac{7}{8} + \frac{5}{4}}$.

41. $\frac{11 - \frac{3}{4} + \frac{1}{8}}{11 + \frac{3}{4} - \frac{1}{8}}$.

43. $\frac{13 - \frac{1}{3} + \frac{5}{6}}{5 + \frac{3}{4} + \frac{1}{8}}$.

45. $\frac{\frac{8}{5} - \frac{7}{10}}{\frac{5}{6}} \cdot \frac{\frac{3}{5} - \frac{1}{8}}{\frac{5}{24}}$.

47. $-5 \cdot \frac{\frac{1}{3} - \frac{1}{8}}{\frac{2}{3} + \frac{1}{10}} \cdot \frac{5}{6}$.

34. $\frac{\frac{-81}{144}}{\frac{18}{24}}$.

36. $-\frac{\frac{81}{42}}{\frac{-9}{13}}$.

38. $\frac{3 + \frac{4}{3}}{14 + \frac{14}{5}}$.

40. $\frac{2 + 3 + \frac{2}{3}}{\frac{2}{3} - \frac{7}{8} + \frac{1}{2}}$.

42. $\frac{3 + \frac{3}{5} - \frac{5}{2}}{1 + \frac{1}{10} - \frac{3}{4}}$.

44. $\frac{\frac{2}{3} + \frac{1}{2}}{7} \cdot \frac{6}{5}$.

46. $\frac{\frac{5}{3} + \frac{1}{5} - \frac{7}{15}}{\frac{1}{10}} \cdot \frac{\frac{3}{5}}{\frac{3}{5}}$.

48. An empty truck weighs 3.5 tons. If the truck has 6 tires, what weight does each tire carry when the truck is carrying 5.5 tons of cargo? Assume equal distribution of weight.

49. The cost of electrical energy is 6 cents for each kilowatt used throughout one hour or 6 cents per kilowatt hour. An electric iron used 0.45 kilowatt. Six lamps consume 0.075 kilowatt each. What is the cost per hour of operating both the iron and lamps? What is the cost of operating them for 3 hours and 20 minutes?

50. Kirchhoff's law states that the sum of the electric currents entering the junction point of several conductors is zero. Consider three wires coming together at a point. One of them carries a current of 4 amperes to the point; another carries 3 amperes away from the point. What is the current in the third wire? Assume that currents going toward the junction are positive, away from the junction negative.

51. The average rainfall in a certain region is 60 in. per year. If three-eighths of the water which falls is evaporated by the sun, how many inches per year are left to be absorbed by the soil and to feed streams?

52. A locomotive pulling a loaded train which weighs 186 tons consumes 104 lb. of coal per square foot of grate surface per hour, when traveling 35 miles per hour. If the grates of the locomotive firebox measure 4 by 8 ft., how many pounds of coal per hour is used? How many pounds of coal per mile? If six-tenths of the weight of the train is pay load, how many pounds of coal are used per ton of pay load per mile?

53. A piece of timber measuring 6 in. by 8 in. in cross section is used as a structural member to support a load. Each square inch of the member carries a load of 20,000 lb. What is the total load on the timber?

54. A steel cable used in a mine hoist has a cross section of 0.8 sq. in. The weight of the cage is 1000 lb. and the load is 2500 lb. What is the total load per square inch on the cable?

55. In a street car, a current of 60 amperes is required for the driving motors, 1.5 amperes for the air pump motor, and 2.2 amperes for the lights. What is the total current drain through the trolley wire?

56. One hundred and forty street cars, each requiring the current given in Exercise 55, are operated on the same system. The generating plant must be able to provide for twice the normal load. How many amperes must the generating plant be able to supply?

57. In a six-tube radio receiver, three tubes require 12 milliamperes, one requires 1.5 milliamperes, and two require 25 milliamperes. What is the total current drain?

58. A fully loaded 10-horsepower motor draws 120 amperes from the line. Assuming that the current drain is proportional to the load, what is the current drain when the motor is operated at 7.5 horsepower?

59. A rectangular tank 10 ft. wide and 22 ft. long supplies boiler feed water at a generating plant. How many cubic feet of water are used if the water level in the tank drops 8 in.? If there are 0.160 cu. ft. of water per gallon, how many gallons of water are used?

60. The weight of cast iron is 0.25 lb. per cubic inch. What is the weight of a cast-iron block 6 in. by 8 in. by 1.2 ft.?

61. A public address system employs a vacuum tube amplifier in which one-fifteenth of the total power supplied to the amplifier is converted to audio power output (sound energy). If 10 watts of audio output are obtained, how many watts does the amplifier consume? What is the cost per hour of operation if electric energy costs 6 cents per kilowatt hour? (See Problem 49.)

1-11. Exponents. Exponents are introduced as an abbreviation for the product of several like factors. For example, in place of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we write 2^5 . The reader will see, by the time he has finished this book, that the exponent idea has become a tool of great usefulness, both in theory and in applications.

If a is any number and n is a positive integer, the product of n of the quantities a is denoted by a^n . The following table gives the definitions needed here.

$$\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} = a^n$$

where n is called the exponent,
 a is called the base.

$$a^n \text{ is read } \begin{cases} \text{"}a \text{ to the exponent } n\text{"}, \\ \text{"}a \text{ to the } n\text{th power,"} \\ \text{"the } n\text{th power of } a\text{."} \end{cases}$$

When a quantity with its sign of quality is written with an exponent, it is necessary to include the quantity in parentheses to avoid ambiguity of meaning; for example, $+2^4 = (+2)^4 = 16$, and $-2^4 = -16$, whereas $(-2)^4 = +16$.

If m and n are positive integers, we have, from our definition,

$$\begin{aligned} a^n \cdot a^m &= \underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} \cdot \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \\ &= \underbrace{a \cdot a \cdots a}_{m+n \text{ factors}}, \end{aligned}$$

whence

$$a^n \cdot a^m = a^{m+n}.$$

Now if $n > m$, we have then

$$\begin{aligned} \frac{a^n}{a^m} &= \frac{(a \cdot a \cdots a) \quad (n \text{ factors})}{(a \cdot a \cdots a) \quad (m \text{ factors})} \\ &= a \cdot a \cdots a \quad (n - m \text{ factors}). \end{aligned}$$

and

$$\frac{a^n}{a^m} = a^{n-m}.$$

The reader may also show very readily that

$$(a^n)^m = a^{m \cdot n}.$$

Thus the addition of exponents takes the place of multiplication; subtraction of exponents takes the place of division; and multiplication of exponents takes the place of raising a quantity to a power.

As a consequence of the definition, if n is a positive integer,

$$\begin{aligned}(a \cdot b)^n &= (a \cdot b) \cdot (a \cdot b) \cdots (a \cdot b) \quad n \text{ factors} \\ &= \underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} \cdot \underbrace{(b \cdot b \cdots b)}_{n \text{ factors}}\end{aligned}$$

which gives

$$(a \cdot b)^n = a^n \cdot b^n.$$

Consider now

$$\frac{a^3}{a^5} = \frac{\phi \cdot \phi \cdot \phi}{\phi \cdot \phi \cdot \phi \cdot a \cdot a} = \frac{1}{a^2}.$$

If we make use formally of the rule of exponents for division, disregarding the requirement that the exponent of the numerator be larger than the exponent of the denominator, we get

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}$$

Now a^{-2} has not yet been defined, but, since $\frac{a^3}{a^5} = \frac{1}{a^2}$, we are led naturally to define $a^{-2} = \frac{1}{a^2}$. More generally, if n is a positive integer, we define

$$a^{-n} = \frac{1}{a^n}.$$

Thus changing the sign of the exponent converts any number to its reciprocal.

Similarly, since on one hand $\frac{a^4}{a^4} = 1$ and on the other $\frac{a^4}{a^4} = a^{4-4} = a^0$, we are led to the general definition,

$$a^0 = 1, \text{ if } a \neq 0.$$

Having a meaning now for negative integral and zero exponents, we may abandon the requirement that $n > m$ in the quotient rule and state simply that

$$\frac{a^n}{a^m} = a^{n-m}.$$

Similarly, the reader will see readily that the requirement that m and n be positive may now be omitted in all the rules.

We have now arrived at an extension of the idea of a positive integral exponent which includes negative integral exponents and zero exponents, and the processes of multiplication, division, and raising to positive and negative integral and zero powers. For present considerations, these concepts and principles will suffice, but a further extension will be made in Chapter 8.

Summarizing:

If a and b are any numbers, and m and n are integers, we have:

Definitions: $\underbrace{a \cdot a \cdots a}_{n \text{ factors}} = a^n$ (if n is positive).

$$a^{-n} = \frac{1}{a^n}$$

$$a^0 = 1, a \neq 0.$$

Rules:

$$a^n \cdot a^m = a^{m+n}.$$

$$\frac{a^n}{a^m} = a^{n-m}.$$

$$(a^n)^m = a^{m \cdot n}.$$

$$(a \cdot b)^n = a^n \cdot b^n.$$

Five other important consequences of these rules are worth stating.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

$$\frac{a^n}{a^m} = \frac{1}{a^{m-n}} = a^{n-m}.$$

$$(a^n \cdot b^m)^q = a^{n \cdot q} \cdot b^{m \cdot q}.$$

$$\frac{a^{-n} \cdot b}{c} = \frac{b}{a^n \cdot c}.$$

$$\frac{a}{b^{-n} \cdot c} = \frac{a \cdot b^n}{c}.$$

Example 1. $2^3 \cdot 3^2 \cdot 2^8 \cdot 3^5 = 2^3 \cdot 2^8 \cdot 3^2 \cdot 3^5 = 2^{3+8} \cdot 3^{2+5} = 2^{11} \cdot 3^7.$

Example 2. $\frac{3^4 \cdot 2^8}{3^7 \cdot 2^2} = \left(\frac{3^4}{3^7}\right) \cdot \left(\frac{2^8}{2^2}\right) = 3^{4-7} \cdot 2^{8-2} = 3^{-3} \cdot 2 = \frac{2}{3^3}.$

Example 3. $\left(\frac{6^2 \cdot 2}{3^4 \cdot 2}\right)^3 = \left(\frac{[2 \cdot 3]^2 \cdot 2}{3^4 \cdot 2}\right)^3 = \left(\frac{2^2 \cdot 3^2 \cdot 2}{3^4 \cdot 2}\right)^3 = \left(\frac{2^3 \cdot 3^2}{2 \cdot 3^4}\right)^3$
 $= [(2^{3-1}) \cdot (3^{2-4})]^3 = (2^2 \cdot 3^{-2})^3 = \left(\frac{2^2}{3^2}\right)^3 = \frac{2^6}{3^6}.$

It is often of considerable help in simplifying an expression involving exponents to reduce all the integers involved to the products of their prime factors. The previous example is an illustration, as well as the following one.

Example 4.

$$\begin{aligned}\frac{12^2 \cdot 5^2 \cdot 60}{3^5 \cdot (-2)^4 \cdot (-5)^3} &= \frac{(2^2 \cdot 3)^2 \cdot 5^2 \cdot (2^2 \cdot 5 \cdot 3)}{3^5 \cdot 2^4 \cdot (-1)^4 \cdot 5^3 \cdot (-1)^3} \\ &= \frac{2^4 \cdot 3^2 \cdot 5^2 \cdot 2^2 \cdot 5 \cdot 3}{3^5 \cdot 2^4 \cdot 5^3 \cdot (-1)} = \frac{2^6 \cdot 3^3 \cdot 5^3}{-2^4 \cdot 3^5 \cdot 5^3} \\ &= -2^{6-4} \cdot 3^{3-5} \cdot 5^{3-3} = -2^2 \cdot 3^{-2} \cdot 5^0 \\ &= -\frac{2^2}{3^2}.\end{aligned}$$

EXERCISES

1. Write the following numbers as powers of 10: 10,000,000; 100; 1; 0.1; 0.0001.
2. Compute the value of the following expressions: 10^8 , 10^3 , 10^{-2} , 10^{-10} .

Simplify, using the laws of exponents. State all your results in terms of prime integers and positive exponents.

- | | |
|---|---|
| 3. $3^5 \cdot 3^2$. | 4. $2^8 \cdot 2^3 \cdot 2^{-6}$. |
| 5. $2^5 \cdot 2^3 \cdot 2^{-9}$. | 6. $3^5 \cdot 3^2 \cdot 3^{11}$. |
| 7. $2^2 \cdot 3^2 \cdot 4^2$. | 8. $5^6 \cdot 4^3 \cdot 2^5 \cdot 5^{-7}$. |
| 9. $\frac{7^4 \cdot 5^3 \cdot 6}{5^2 \cdot 7^3 \cdot 2}$. | 10. $\frac{3^7 \cdot 4^6 \cdot 5^5 \cdot 6^4}{5^5 \cdot 2^{12} \cdot 3^7 \cdot 36^2}$. |
| 11. $\frac{(-2)^3 \cdot 8 \cdot 18 \cdot 3^6}{2^8 \cdot 3^5}$. | 12. $\frac{8 \cdot 15 \cdot 10^8}{10^6 \cdot 0.25 \cdot 2^3}$. |
| 13. $\frac{(25)^{-3} \cdot (-25)^3}{10^2}$. | 14. $\frac{49 \cdot 64}{14^3}$. |
| 15. $\frac{64 \cdot 10^{-5}}{(25)^{-3}}$. | 16. $\frac{10^8 \cdot 10^{-6} \cdot 3^3}{-10^2 \cdot 27}$. |
| 17. $\frac{124 \cdot 2^5}{-31 \cdot 2^{10}}$. | 18. $\frac{-144 \cdot 128}{2^{15} \cdot 3^3}$. |
| 19. $\frac{(-18)^{-4} \cdot 27^2}{2^{-5} \cdot 3}$. | 20. $\frac{36 \cdot 144}{12^5}$. |
| 21. $\frac{12^{-3} \cdot (-25)^3}{6^{-4} \cdot 100^2}$. | 22. $\frac{(-60)^3}{125 \cdot 15^{-3} \cdot 3^4}$. |
| 23. $\frac{14^{-3} \cdot 2^5}{(-7)^{-4} \cdot 3}$. | 24. $\frac{-6 \cdot (-18) \cdot 48 \cdot 144}{(-3)^8 \cdot 2^{10}}$. |
| 25. $\frac{(-35)^2 \cdot 49}{7^5 \cdot 5^2}$. | 26. $\frac{128 \cdot 144 \cdot (-12)^{-2}}{(-2)^8}$. |
| 27. $\frac{28 \cdot 56 \cdot 108}{2^5 \cdot 7 \cdot 3^6}$. | |

1-12. Scientific and Engineering Notation of Numbers. Very large or very small numbers are frequently encountered in scientific work. The distance of the earth from the sun is 93,000,000 miles. Light travels at a speed of about 300,000,000 meters per second. The number of molecules in a cubic centimeter of gas at a temperature of 0° Centigrade and a pressure of 76 centimeters of mercury is about 27,050,000,000,000,000,000. The thickness of an oil film on water is 0.000,000,2 in. Scientific notation provides a convenient scheme for representation of such quantities.

A positive number is expressed in scientific notation when it is written as the product of an integral power of 10 and a number between 1 and 10. A number in scientific notation has the form

$$M \times 10^n$$

where M is a number between 1 and 10, and n is an integer.

The quantities already mentioned, which are in ordinary (or **positional**) notation, may be written in scientific notation as follows.

$$93,000,000 = 9.3 \times 10,000,000 = 9.3 \times 10^7,$$

$$300,000,000 = 3 \times 100,000,000 = 3 \times 10^8,$$

$$\begin{aligned} 27,050,000,000,000,000,000 &= 2.705 \times 10,000,000,000,000,000,000 \\ &= 2.705 \times 10^{19}, \end{aligned}$$

$$0.000,000,2 = 2 \times 0.000,000,1 = 2 \times 10^{-7}.$$

The student may verify the following practical rule.

To change the form of a number from positional to scientific notation:

(a) Move the decimal point to a position such that only one non-zero digit appears to its left, thus obtaining a number between 1 and 10.

(b) Multiply this number by 10^n , where $|n|$ equals the number of places the decimal point has been moved, n being positive if the decimal point has been moved to the left, and n being negative if the decimal point has been moved to the right.

A second rule for determining the value of the exponent n is often used:

(a) If the number is greater than 1, then n is 1 less than the number of digits to the left of the decimal point.

(b) If the number is less than 1, then n is negative and its absolute value is 1 greater than the number of zeros between the decimal point and the left-most non-zero digit.

The rule for writing in positional notation a number expressed in scientific notation can be readily inferred from the rules given above.

Multiplication and division of very large or very small numbers can be simplified by the use of scientific notation and the laws of exponents. Scientific notation can also be used to advantage in slide rule computations.

Table 8 in the Appendix lists some of the units which are used in scientific and engineering practice.

EXERCISES

Express each of the quantities in the following statements in scientific notation, dropping all the zeros after the right-most non-zero digit.

1. The age of the earth is estimated to be about 694,000,000,000 days.
2. The diameter of the earth is approximately 7930 miles.
3. The number of atoms in 1.008 grams of hydrogen is estimated to be 606,000,000,000,000,000,000,000.
4. A light-year, the distance that light travels in a year, is approximately 5,870,000,000,000 miles.
5. The mass of a water molecule is estimated to be 0.000,000,000,000,000,000,03 gram.
6. The distance from the earth to the moon is 240,000 miles.
7. The diameter of the smallest visible particle is about 0.005 cm.
8. The thickness of an oil film is about 0.000,000,5 cm.
9. The diameter of the Einstein universe according to the theory of relativity is 2,000,000,000 light-years.
10. The distance from the earth to the sun is approximately 149,000,000 km.

Express each of the quantities in the following statements in positional notation:

11. The mass of the earth is approximately 6.6×10^{21} tons.
12. The charge on the electron is about 4.77×10^{-10} electrostatic unit.
13. The diameter of an electron is estimated to be about 4×10^{-13} cm.
14. The mass of the electron is indicated by experimental evidence to be 9×10^{-28} gram.
15. It has been estimated that a pair of electrons placed at a distance of 1 cm. from each other repel each other with force of 2.27×10^{-19} dyne.
16. The proton weighs about 1.66×10^{-24} gram.
17. There are about 6×10^{23} protons in every gram of matter.
18. The estimated diameter of the average atom is 2×10^{-8} cm.
19. It has been estimated that in ordinary air every atom collides with another about 6×10^9 times per second.
20. The length of time that a motion picture image is on the screen is approximately 6.4×10^{-2} second.
21. The diameter of the average red blood corpuscle is 8×10^{-5} cm.
22. The speed of the earth in its orbit is approximately 9.768×10^4 ft. per second.
23. The speed of the planet Mercury in its orbit is approximately 2.76×10^4 km. per second.
24. The greatest rate of plant growth is about 3×10^{-2} mm. per second.
25. The radius of the star Betelgeuse is about 1.12×10^8 miles.
26. The radius of the sun is approximately 6.9×10^5 km.

1-13. Significant Figures. The **digits** are the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. **Figure** is synonymous with digit.

A certain switchboard voltmeter has primary and secondary division marks on its scale; the distance between two adjacent primary marks corresponds to a difference of 10 volts; and the difference between two adjacent secondary marks corresponds to a difference of 5 volts.

Let us suppose that a reading of 220 volts is taken on this meter by simply noting the number corresponding to the primary division mark nearest the needle. By this rough reading we are assured only that the actual value is closer to 220 volts than to 230 volts or to 210 volts. We know that no matter how the reading is refined the digit 2 in the hun-

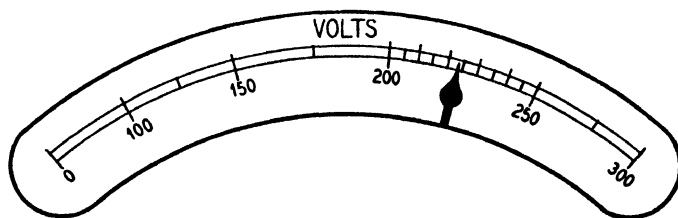


FIG. 1-3.

dred's place is accurate. In general, *a digit in an observed quantity which will not be changed by a refinement of the observation is called an **accurate digit***. Since by the reading taken on the meter we know only that the correct value is closer to 220 than to 210 or to 230, a refinement of the reading might give any number from 215 to 225. Thus the digit in the ten's place might become a 1 instead of a 2, and the digit in the unit's place might be any of the digits. We say that these digits are **inaccurate**. In general, *a digit in an observed quantity which may be changed by a refinement of the observation is called an **inaccurate digit***.

As we have made the reading, the only digits in the number 220 of any significance for computations are the first two. The zero merely serves as an occupant of the unit's place.

Now suppose we inspect the meter scale more carefully, estimating to the nearest unit where the needle lies between secondary scale marks, and obtain a reading of 224 volts. Here all three digits have significance, but the reading means only that the actual value is closer to 224 volts than to 223 volts or to 225 volts. In the reading 224, the digits in the ten's and hundred's places are accurate, and the unit's place digit is inaccurate.

Observational data in this way is never absolutely accurate, many possibilities of error entering into almost any given experiment. Num-

bers obtained from experiments thus contain accurate digits followed by inaccurate digits. Also, as we shall see later, computations with numbers of limited accuracy contain errors which limit the accuracy of results. Since it is important to know which digits of a result are accurate and which are inaccurate, we shall investigate the influence of the accuracy of given data upon results computed from this data.

In the paragraphs above we have called certain digits significant. In order to be a little more precise, we shall agree upon the following criterion, which will be used throughout this discussion to distinguish between the significant and non-significant digits of quantities which are of limited accuracy.

A digit is significant if the maximum error in the number in which it is contained is less than or at most equal to one-half of one unit in the place which the digit occupies.

Thus in the reading of 220 volts given above, the maximum error is 5 volts, which is $\frac{1}{2} \times 10$, making the 2 in the ten's place significant, and also the 2 in the hundred's place. In the reading of 224 volts the maximum error is $\frac{1}{2} \times 1$, which makes the 4 significant, as well as all the integers preceding it.

All the accurate digits are significant. Usually, as in the examples above, the *right-most significant digit is inaccurate*. Consequently, the right-most significant digit will be called the *inaccurate significant digit*, and sometimes for the sake of simplicity, when only significant digits are under consideration, the *inaccurate digit*.

It is worth while to establish notational conventions which make clear how many digits of a given number are significant. Two such agreements are in common use.

1. *The significant figures in a number in positional notation consist of:*

a. *All non-zero digits.*

b. *Zero digits which:*

(1) *Lie between significant digits;*

(2) *Lie to the right of the decimal point, and at the same time lie to the right of a non-zero digit;*

(3) *Are specifically indicated by the context to be significant.*

2. *The significant figures in a number written in scientific notation ($M \times 10^n$) consist of all the digits expressed explicitly in M .*

Significant figures are counted from left to right, starting with the left-most non-zero digit.

Example.

NUMBER	SIGNIFICANT FIGURES	NUMBER OF SIGNIFICANT FIGURES
35.62	3, 5, 6, 2	4
5.600	5, 6, 0, 0	4
3020	3, 0, 2	3
0.00046	4, 6	2
0.000850	8, 5, 0	3
3.0080	3, 0, 0, 8, 0	5
5.00	5, 0, 0	3
9.3×10^7	9, 3	2
5.00×10^8	5, 0, 0	3
2.705×10^{10}	2, 7, 0, 5	4
2×10^{-7}	2	1

EXERCISES

State the number of significant figures, according to the agreements of this section, in the numbers given in the exercises following Sec. 1-12.

1-14. Accuracy in Computations. In computations involving observed data accurate only to a certain number of significant figures, it is important to be able to judge how many figures of the result are significant. Simple considerations form the basis of such a judgment. These will be pointed out here through examples, and specific rules will be given. However, the student should regard these rules not as inflexible, but simply as the most convenient expression of the common sense which should govern any computation in practical work.

Given three numbers 2.88×10^3 , 4.346×10 , and 1.376×10^{-1} which are assumed to have been obtained by measurement or experimental means and are therefore accurate only to the number of figures indicated in accordance with the agreements of the preceding section. Arithmetically, the sum of these numbers is given by:

$$\begin{array}{r}
 2.88 \times 10^3 = 2,880 \\
 4.346 \times 10 = 43.46 \\
 1.376 \times 10^{-1} = 0.1376 \\
 \hline
 2,923.5976
 \end{array}$$

At greatest, the error in 2880 is 5, the error in 43.46 is 0.005, and the error in 0.1376 is 0.00005. Thus the greatest possible total error in the sum would be 5.00505. However, the second two errors are so small compared with the first that they may well be ignored. Hence the greatest error in the sum is about 5.

Since the error in the sum may be as much as 5, the sum may be as much as 2928 or as little as 2918. Hence the last six digits of the sum are inaccurate. The last five digits are not significant and may be discarded by *rounding off* the sum to its three significant figures, expressing the result as a number of three significant figures nearest to 2,923.5976. This number is 2920, or 2.92×10^3 .

For addition and subtraction we may state the following rule.

The right-most significant figure in a sum (or difference) occurs in the left-most place at which an inaccurate digit occurs in any of the numbers entering into the sum (or difference).

Given two numbers 3.71×10^2 and 8.216×10^3 which are accurate only to the number of significant figures indicated in accordance with the previous agreements. The product is $3.71 \times 10^2 \times 8.216 \times 10^3 = 3.71 \times 8.216 \times 10^5$. This multiplication is carried out by the ordinary process of arithmetic as follows.

$$\begin{array}{r}
 8.216 \\
 \underline{3.71} \\
 8216 \qquad (1) \\
 57512 \qquad (2) \\
 \underline{24648} \qquad (3) \\
 30.48136 \qquad (4)
 \end{array}$$

Since line (1) is really the product of 8.216 and 0.01, line (2) is the product of 8.216 and 0.7, and line (3) is the product of 8.216 and 3, this multiplication can be written:

$$\begin{array}{r}
 8.216 \\
 \underline{3.71} \\
 0.08216 \qquad (5) \\
 \underline{5.7512} \qquad (6) \\
 24.648 \qquad (7) \\
 \hline
 30.48136 \qquad (8)
 \end{array}$$

Now in line (5) the inaccuracy of 3.71 has been multiplied by 8.216. At worst, if the inaccuracy of 3.71 is 0.005, the inaccuracy of the product in line (5) is about $8.216 \times 0.005 = 0.041080$, whence the number in line (5) could be as little as 0.04108 or as much as 0.12324. Thus the digit in the tenth's place in line (5) is inaccurate, and all the underlined digits in line (5) are inaccurate. Similarly, at worst the inaccuracy of 8.216 is 0.0005, making the inaccuracy in line (6) about $0.0005 \times 0.7 = 0.00035$, and making the inaccuracy in line (7) about $0.0005 \times 3 =$

0.0015. Arguing as above, we can show that the underlined digits in lines (6) and (7) are inaccurate. Since line (8) is the sum of the lines (5) (6), and (7), its inaccuracy may be as much as $0.04108 + 0.00035 + 0.0015 = 0.04293$, or slightly less than 0.05. It follows then that 30.48136 is inaccurate in the first place after the decimal point, and that only the first three digits of the product are significant. Rounding off to three significant figures, the result is 30.5. The result of the original problem is then

$$\begin{aligned} 3.71 \times 10^2 \times 8.126 \times 10^3 &= 3.71 \times 8.126 \times 10^5 \\ &= 30.5 \times 10^5 = 3.05 \times 10^6. \end{aligned}$$

The reader may study what happens in division in like manner, and will find that the following rule is generally valid.

The product (or quotient) of numbers is accurate at most to the number of significant figures contained in the least accurate factor. The least accurate factor is the number entering into the computation which has the least number of significant figures.

In the preceding example the result was accurate at most to three figures. The product of 6.8×10^2 and 4.5186×10 will be accurate at most to two figures, since 6.8×10^2 is accurate only to two figures.

Examples can be easily constructed which deviate from the rules given above, but for the usual numerical computations used in engineering these rules are valid, or very nearly valid. In order to avoid a tedious analysis of every computation, it is convenient to adopt these rules as a practical working criterion for the accuracy of results.

1-15. Rounding off Numbers. In rounding off a number after a computation, the number is chosen which has the required number of significant figures and which is closest to the number to be rounded off. *When either of two numbers equally close to the number to be rounded off can be chosen, we shall adopt the convention of choosing the one ending in an even digit.*

Example.

Number	Rounded off to		
	Three figures	Four figures	Five figures
0.666666	0.667	0.6667	0.66667
0.312341	0.312	0.3123	0.31234
51.2155	51.2	51.22	51.216
26.5455	26.5	26.55	26.546
18.3545	18.4	18.35	18.354

1-16. Computation with Numbers of Limited Accuracy. In order to achieve in numerical computations an economy of effort and a uniform accuracy commensurable with the accuracy of the numbers given, the directions given below should be followed.

1. Addition and subtraction. Perform all the operations with the complete numbers given, and then round off the final result. Round off the result so that its right-most significant figure appears in the left-most place at which an inaccurate figure occurs in any number entering into the sum or difference.

2. Multiplication. Perform all the operations with the complete numbers given, and then round off the final result to the number of significant digits in the least accurate factor.

3. Division. Perform the operations with the complete quantities given, obtaining for each division an answer to one more digit than the number of significant digits in the least accurate given quantity. Then round off the final result to the number of significant figures in the least accurate given quantity.

4. Combined operations. When multiplications and divisions both occur in an expression, perform the multiplications first.

EXERCISES

In the following exercises assume that the numbers are accurate only to the number of digits indicated in accordance with the agreements of Sec. 1-13. Round off your results in accordance with the rules of Sec. 1-16, and state your results in scientific notation.

Add:

- | | |
|-----------------------------------|----------------------------------|
| 1. 897, 6.92, and 86.4. | 2. 96, 8.967, and -672 . |
| 3. 89, 67.2, and -865 . | 4. 483, 97.2, and 87.6. |
| 5. 0.00864, 0.8946, and 0.006725. | 6. 0.08694, 0.0964, and 84,689. |
| 7. 46.8, -36.42 , and 56.12. | 8. 18.37, -16.9 , and 890. |
| 9. 8900, 360, and -672 . | 10. 0.00500, 0.8760, and 0.8765. |

Multiply:

- | | |
|-----------------------|-------------------------|
| 11. 8.6 by 9.23. | 12. 9.3 by 18.7. |
| 13. 2.31 by 1.121. | 14. -3.76 by 0.00801. |
| 15. 0.00300 by 0.960. | 16. 3600 by 0.050. |
| 17. 18,000 by 0.051. | 18. 976 by 0.02. |
| 19. 876 by 0.672. | 20. 19.6 by 1.2. |

Divide:

- | | |
|---------------------|--------------------|
| 21. 8.2 by 1.12. | 22. 3.2 by 19.68. |
| 23. 18.6 by 6.2. | 24. 97.6 by 0.061. |
| 25. 8900 by 0.012. | 26. 672 by 0.97. |
| 27. 36,000 by 2800. | 28. 27 by 18.2. |
| 29. 36 by 97.2. | 30. 929 by 0.18. |

Add:

31. 1.87×10^2 , 1.89×10^3 , and 1.672 .
32. 1.62×10^3 , 3.69×10^2 , and 1.02×10^4 .
33. -8.9×10 , 6.7×10^2 , and -3.25×10^2 .
34. 9.986×10^2 , 1.23×10^{-1} , and 5.67×10^{-2} .
35. 8.86×10^3 , -8.967×10^2 , and -9.67×10^5 .
36. 3.32×10^5 , -1.25×10^{-4} , and -3.26×10^{-4} .
37. 8.967×10 , -3.462 , and 8.814×10^2 .
38. 3.267×10^2 , -8.94×10^3 , and -3.46×10^3 .
39. 4.765×10^{-3} , -8.32×10^{-4} , and 8.964×10^{-4} .
40. 5.672×10^5 , 3.15×10^3 , and 4.672×10^3 .

Perform the indicated operations in accordance with the general directions above.

- | | |
|---|---|
| 41. $\frac{(1.23 \times 10^{-8}) \cdot (1.5 \times 10^6)}{3.2 \times 10^5}$. | 42. $\frac{1.56 \times 10^3}{8.9 \times 10^{-3}}$. |
| 43. $\frac{8.97 \times 10^5}{(1.2 \times 10^3) \cdot (1.36 \times 10^7)}$. | 44. $\frac{8.67 \times 10^3}{1.3 \times 10^6}$. |
| 45. $\frac{6.72 \times 10^5}{(8.1 \times 10^2) \cdot (3.2 \times 10^3)}$. | 46. $\frac{(1.987) \cdot (8.6 \times 10^3)}{1.26 \times 10^5}$. |
| 47. $(1.96 \times 10^3) \cdot (1.83 \times 10^6)$. | 48. $\frac{(6.72 \times 10^3) \cdot (8.91 \times 10^{-5})}{1.6 \times 10^{-2}}$. |
| 49. $\frac{(1.378 \times 10^2) \cdot (8.967)}{3942}$. | 50. $\frac{896 \times 6.759}{3160}$. |

1-17. Description of the Slide Rule. The slide rule is an instrument on which the processes of multiplication, division, finding squares, and finding of square roots may be performed to three-figure accuracy simply and quickly. Other computations may also be performed, as will be seen later, but in this chapter we will be concerned only with the four processes mentioned. However, addition and subtraction may not be performed on the slide rule. The theory governing the construction of a slide rule will be found in Chapter 9; only its operation will be discussed here.

We shall discuss the 10-in. Mannheim type slide rule, named after its inventor Lieutenant Mannheim of the French artillery, who devised it in 1859. However, the techniques discussed here may be applied to any rule which the reader has in his possession.

The central sliding part of the rule is the **slide**; the part surrounding it is the **body**. The glass runner is the **indicator**, and the fine line on the indicator is the **hairline**. The mark associated with the primary number 1 on any scale is the **index** of the scale. Two positions on different scales are **opposite** for a given position of the slide if the hairline may be brought to cover both positions simultaneously, without moving the slide.

The Mannheim rule has the four scales *A*, *B*, *C*, and *D*, on one side. The *A* and *B* scales are identical, and the *C* and *D* scales are identical.

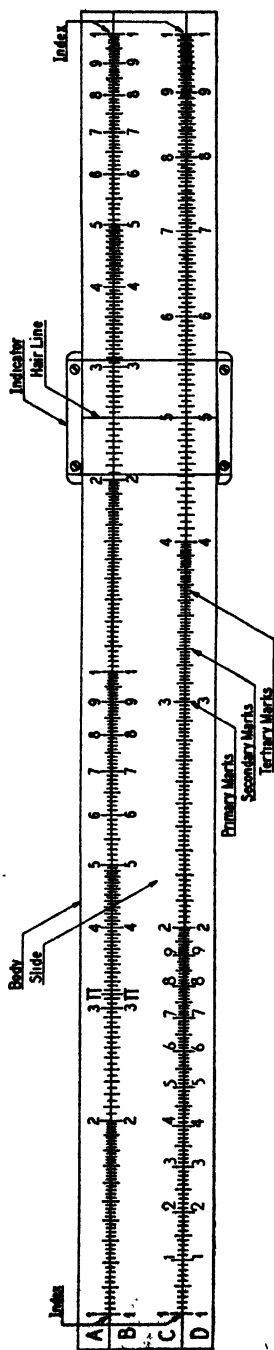


FIG. 1-4.

The *B* and *C* scales are on a slide which can be moved back and forth (see Fig. 1-4). The distances between successive digits are unequal. The distance between successive integers on the *C* and *D* scales is double the distance between the same integers on the *A* and *B* scales.

Since such wide variations in scales exist (even among different makes of Mannheim rules), the following remarks are general suggestions which each user may follow in interpreting his own slide rule.

1. Every *C* or *D* scale is divided into nine principal divisions by **primary marks** numbered 1, 2, 3, ..., 9, 1. The space between any two primary marks is divided into ten parts by nine **secondary marks**. These are not numbered except between the primary marks 1 and 2. Between the secondary marks appear unnumbered **tertiary marks**; some spaces between secondary marks are divided into ten parts, some into five parts, and others into two.

2. Every *A* or *B* scale has two portions, identical with each other, which are each divided into nine principal divisions by primary marks numbered as on the *C* and *D* scales. Unnumbered secondary and tertiary marks also appear on these scales.

3. Every *K* scale has three portions, identical with each other, divided much like the two portions of the *A* and *B* scales.

On none of these scales is the distance between consecutive numbers equal, the distances being greater on the left end of the scales. Thus there are more subdivisions toward the left. The following points should be carefully noted.

1. Where there are ten subdivisions of a given division, each subdivision counts 0.1 of the value of the given division.
2. Where there are five subdivisions of a given division, each subdivision counts 0.2 of the value of the given division.

3. Where there are two subdivisions of a given division, each subdivision counts 0.5 of the value of the given division.

Many errors are made in reading the slide rule because of failure to observe these facts. Practice, which leads to the ability to recognize at a glance the values of the different markings, is the only answer. As we shall see, the reading between any successive tertiary marks must be based on an estimate. Hence in making these estimates it is necessary to have firmly in mind the number associated with the space under consideration.

1-18. Locating Numbers on the Slide Rule. It is important to observe that the decimal point has no bearing upon the position associated with a number on the *C* and *D* scales, and, in certain operations, on the *A* and *B* scales.

The following examples indicate how to locate three- and four-digit numbers on the *D* scale. Only that portion of the scale is shown which contains the number in question. The subdivisions are shown as usually found on a 10-in. rule *at that place on it*. The reader will see immediately the application of these ideas to the location of numbers on the other scales.

Example 1. Locate 151.

This is easily done since in this region the space between 1 and 2 is divided into ten parts and each of these into ten more parts.

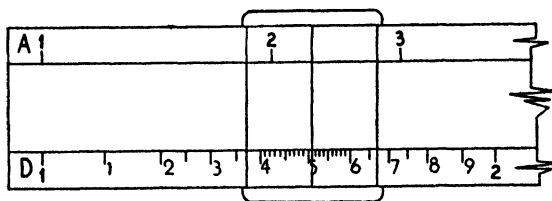


FIG. 1-5.

Example 2. Locate 251.

This is a little less easily done since the last subdivision consists of only five marks, each of which counts two units. Since our last digit is 1, we must estimate halfway between two marks.

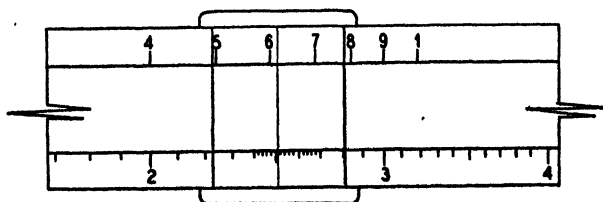


FIG. 1-6.

Example 3. Locate 1516.

The number 1510 is located as 151 was located in Example 1. To locate 1516 we estimate 0.6 of the distance from 1510 to 1520.

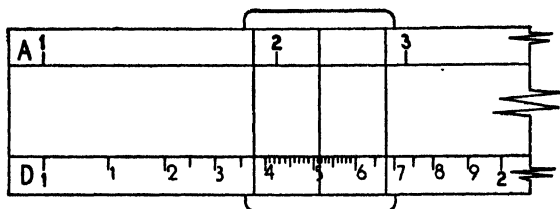


FIG. 1-7.

EXERCISES

Locate the following numbers on the *D* scale. Then locate each one on the *A* scale.

- | | | | |
|------------|-----------|-----------|------------|
| 1. 20. | 2. 25. | 3. 350. | 4. 124. |
| 5. 11. | 6. 341. | 7. 945. | 8. 846. |
| 9. 317. | 10. 246. | 11. 3.46. | 12. 0.985. |
| 13. 7.65. | 14. 2.45. | 15. 1265. | 16. 137.5. |
| 17. 18.60. | 18. 676. | 19. 8.97. | 20. 12.68. |
| 21. 34.7. | 22. 969. | 23. 465. | 24. 1246. |

1-19. Accuracy of the Slide Rule. Between the primary marks 1 and 4 on the *D* scale of a 10-in. rule, it is possible to read four figures, although the last figure is very inaccurate between 2 and 4. No attempt should be made to read more than three digits to the right of the primary mark numbered 4. This means roughly that the maximum attainable accuracy is one part in 1000. Hence the right-hand end of the scale gives computational accuracy to three significant figures, the left end to four significant figures. But in general, computations usually involve both ends of the scale and we are limited to an accuracy of three significant figures.

These remarks must be modified by the reader at his own judgment to fit the rule in his possession, depending on its length, the care used in its manufacture, and other factors. The accuracy of a rule is nearly proportional to the length of the scale.

1-20. Multiplication by the Slide Rule. Two scales are used in conjunction. Either the *C* and *D* scales or the *A* and *B* may be used. The *C* and *D* scales, having larger scale divisions, are more accurate. Let us use them on a simple example.

Example 1. Multiply 2×3 by the slide rule.

The multiplicand, 2, is located on the *D* scale and the left 1 (the left index) of the *C* scale is brought even with it. The hairline may be used to assist in this alignment

if desired. Then the hairline is moved along the *C* scale to the multiplier, 3. The product, 6, is directly below the 3 on the *D* scale.

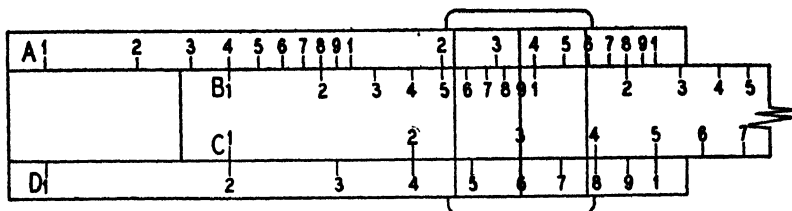


FIG. 1-8.

While the rule is in this adjustment it might be noted that the products of 2 by the integers 2, 4, and 5 may also be obtained by looking on the *D* scale below each of these on the *C* scale, without changing the adjustment of the slide. But the multiplication of 2 by the integers 6, 7, 8, and 9 is impossible, as the slide is now set, because the *D* scale does not extend far enough. Multiplication by these numbers is accomplished by putting the right index, rather than the left, even with the 2 as shown in Fig. 1-9.

Example 2. Multiply 2×6 by the slide rule.

The product, 12, is found to be two tenths of the way between the large 1 and 2, which, in this case, is interpreted as being two tenths of the way between 10 and 20, that is, 12. Had we been using the left half of the A and B scales for the multiplication, the use of the right index could have been avoided, since the A scale (used in the same manner as the *D* scale) is continuous and can take care of products greater than 10. And, for these simple numbers, the A and B scales would have been sufficiently accurate.

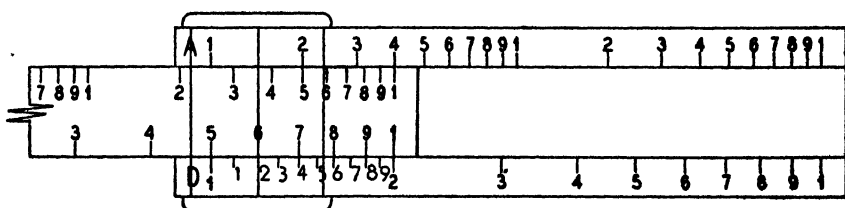


FIG. 1-9.

In order to avoid the trial and error process of deciding which index to use when multiplying numbers with 2 or more digits on the *C* and *D* scales, the following suggestion will be found useful: *When the product of the first digits of each of the two factors is less than 10, use the left index; when it is greater than 10 use the right index.*

While learning to multiply on the slide rule, it is well to check the rules by applying them to a number of simple problems for which the correct answer is known. For products of more than two numbers, consider the product of the first two (which you will mark with the hairline without reading it) as the multiplicand for the next multiplication.

The slide rule merely gives the significant digits of the product; the position of the decimal point must be determined by other methods.

Example 3. Multiply $12.6 \times 35.9 \times 187$ on the slide rule.

The significant digits for 12.6×35.9 are 452, and for the entire product they are 846. Now rounding off the numbers, the product must be in the neighborhood of $10 \times 35 \times 200 = 70,000$. Hence the proper result is $84,600 = 8.46 \times 10^4$.

The position of the decimal point in a slide rule computation can be determined by a mental computation with the simple numbers obtained from the given numbers by rounding them off to one or two significant figures.

An alternative method to be used in complicated computations will be given later, but many persons prefer to use the method shown above in all their work. The authors suggest that the student adopt whatever method he finds to be the easier and faster.

We have, then, this general procedure.

To multiply two numbers:

1. Find the multiplicand on the D (or A) scale.
2. Bring proper index of C (or B) scale even with it.
3. Locate the multiplier on the C (or B) scale.
4. With the aid of the hairline find the number on the D (or A) scale opposite the multiplier. This number gives the significant digits of the product.
5. Determine the decimal point by means of a mental computation with the given numbers rounded off.

EXERCISES

Perform the following multiplications, obtaining results accurate to as many places as your slide rule will allow, and express these results in scientific notation.

- | | |
|---|---------------------------------------|
| 1. 5.00×16.00 . | 2. 11.00×4.00 . |
| 3. 17.00×4.32 . | 4. 14.00×12.00 . |
| 5. 13.00×5.00 . | 6. 36.0×1.020 . |
| 7. 5.00×2.99 . | 8. 5.12×286 . |
| 9. 6.37×10.88 . | 10. 87.6×96.2 . |
| 11. 96.8×89.42 . | 12. 86.4×11.65 . |
| 13. 19.87×98.46 . | 14. 0.00865×85.9 . |
| 15. 0.00972×0.00864 . | 16. 0.876×1.365 . |
| 17. 9.87×4560 . | 18. 36.4×18.41 . |
| 19. 998×0.00672 . | 20. 89.4×67.3 . |
| 21. $16.82 \times 18.96 \times 65.8$. | 22. $79.2 \times 67.3 \times 61.4$. |
| 23. $68.4 \times 5.92 \times 0.00672$. | 24. $98.3 \times 1.672 \times 98.6$. |
| 25. $68.2 \times 0.00892 \times 0.0672$. | |

1-21. Division by the Slide Rule. Division is easily understood after multiplication has been studied. The following example, with Fig. 1-10, will suffice to explain the procedure.

Example. Divide 8 by 4 by the slide rule.

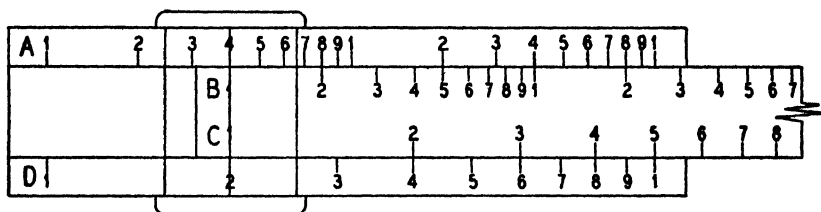


FIG. 1-10.

To divide one number by another:

1. Locate the dividend on the D (or A) scale with the aid of the hairline.
2. Locate the divisor on the C (or B) scale and bring it opposite the dividend at the hairline.
3. With the aid of the hairline find the number on the D (or A) scale opposite the index of the slide. This number gives the significant digits of the quotient.
4. Determine the position of the decimal point by a mental computation with the given figures rounded off.

EXERCISES

Perform the following computations, obtaining results accurate to as many places as your slide rule will allow, and express these results in scientific notation.

- | | |
|--|---|
| 1. $12.00 \div 4.00.$ | 2. $16.00 \div 5.00.$ |
| 3. $8.0 \div 9.00.$ | 4. $64.0 \div 8.00.$ |
| 5. $65.2 \div 3.10.$ | 6. $876 \div 67.3.$ |
| 7. $983 \div 5.69.$ | 8. $0.00864 \div 0.695.$ |
| 9. $0.0897 \div 679.$ | 10. $11.46 \div 96.5.$ |
| 11. $14.67 \div 59.6.$ | 12. $15.89 \div 69.4.$ |
| 13. $48.6 \div 12.64.$ | 14. $67.3 \div 596.$ |
| 15. $487 \div 965.$ | 16. $957 \div 48.6.$ |
| 17. $859 \div 0.00762.$ | 18. $0.000753 \div 56.7.$ |
| 19. $896 \div 672.$ | 20. $48.6 \div 573.$ |
| 21. $\frac{69.4 \times 86.5}{6750}.$ | 22. $\frac{46.8 \times 12.58}{1892}.$ |
| 23. $\frac{0.00876 \times 0.00306}{0.0892}.$ | 24. $\frac{67.3 \times 18.64}{0.0960}.$ |
| 25. $\frac{89.6 \times 47.8}{86.4 \times 99.4}.$ | |

1-22. Finding Squares by the Slide Rule.**To find the square of a number:**

1. *Locate the number to be squared on the D scale.*
2. *By the aid of the hairline find the number opposite it on the A scale.*
This gives the significant figures of the square of the original number.
3. *Determine the position of the decimal point by a mental computation with the given numbers rounded off.*

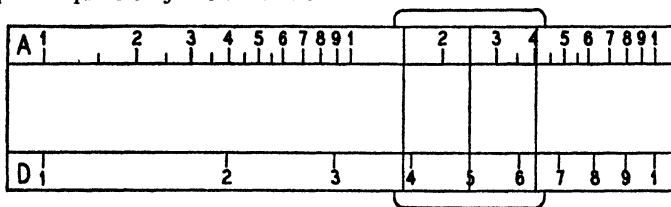
Example. Square 5 by the slide rule.

FIG. 1-11.

The slide is not used in this operation and therefore is not shown in the figure.

Occasionally a rule, other than a Mannheim, may be found which does not have the *A* and *D* scales on the same side. However, these rules have the *A* and *D* scales matched and a hairline on both sides so that the process is the same even though the rule has to be turned over.

For accurate readings, a hairline which is exactly perpendicular to the scales is essential. This is true for any operation, but it is more important here because the *A* and *D* scales are widely separated. A good check on the perpendicularity of the hairline is to see if it gives the exact square of some simple number such as 2 or 3. If found in error, the hairline should be adjusted or replaced.

EXERCISES

Find the squares of the following numbers, obtaining results accurate to as many places as your slide rule will allow, and express these results in scientific notation.

- | | |
|--------------|--------------|
| 1. 16.00. | 2. 76.0. |
| 3. 89.0. | 4. 67.8. |
| 5. 98.6. | 6. 112.2. |
| 7. 89.8. | 8. 67.2. |
| 9. 114.8. | 10. 69.2. |
| 11. 48.6. | 12. 89,600. |
| 13. 0.00672. | 14. 0.968. |
| 15. 0.876. | 16. 48.3. |
| 17. 89.6. | 18. 196.2. |
| 19. 469. | 20. 675. |
| 21. 896. | 22. 0.01234. |
| 23. 0.00968. | 24. 0.467. |

1-23. Finding Square Roots by the Slide Rule. This process is just the reverse of that of the previous section. However, certain care must be taken since the *A* scale has two identical halves. The question is which to use in a given case. The process of taking the square root in arithmetic requires that the number whose root is to be taken be marked off in periods of two digits beginning at the decimal point, as for example, 4 56, 35 69.32, 4 59.63 18, or 0.08 17 1. The first digit of the root is different when the left-most period contains one digit from what it is when the left-most period contains two digits even though the sequence of digits is the same. The result by slide rule must, of course, differ correspondingly. We frame the procedure as follows.

To find the square root of a number:

1. Mark off the number into periods of two digits so that the decimal point is a point of division.

2. (a) If the number whose square root is sought has one digit in the left-most period, locate it on the left half of the *A* scale.

(b) If the number whose square root is sought has two digits in the left-most period, locate it on the right half of the *A* scale.

These rules apply to decimal fractions less than one if we consider non-significant zeros to the right of the decimal point as not being digits.

3. The significant digits of the square root of the number will be found opposite it on the *D* scale.

4. Determine the position of the decimal point by a mental computation with the given number rounded off.

Example 1. Find the square root of 156.2.

Dividing the number into periods, we obtain 1 56.2. By 2(a), we locate 1562 on the left *A* scale, and opposite on the *D* scale we read 125. Since the required root, by mental computation, is between 12 and 13, the result is $12.5 = 1.25 \times 10$.

Example 2. Find the square root of 0.00869.

Dividing the number into periods, we obtain 0.00 86 9. By 2(b) we locate 869 on the right hand *A* scale, and opposite on the *D* scale we read 932. Since the required root, by mental computation, is between 0.09 and 0.10, the result is $0.0932 = 9.32 \times 10^{-2}$.

EXERCISES

Find the square roots of the following numbers, obtaining results accurate to as many places as your slide rule will allow, and express these results in scientific notation.

1. 25.0.

2. 250.

3. 2500.

4. 3.60.

5. 814.

6. 1156.

7. 11.56.

8. 8.94.

9. 89.4.

10. 894.

11. 967.	12. 18,400.
13. 0.0964.	14. 0.817.
15. 0.000632.	16. 89.6.
17. 8.04.	18. 0.00001894.
19. 64,500.	20. 895,000.
21. 798,000.	22. 14.82.
23. 495.	24. 6,870,000.
25. 14.62.	

1-24. Finding Cubes and Cube Roots by the Slide Rule. If the reader possesses a rule with a K scale, he will discover quickly how to find cubes, using the K and D scales. The rule for finding cube roots is similar to that for finding square roots, but the given numbers must be marked off into periods of three digits instead of two.

1-25. The Use of the Slide Rule in Extended Computations. Several useful hints on the use of the slide rule will appear in connection with the following computation:

$$\frac{346 \times 17.1}{221}.$$

We may multiply 346 by 17.1 and divide this result by 221, or we may divide 346 by 221 and then multiply the result by 17.1. The latter is the shorter of the two methods since one setting of the slide will work the entire problem. To carry out the second process, first find 346 on the D scale and bring 221 on the C scale opposite it by the aid of the hairline. The result of this division appears on the D scale directly below the left index (at the digits 156), but *we need not read this because whatever it reads is merely the point at which the index should be set for the ensuing multiplication.* So, *without moving the slide or even reading what is under the index, move the hairline to the next multiplier (17.1) on the C scale.* The result is the digits 268 on the D scale. Although the first method is not quite so easy, let us follow it through as a check. First multiply 346×17.1 . This gives the digits 592. Then divide this by 221. Again we obtain the digits 268. In general, *where several multiplications and divisions are to be performed, the easiest and most accurate method is to alternate between multiplication and division, rather than doing all the multiplying and then all the dividing.* The reason is that, in general, fewer moves of the slide are required, and therefore there is less chance for error. It is very good practice to work first by one method and then by the other, as a check. Since

$$\frac{350 \times 20}{200} = 35,$$

the result of the computation is 26.8.

It should be observed that this method may fail, because the final reading may be beyond the limits of scale D . If we replace, for instance, the factor 17.1 by 7.60 using the first method, we shall find that the multiplier 7.60 is outside the body of the slide rule. In order to read the product, the slide must be reset so that the right index of it takes the place of the left index, which has been marked with the hairline. Then, after the resetting of the slide, we move the hairline to the multiplier 7.60 on C and read 11.9 on scale D .

The necessity of resetting the slide can be avoided by the use of the scales A and B , if the accuracy of these scales is sufficient.

Scientific notation is sometimes useful in determining the position of the decimal point, especially if the numbers are very large or very small. For example,

$$\frac{3,460,000 \times 0.000171}{22,100}$$

can be written as

$$\begin{aligned} \frac{3.46 \times 10^6 \times 1.71 \times 10^{-4}}{2.21 \times 10^4} &= \frac{3.46 \times 1.71}{2.21} \times 10^{6-4-4} \\ &= \frac{3.46 \times 1.71}{2.21} \times 10^{-2}. \end{aligned}$$

Performing the slide rule computations on

$$\frac{3.46 \times 1.71}{2.21}$$

we obtain the significant digits 268, and the decimal point is easily de-

termined from $\frac{3.5 \times 2}{2} = 3.5$. Thus the result of the computation is

$$2.68 \times 10^{-2} = 0.0268.$$

EXERCISES

Perform the following computations on the slide rule, obtaining results accurate to as many places as your slide rule will allow, and express these results in scientific notation.

1. $\frac{18.63 \times 49.5}{893}$.

2. $\frac{695 \times 10.942}{456}$.

3. $\frac{1876 \times 976}{6430}$.

4. $\frac{0.000469 \times 1846}{653}$.

5. $\frac{1896}{43.6 \times 0.0872}$.

6. $\frac{675}{0.0259 \times 6.34}$.

7. $\left(\frac{86.4}{63.5}\right)^2.$

9. $\frac{(86.5)^2 \times 5.67}{6.54 \times 10^5}.$

11. $\left(\frac{389 \times 465}{85.3}\right)^2.$

13. $\sqrt{\frac{976 \times 0.00864}{0.0943}}.$

15. $\frac{759 \times 853}{\sqrt{325}}.$

17. $\left(\sqrt{\frac{498 \times 9.56}{43.6}}\right) \cdot 5.65.$

19. $\sqrt{496} \cdot \sqrt{46.5 \times 0.964}.$

8. $\left(\frac{963}{85.2}\right)^2.$

10. $\left(\frac{497 \times 856}{5.68 \times 10^6}\right)^2.$

12. $\sqrt{\frac{469 \times 856}{672}}.$

14. $\frac{\sqrt{856 \times 962}}{7.59}.$

16. $\frac{487 \times 6.59}{5.46 \times 10^8}.$

18. $\frac{\sqrt{659 \times 897}}{32.5}.$

20. $\left(\sqrt{\frac{45.6}{91.8}}\right) \cdot (85.7)^2.$

PROGRESS REPORT

The purpose of this chapter was to introduce numbers and to explain how to operate with them. The necessity of counting led man to the invention of positive integers. Practical problems made it necessary to widen the system of integers and to introduce positive, negative, fractional and irrational numbers. All these numbers together form the real number system and by means of them engineering computations required in practical problems can be performed.

In practical applications, numbers which are encountered have a limited degree of accuracy. This led to the definition of significant digits and to the development of rules for determining the accuracy of results computed from numbers of limited accuracy.

The necessity of performing numerical computations led to the introduction of scientific and engineering notation and to the discussion of the slide rule, a tool for speeding and simplifying numerical work.

The student should not only understand the principles introduced here but should also acquire skill in computation by continuous practice.

CHAPTER 2

SIMPLE ALGEBRAIC OPERATIONS

Engineering formulas give the relationships between physical quantities. These quantities are designated by letters of the alphabet, by Greek letters, or other symbols. Frequently the engineer uses his own notation which has been standardized by the particular engineering field in which the specific problem occurs. Thus, a force which may act on a structural member is designated by F , while a voltage acting in a circuit is designated by E or e . Mathematicians, because they deal with abstract or perfectly general quantities, prefer to use such letters as a , b , c , x , y , and z , which have no particular reference to engineering symbols. In order to arrive at as simple formulas as possible, it is quite necessary that the engineer be able to manipulate expressions so as to obtain them in concise and usable forms.

2-1. The Nature of Algebra. Algebra is an extension of arithmetic. Each statement of arithmetic deals with particular numbers: the statement $(20 + 4)^2 = 20^2 + 2 \cdot 20 \cdot 4 + 4^2 = 576$ explains how the square of the sum of the two numbers, 20 and 4, may be computed. It can be shown that the same procedure applies if the numbers 20 and 4 are replaced by any two other numbers. In order to state the general rule, we write symbols, ordinarily letters, instead of particular numbers. Let the number 20 be replaced by the symbol a , which may denote any number, and the number 4 by the symbol b . Then the statement is true that the square of the sum of any two numbers a and b can be computed by the rule

$$(a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2.$$

This is a general rule which remains true no matter what particular numbers may replace the symbols a and b . A rule of this kind is often called a **formula**.

Algebra is the system of rules concerning the operations with numbers. These rules can be most easily stated as formulas in terms of letters, like the rule given above for squaring the sum of two numbers.

The outstanding characteristic of algebra is the use of letters to represent numbers. *Since the letters used represent numbers, all the laws of arithmetic given in Chapter I hold for operations with letters.*

In the same way, all the signs which have been introduced to denote relations between numbers and the operations with them are likewise used with letters.

For convenience the operation of multiplication is generally denoted by a dot or by placing the letters adjacent to each other. For example $a \cdot b$ is written simply as ab .

The operations of addition, subtraction, multiplication, division, and extracting roots are called **algebraic operations**. Any symbol or combination of symbols obtained by employing algebraic operations is called an **algebraic expression**. Examples of algebraic expressions are given by

$$2x + 7y - \sqrt{x + y}, \quad 2a^3b + 7ab.$$

When an algebraic expression consists of several parts connected by + or - signs, each part with its preceding sign is called a **term**. In the above examples $2x$, $7y$, and $-\sqrt{x + y}$ are terms of the expression $2x + 7y - \sqrt{x + y}$, and $2a^3b$ and $7ab$ are terms of the expression $2a^3b + 7ab$.

In order to represent different quantities, different letters are used. Although these letters may be picked at random in any given problem, it is usually found convenient in engineering work to set aside a certain letter which is always used to represent physical quantities of a certain class. If there are several quantities in the same class, these may be denoted by subscripts. Thus E refers to voltage in electrical problems, and when several voltages occur in a given problem they are denoted by

$$E_1, E_2, E_3, \dots, E_n,$$

read E sub-one, E sub-two, \dots , E sub- n , or, if no misunderstanding is possible,

$$E\text{-one}, E\text{-two}, \dots, E\text{-}n.$$

A similar type of notation employs accents for the same purpose. Thus:

$$I', I'', I''', \dots$$

is read I -prime, I -two prime (or double prime), I -three prime, etc.

In general, any type of symbol may be attached to a letter if it forms a convenient representation of the quantities involved. Such forms as

$$E_0, E^*, \bar{E}, V_R, F_x, V_{AB}, \text{etc.},$$

will be frequently encountered.

2-2. Addition and Subtraction. In adding and subtracting algebraic expressions, we use the commutative and associative laws of addition (Sec. 1-8) and, through the rules for removing and inserting signs of grouping (Sec. 1-9), the distributive law.

Example 1. Add $4E_1 + 7E_2$, $5E_1 - 3E_2$, $2E_2 - E_1$.

The sum can be written as:

$$\begin{aligned}(4E_1 + 7E_2) + (5E_1 - 3E_2) + (2E_2 - E_1) \\ = 4E_1 + 7E_2 + 5E_1 - 3E_2 + 2E_2 - E_1 = 8E_1 + 6E_2.\end{aligned}$$

Thus, *algebraic expressions may be given a simpler form by combining similar terms*. Two terms are called **similar**, if they differ only in their numerical factor (called a **coefficient**).

Example 2. Simplify the expression

$$2a + \{5a - [8a - 3b - (2a - b) + 3c] + 10c\}.$$

Removing successively the parentheses, brackets, and braces, we obtain

$$\begin{aligned}2a + \{5a - [8a - 3b - 2a + b + 3c] + 10c\} \\ = 2a + \{5a - 8a + 3b + 2a - b - 3c + 10c\} \\ = 2a + 5a - 8a + 3b + 2a - b - 3c + 10c \\ = a + 2b + 7c.\end{aligned}$$

The last example shows that *in removing several signs of grouping, it is more convenient to remove the innermost first*.

EXERCISES

Add:

1. $5a, -7a, 11a, -3a$.
2. $5xy, -2xy, 7xy$.
3. $3x + 2y, 5x - 3y, 7y - 2x$.
4. $2a^2b + 3ab^2, 5a^2b - a^2b$.
5. $6E_1 + 3E_2 - 7E_3, 2E_1 - 5E_2 - E_3$.
6. $6ab - 3ac + 4bc, 3bc - 2ab + 5ac$.
7. $V_1 - 2V_2 + V_3, -4V_1 - V_2 + 6V_3$.
8. $E_1 - 5I_1 + 6I_2, -E_1 - 2I_1 + I_2$.
9. $5x - 3y, -2x + 4y, 5y - x$.
10. $4E - 6IX, -6E + 10IX, 3E - 2IX$.
11. $3ab - 4cd, 2ab + 3cd, 2cd - 4ab, 6cd + 2ab$.
12. $2x^4 + 3x^2y^2 + 5y^4, -x^4 - 5x^2y^2 - 2y^4, 3x^4 - 2x^2y^2 + y^4$.
13. $5R_1 - 3R_2 + R_3, -3R_1 + 6R_2 - 2R_3, 2R_1 - R_2 + 6R_3$.
14. $3I_1 + 5I_2 - 3I_3, -2I_1 - 2I_2 + 6I_3, I_1 + I_2 - I_3$.
15. $E - 3RI - 16ZI, 7RI + 25ZI, -3E - 2RI + 12ZI$.
16. $2W + 3EI - 10I^2R, W + 2EI + 5I^2R, -EI + 2I^2R, -W - EI$.
17. $3R_1 + 6R_2 - 2R_3 + R_4, -R_1 - 2R_2 + R_4, R_1 - 3R_3 - 3R_4, R_2 + 5R_3 + R_4$.
18. $2I_1 + 6I_2 + 3I_3 - 5I_4, -I_1 + 2I_2 + 3I_4, -I_2 - I_3 + 3I_4, I_1 - I_2 + I_3$.
19. $3x + 5y - 3z + 4w, 2y + 4w, 2x - 2z, -4x + 6z - 2w$.
20. $10a - 4b + 3c + 6d, 2b - 5a + c, 3d - 2b - c, 6c - 4a - d + 5c$.

Subtract the second expression from the first:

21. $10V, 3V$.
22. $10ax, -3ax$.
23. $3a + 5b, 2a + 3b$.
24. $6x - 8y, 3x - 2y$.
25. $5E - 3IX, 7E - 5IX$.
26. $7E + 3RI + 6XI, 2E + RI - 3XI$.

27. $12ax + 17by - 9cz, -8ax - 3by + cz$.
 28. $3W + 2EI - 12I^2R, 5EI - W - 5I^2R$.
 29. $3x^2 + 6xy - 4y^2, x^2 - 2xy + 2y^2$.
 30. $6I_3 - 5I_1 + 8I_2, 7I_3 - 2I_1 - 3I_2$.
 31. $7I_1 - 4I_2 + 6I_3, 9I_3 + 3I_1 - 2I_2$.
 32. $12x^2 - 24xy + 2y^2, 8x^2 + 10xy - 4y^2$.
 33. $3Z_1 - 3Z_2 - 10, 6Z_1 + 5Z_2 + 5$.
 34. $9x + 6y - 7z - 10w, 12w - 3y - 3z + 4x$.
 35. $3R_1 + 6R_2 - 8R_3 - R_4, 5R_3 - 2R_2 + 3R_4$.

Simplify:

36. $(a + 6) - (a - 7) + (2a + 1)$. 37. $(2x + 3y) - (x - 2y) + (x + y)$.
 38. $E + 3IR - (2E + IR)$. 39. $3ac - \{3by + (ac - by)\}$.
 40. $6p - 2q - [(3p + 2q) - (3p - 2q)]$.
 41. $a + b - \{a - b + [a + b - (a - b)]\}$.
 42. $e - (ir + e) - \{e - [ir - (e + ir)] - e\}$.
 43. $E - (6I_1R_1 + 2I_2R_2) - \{2E - (3I_1R_1 + I_2R_2)\}$.
 44. $W - \{2W - EI - [I^2R - (-3EI)] + 4I^2R\}$.
 45. $4x - \{6x - 2y + [2x - y + 43] - (z + 2y)\}$.
 46. $3I_1 - \{2I_1 - [3I_2 + (I_2 - I_3)] + 4I_3\}$.
 47. $6y - \{5x - [2x - (3y - x) + 4y]\} - 3x$.
 48. $- \{ - [- (a - b - c) + 29] - 39 + 2b - c \}$.
 49. $3E - (2RI + IZ) - [2E - (3RI - 2IZ)]$.
 50. $6R_1 - \{R_2 + [R_1 - (R_2 + 3R_3)] - R_3\}$.

2-3. Multiplication. In multiplying algebraic expressions consisting of one term, we use the commutative and associative laws (Sec. 1-8) and the laws of exponents (Sec. 1-11). An algebraic expression of one term is a **monomial**.

Example 1.

$$(5a^2x^3)(3a^4x^2) = 5 \cdot 3 \cdot a^2 \cdot a^4 \cdot x^3 \cdot x^2 = 15a^{2+4}x^{3+2} = 15a^6x^5.$$

Example 2.

$$(2ab^2c)(3a^2bc)(5abc^2) = 2 \cdot 3 \cdot 5 \cdot a \cdot a^2 \cdot a \cdot b^2 \cdot b \cdot b \cdot c \cdot c \cdot c^2 \\ = 30a^{1+2+1}b^{2+1+1}c^{1+1+2} = 30a^4b^4c^4.$$

Example 3.

$$(2x^2y^3)^2 = 2^2(x^2)^2(y^3)^2 = 4x^4y^6.$$

Example 4.

$$(3a^4b^3c)^3(2abc^3)^2 = 3^3a^{12}b^9c^3 \cdot 2^2a^2b^2c^6 = 108a^{14}b^{11}c^9.$$

EXERCISES

Perform the indicated operations.

- $-3xy \cdot 2xy \cdot 5x$.
- $2n^3 \cdot 3n^2 \cdot 5n$.
- $(3a^5)(-8a^4)(-2a)$.
- $(-2a^2b)(-3ab^2)$.
- $5IR \cdot 3I$.
- $(3xyz)(4yz)$.

- | | |
|---|---|
| 7. $6ab^2cd^2 \cdot 9a^2bc^2d$. | 8. $(3I^2R)(2IX)$. |
| 9. $7I^2R \cdot 3EI$. | 10. $12xy^2z \cdot 5x^2yz^2$. |
| 11. $(-3W^2) \cdot 2WX^2Y \cdot XY^2$. | 12. $3IZ^2(-2I)(-3ZR^2)$. |
| 13. $3R_1I \cdot 2R_2I(-5R_3I)$. | 14. $6IRZ \cdot 7RZ^2 \cdot 3I^2R$. |
| 15. $4it^2 \cdot 2irs \cdot 8rt^2s$. | 16. $5RL \cdot 2WR \cdot 3RC$. |
| 17. $6xyz(-7yzw) \cdot 3xzw$. | 18. $12EI \cdot 6I^2R \cdot 3E$. |
| 19. $14Irt \cdot 3I \cdot 6Irt$. | 20. $15pqr \cdot 5prt \cdot 2qrs$. |
| 21. $(-9x^2yz) \cdot 3xy^2z \cdot 6xyz^2$. | 22. $12mnp \cdot 3npr(-5mprs)$. |
| 23. $2ab^2c \cdot 3abc^2d \cdot 4abcd^2e \cdot abcde$. | 24. $(5xyz)(6xzw)(2yzwu)(xzwuv)$. |
| 25. $7x^3y(-8xy^4) \cdot (-x^2y)$. | 26. $5W^2L \cdot (-2WL^2) \cdot (-3WL^2)$. |
| 27. $2a^3b \cdot 3a^2b^3 \cdot 5a^2b^3$. | 28. $2x^2yz \cdot 3xy^2z^2 \cdot 4x^2y^2z^2$. |
| 29. $R_a^2R_b \cdot R_aR_c^2 \cdot R_b^2R_c$. | 30. $(-Z_1^3Z_2^2) \cdot Z_1^3Z_3^2(-Z_2Z_3)$. |

2-4. Multiplication of Multinomials. Algebraic expressions consisting of more than one term are called **multinomials**. In particular, an expression of two terms is a **binomial**, an expression of three terms is a **trinomial**. In finding the product of multinomials, we make use of the distributive law (Sec. 1-8).

Rule 1. To multiply a multinomial by a monomial, multiply each term of the multinomial by the monomial and add the resulting products.

Example 1.

$$5a^3b(7a^2b^2 - 3ab^3 + 2a^4b) = 35a^5b^3 - 15a^4b^4 + 10a^7b^2.$$

Rule 2. To find the product of two multinomials, take the sum of the products which result from multiplying each term of one multinomial by each term of the other.

Example 2.

$$\begin{aligned} (x + y)(3x^2 - 2xy + 5y^2) &= x(3x^2 - 2xy + 5y^2) + y(3x^2 - 2xy + 5y^2) \\ &= (3x^3 - 2x^2y + 5xy^2) + (3x^2y - 2xy^2 + 5y^3) \\ &= 3x^3 - 2x^2y + 5xy^2 + 3x^2y - 2xy^2 + 5y^3 \\ &= 3x^3 + x^2y + 3xy^2 + 5y^3. \end{aligned}$$

EXERCISES

Perform the indicated operations:

- | | |
|---|--|
| 1. $x(x + 2)$. | 2. $(3x - y)(x + 4y)$. |
| 3. $(3R_1 + 5)(2R_2 + 1)$. | 4. $2a^3b^2(a^2 - 3ab^3 + b)$. |
| 5. $3I^2R(X + Z)$. | 6. $2ac(ab + cd)$. |
| 7. $(a + b)(a - b)$. | 8. $(x - y)(x + y)$. |
| 9. $(3I + 4)(5I - 2)$. | 10. $(6ei - 4)(7ei - 3)$. |
| 11. $(6r^2 + 3Z)(8r^2 - 5Z)$. | 12. $4IZ(3IR + 7IX + 5XL)$. |
| 13. $7xy^2z(2xy + 3yz + 5xz)$. | 14. $5E(2IR_1 + 3IR_2 + 6IR_3)$. |
| 15. $6ab(2ac + 3ad + 7bc + 9bd)$. | 16. $7ei(3t_1 + 4t_2 - 7t_1 + 6t_3)$. |
| 17. $16RI_1(RI_2 + 3RI_1 - 2RI_2 + 5E_0)$. | 18. $(a + b)(a^2 - ab + b^2)$. |
| 19. $(3P + 5)(5P^2 + 2P + 2)$. | 20. $(a + b)(a^2 + 2ab + b^2)$. |

21. $(2I + 4)(3I^2R + 2R^2 + 3)$.
 22. $(x^2 - y^2)(x^4 + x^2y^2 + y^4)$.
 23. $(a - b + c)(a - b - c)$.
 24. $(3Z + 4R - 5)(5Z - 2R + 7)$.
 25. $(3x^2 + 2x + 1)(x^2 - 4x + 2)$.
 26. $(6EI - 5I^2R + 2)(2EI + 3I^2R + 5)$.
 27. $(3mn + 2np + 5mp)(2mn - 7np - 3mp)$.
 28. $(2I_1R_1 + 3E - 2I_2R_2)\left(\frac{E}{R} + I_1 - I_2\right)$.
 29. $(2x + 3y)(3x^2 + 2x^2y + 3xy^2 + y^2)$.
 30. $(4T + 5)\left(3RI^2 + 2EI + 3\frac{E^2}{R} + 5\right)$.

2-5. Division of Monomials. In dividing monomials the laws of exponents (Sec. 1-11) are used.

Example 1.

$$5a^2b \div a = \frac{5a^2b}{a} = 5ab.$$

Example 2.

$$\frac{3m^3lt^{-5}}{5m^2l^2t^{-4}} = \frac{3m}{5l^2t}.$$

EXERCISES

Perform the indicated operations:

- $5a^{10} \div a^6$.
- $-15b^7c^3 \div bc^2$.
- $(T^2 + 7Y^4)a \div 4Z(T^2 + 7Y^4)$.
- $\frac{(7X_c^2 - 4)^3ZL^2}{R_TL(7X_c^2 - 4)^3}$.
- $\frac{a^{n+2}}{a^n}$.
- $\frac{R^{2n}}{LR^{n-1}}$.
- $21A^{10}B^2C \div 7A^6B^2C^3$.
- $5kT^2(E + E_0)^3 \div kT(E + E_0)^2$.
- $L^{x+2}(x - y)^2 \div 4L^{x+1}(x - y)$.
- $\frac{(x^{10} + 7x^2)y^4Z^3}{xy^2Z}$.
- $\frac{(a + b)^3ayt}{a(a + b)^2}$.
- $\frac{17Z^3 \cdot 4bZ \cdot 2T}{34bZT}$.
- $\frac{14a^2 \cdot 3b^3c \cdot 4d(b + 1)}{24b(b + 1)}$.
- $4P^2K^m(Z - Z_0)^{m-1} \div 2PK^{m-1}(Z - Z_0)^m$.
- $\frac{(R^2 + 2X - 7)^4T^{n-1}}{RT^2(R^2 + 2X - 7)}$.
- $\frac{(r + s)(r - s)(K^2 + rs)}{K(r - s)}$.
- $\frac{N^{m+2}Q^nR^7}{N^mR^4Q^{1-n}}$.
- $\frac{(f + g^2)kL^2 \cdot 2(f - g^2)^2}{(f - g^2)(f + g)(f + g^2)L}$.
- $\frac{M^{n+1}(NQ)^4}{N^{4-n}(QM)^3}$.
- $\frac{AB^4(C + 1)^2}{(A + C)^3B^2(C + 1)}$.

2-6. Division of a Multinomial by a Monomial. According to the laws for division (Sec. 1-10), a multinomial is divided by a monomial by dividing each term separately.

Example 1.

$$\frac{7a^2 + 14ab - 21a^3}{7a} = \frac{7a^2}{7a} + \frac{14ab}{7a} - \frac{21a^3}{7a} = a + 2b - 3a^2.$$

Example 2.

$$\begin{aligned} \frac{4Z(P+W) - Z^3(P+W)^2 + XZ^2(P+W)^3}{Z^2(P+W)} \\ = \frac{4Z(P+W)}{Z^2(P+W)} - \frac{Z^3(P+W)^2}{Z^2(P+W)} + \frac{XZ^2(P+W)^3}{Z^2(P+W)} \\ = \frac{4}{Z} - Z(P+W) + X(P+W)^2. \end{aligned}$$

EXERCISES

Perform the indicated operations:

1. $\frac{R^2T + 4TV}{T}$.
2. $\frac{a^2bc^3 + 27a^3bc^2}{abc}$.
3. $\frac{IR_1 + I^2R_1 - IE}{IR_1}$.
4. $\frac{ZX^2 + XZ^3 - 4XZ^2}{XZ}$.
5. $\frac{2\pi fL - \pi fR^2}{\pi fR}$.
6. $\frac{7(a+b) - 4(a+b)^2(a+2b)}{4(a+b)}$.
7. $\frac{4y^2x^n + 8x^{n+1} - 7y^{-2}Z^3}{x^{2n}y^nZ}$.
8. $\frac{IR^2 + I^2R^5 - I^4RL}{2I^2R}$.
9. $\frac{k(a+c)^4 + 4(a+c)^3 + (a+c)}{4k(a+c)}$.
10. $\frac{(r-1)b - (r-1)c}{(r-1)bc}$.
11. $\frac{14KP^{2+n} + 28MP^n + 7KM}{14KP^2M}$.
12. $\frac{5I_p(E_p + e_s) - (E_p + e_s)}{E_p + e_s}$.
13. $\frac{7qx^{10} - 6p^3x^5 + 4q^3x^2}{(6pq^2)(4x^2)}$.
14. $\frac{7RV + MV^2(k+1)^2 - 14R(k+1)}{(k+1)(RV^2)}$.

Divide $18y^4Z^2(p+1)^m - 9x^2yZ^3 + x^m(p+1)$ by:

15. $3y^2Zx^m$.
16. $9mZ(p+1)^2$.
17. $18yx^{-2}(p+1)^{-m}$.

Divide $12R^6Z^5L^4 + 16RZ^2 - 8(RZL)^3$ by:

18. R^5Z^{-2} .
19. $2(RZL)^5$.
20. $4Z^{-6}RL$.

2-7. Special Products. Certain special products occur so frequently in algebra that it is well to become familiar with them. They are given below and should be verified by the student.

1. $(a + b)(a - b) = a^2 - b^2$.
2. $(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$.
3. $(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2$.
4. $a(x + y - z) = ax + ay - az$.
5. $(a + b)(x + y) = x(a + b) + y(a + b) = ax + bx + ay + by$.
6. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

Formula 1 states that *the product of the sum and the difference of two numbers is equal to the difference of their squares*.

Formula 2 states that *the square of the sum of two numbers equals the square of the first, plus twice the product of the numbers, plus the square of the second number*.

Formula 3 states that *the square of the difference of two numbers equals the square of the first, minus twice the product of the numbers, plus the square of the second number*.

Similar statements may be supplied by the reader for formulas 4, 5, and 6.

Algebraic computations may be shortened by the use of the above formulas, as explained by the following examples:

Example 1. Find the product $(x - 2)(x + 2)$.

Substituting x for a and 2 for b in formula 1, we obtain:

$$(x - 2)(x + 2) = x^2 - 2^2 = x^2 - 4.$$

Example 2. Find the product $(4y + c)^2$.

Substituting $4y$ for a and c for b in formula 2, we obtain:

$$(4y + c)^2 = 16y^2 + 8yc + c^2.$$

Example 3. Find the product $(R_1 + 5)(R_1 - 2)$.

Substituting R_1 for x , 5 for a and -2 for b in formula 6 we obtain:

$$(R_1 + 5)(R_1 - 2) = R_1^2 + (5 - 2)R_1 + 5(-2) = R_1^2 + 3R_1 - 10.$$

EXERCISES

Perform the indicated operations.

- | | |
|-----------------------------|--------------------------|
| 1. $2(x + 3)$. | 5. $-5x(2y - 3x)$. |
| 3. $e(e^2 + e - e^3 + 2)$. | 4. $(p + 2)^2$. |
| 5. $(\lambda f + v)^2$. | 6. $(R^2 + x^2)^2$. |
| 7. $(yb - 2m)^2$. | 8. $(r^2 - t^2)^2$. |
| 9. $(R - \frac{1}{2})^2$. | 10. $(D + 2K)(D - 2K)$. |
| 11. $(IR + 1)(IR - 1)$. | 12. $(7x + 4)(7x - 4)$. |
| 13. $(r + 3)(r + 5)$. | 14. $(E + 10)(E - 3)$. |
| 15. $(a - 12)(a - 4)$. | 16. $(x + 3y)(n - k)$. |

17. $(2w + a)(c + b)$.
19. $(2c - D)^2 + 4cD - D^2$.
21. $(\mu + 10)^2$.
23. $3a(x^2 + 2xy + y^2)$.
25. $[2(x - y)]^2$.
27. $(R^2 - x)(R^2 + x)$.
29. $(\phi^2 + \frac{1}{10})(\phi^2 + \frac{1}{5})$.
31. $(x - 1)^2$.
33. $\left(\frac{M}{X_p} - i_p\right)^2$.
35. $\left(\frac{1}{R} + \frac{1}{r}\right)\left(\frac{1}{R} - \frac{1}{r}\right)$.
37. $(\theta - \frac{1}{3})(\theta + \frac{1}{3})$.
39. $[(Z + 6) + 5][(Z + 6) - 5]$.
41. $[(a + c) - (a - c)][(a + c) + (a - c)]$.
42. $\left(\frac{1}{c} + \frac{v}{q}\right)\left(\frac{1}{c} - \frac{v}{q}\right)$.
43. $[5(2c - d)]^2$.
45. $(t + 3v)(3v - t)$.
47. $\left(\frac{E}{I} + \frac{9}{i}\right)\left(\frac{E}{I} - \frac{9}{i}\right)$.
49. $\left(\frac{R^2}{6} + \frac{3Z^2}{10}\right)\left(\frac{R^2}{6} - \frac{3Z^2}{10}\right)$.
18. $(2 + a)(x + w)$.
20. $(4G - 2)(4G + 2)$.
22. $(F - f)^2$.
24. $(4x^2 + 9)^2$.
26. $5abc(6a^3bc^2d - 45ac^3d^2 + 9)$.
28. $(e + 3)^2$.
30. $\left(\frac{E^2}{R} + 1\right)^2$.
32. $(\beta + 6)(\beta - 12)$.
34. $(Z - \frac{2}{3})^2$.
36. $(c - \frac{1}{4})^2$.
38. $(3Z - 2R)(3Z + 2R)$.
40. $[3c + (a + b)][3c - (a + b)]$.
44. $(5p - 6qr)^2$.
46. $\left(\frac{E^2}{R} + I^2R\right)\left(\frac{E^2}{R} - I^2R\right)$.
48. $(\frac{1}{9} + E)(E - \frac{1}{9})$.
50. $(3i^2R + ie)(ie - 4i^2R)$.

2-8. Simple Factoring. In using the previous six formulas going from the left side of the equations to the right side, the process known as **expansion** was performed. If this process is reversed, going from the right side of the equations to the left, the operation is called **factoring**. In arithmetic the analogous process is that of resolving integers into the products of prime factors.

Typical procedures in simple factoring are indicated in the following examples.

Example 1. Factor $4R^2 - Z^2$.

Using formula 1 of Sec. 2-7 we have:

$$(4R^2 - Z^2) = (2R)^2 - Z^2 = (2R + Z)(2R - Z).$$

Example 2. Find the factors of $9x^2 - 12x + 4$.

Using formula 3 of Sec. 2-7 we have:

$$9x^2 - 12x + 4 = (3x)^2 - 2 \cdot (2 \cdot 3x) + 2^2 = (3x - 2)^2.$$

Example 3. Factor $2r^2s - 4rs^2t + 8r^2s^2t$.

Using formula 4 of Sec. 2-7 we have:

$$2r^2s - 4rs^2t + 8r^2s^2t = 2rs(r - 2st + 4rst).$$

The last example illustrates the fact that a *monomial factor of each term of a multinomial is a factor of the multinomial*.

Grouping of terms as in formula 5 of Sec. 2-7 sometimes allows a common factor to be found for all the groups. This is illustrated by the following.

Example 4. Factor $d^3 - 3d^2 + 6d - 18$.

Grouping terms we have:

$$\begin{aligned} d^3 - 3d^2 + 6d - 18 &= (d^3 - 3d^2) + (6d - 18) \\ &= d^2(d - 3) + 6(d - 3) \\ &= (d - 3)(d^2 + 6). \end{aligned}$$

Formula 6 of Sec. 2-7 enables us to factor trinomials like $x^2 + px + q$. Such expressions can sometimes be factored by inspection into two binomials $(x + a)(x + b)$ where $a + b = p$ and $ab = q$.

Example 5. Factor $x^2 - 7x + 10$.

Following the explanation just given, we search for two numbers a and b whose sum is -7 and whose product is 10 . By inspection, these numbers are found to be -5 and -2 . Hence

$$x^2 - 7x + 10 = (x - 5)(x - 2)$$

where $x - 5$ and $x - 2$ are the factors of the given expression.

Example 6. Factor $5 + 6Z + Z^2$.

As in the previous example we have:

$$5 + 6Z + Z^2 = Z^2 + 6Z + 5 = (Z + 5)(Z + 1).$$

A few suggestions will help the student in factoring.

1. Remove *all* monomial factors first.
2. Factor, if possible, the remaining expression; be sure that the factors are simple.
3. Check by multiplying the factors.

EXERCISES

Factor the following by removing a monomial factor.

- | | |
|-----------------------------------|---|
| 1. $ay^2 + ax^2 + az^2$. | 5. $5a - 10b$. |
| 3. $17x^2 - 289x^3$. | 4. $12y^2 + 2xy$. |
| 5. $4L^3 - 8L^2R$. | 6. $10m^4n^2 - 15m^3n^3$. |
| 7. $2ab^3 + 4a^2b - 8a^2b^2$. | 8. $4r^3st - r^2s^2t^2 + 2rs^3t^3$. |
| 9. $a^3t^2 + 3a^2t^3 - 5a^2t^2$. | 10. $16x^2 - 2abx$. |
| 11. $\frac{WL}{R} + WC - W^2$. | 12. $\frac{2\pi}{T} + 8\pi^2C - 6\pi T$. |

- | | |
|---------------------------------------|---|
| 13. $x^4yZ^2 - xy^4Z^3 - x^2y^2Z^2$. | 14. $2\pi rh - \pi r^2h + 3rh^2$. |
| 15. $3m^5 - 12m^3n^2 + 6mn^4$. | 16. $3a^4 - 6a^4b + a^4b^2$. |
| 17. $13R^3L^3 - 13R^2L^2 + 39RL$. | 18. $14x^2y^3z^3 - 7x^3y^2z^2 + 8xy^2z^2$. |
| 19. $y^2(a-b)^2 - 5(a-b)^2$. | 20. $5X(Z-L)^3 + 4Y(Z-L)^3$. |

Factor the following by grouping terms.

- | | |
|---|--|
| 21. $x^4 + x^3 + 2x + 2$. | 22. $ax + a + 2x + 2$. |
| 23. $4p + 4prt + pq + pqrt$. | 24. $u^2l - 2w^2m - v^2l + 2v^2m$. |
| 25. $4y + 4b - cy - cb$. | 26. $2G_mE_p - E_pI_p - 2G_mI_p + I_p^2$. |
| 27. $a^2 + ab + ay + yb - 2a - 2b$. | 28. $18K_1 + 21 - 6X_cK_1 - 7X_c$. |
| 29. $2xy - 6y + 2xz - 6z$. | 30. $ab + acd - b^2 - bcd$. |
| 31. $a^3b^3 + a^4b^2 + a^3b + a^2b^2$. | 32. $a^2X^2 + a^2Y + bX^2 + bY$. |
| 33. $x^2Z^2 + x^2w - 2Z^2 - 2w$. | 34. $eg + fh + fg + eh$. |
| 35. $P^2RL + P^2EL + P^2RI + P^2EI$. | 36. $y^2 + ny - my - mn$. |

Factor the following.

- | | |
|---|--------------------------------------|
| 37. $k^2 + 8k + 16$. | 38. $a^2 + 4a + 4$. |
| 39. $N^2 - 2NP + P^2$. | 40. $25R^4 - 1$. |
| 41. $a^4b^2 + 2a^2b + 1$. | 42. $x^2 - 26x + 169$. |
| 43. $\frac{V_t^2}{4} - \frac{V_0^2}{4}$. | 44. $9E^2 - 42E + 49$. |
| 45. $X_L^2 - 16X_c^2$. | 46. $X^4 - 16Z^2$. |
| 47. $H^2L^2I^2 - R^2$. | 48. $a^4b^2 - x^2y^4$. |
| 49. $C^2d^4 - C^2$. | 50. $9p^2 + 6p + 1$. |
| 51. $4Z^2 - 12Z + 9$. | 52. $9R^2 - 24RL + 16L^2$. |
| 53. $25X_L^2 + 20X_cX_L + 4X_c^2$. | 54. $16x^4 - 81x^4y^4Z^4$. |
| 55. $x^2 - 4x + 3$. | 56. $y^2 - 8y + 12$. |
| 57. $x^2 - 3x - 4$. | 58. $p^2 + 16p + 28$. |
| 59. $t^2 - 13t - 68$. | 60. $2K - 24 + K^2$. |
| 61. $a^2 + 7ad + 12d^2$. | 62. $R^2 + 3RW^2L^2 + 2W^4L^4$. |
| 63. $W^2 - 2W - 35$. | 64. $r^2s^2 - 9r^2s + 8r^2$. |
| 65. $ad^2 + 5ad - 14a$. | 66. $3x^2 - 9x - 84$. |
| 67. $9L^2 + 6L + 1$. | 68. $i^2k + 2\pi ijk + \pi^2j^2k$. |
| 69. $30P^5R^7 - 25P^4R^6 + 5P^3R^5$. | 70. $WC_r p^2 + 5WC_r pZ + 6WCZ^2$. |
| 71. $E^2 + 11ER - 26R^2$. | 72. $m^2x^4 + 12mx^2 + 35$. |
| 73. $Z^4 - 15Z^2 - 100$. | 74. $2x^2 + 6x + 4$. |
| 75. $5L^2 - 25L + 30$. | 76. $2R^4 + 32R^2 + 110$. |
| 77. $x^4y^2 + 4x^2y + 3$. | 78. $P^2R - 16PR - 225R$. |
| 79. $\pi a^2L^2 + 17\pi aL + 66\pi$. | 80. $a^6R^2 - 5a^3R - 14$. |

2-9. Lowest Common Multiple. In adding algebraic fractions a common denominator must first be obtained. *The lowest common multiple (designated as L.C.M.) of two or more expressions is the product of all their different factors, each taken the greatest number of times that it occurs as a factor in any one of the expressions.*

The method of finding the L.C.M. is illustrated by the following examples.

Example 1. Find the L.C.M. of:

$$ax + 3a, \quad x^3 + 2x^2, \quad x^2 + 5x + 6.$$

Factoring each, we have:

$$a(x + 3), \quad x^2(x + 2), \quad (x + 2)(x + 3).$$

The factors a , $(x + 2)$, $(x + 3)$ each occur at most once in any expression; x occurs at most twice. Hence the L.C.M. is $ax^2(x + 2)(x + 3)$.

Example 2. Find the L.C.M. of:

$$15a^3b^2c, \quad 12a^2b^3c^2, \quad 18b^4c^3, \quad \text{and} \quad 20a^2b^4.$$

The required L.C.M. is $180a^3b^4c^3$, because 180 is the least common multiple of the numbers 15, 12, 18, 20 and no lower powers than a^3 , b^4 , or c^3 will contain as factors the powers of a , b , c in all the given monomials.

EXERCISES

Find the L.C.M. of the expressions in the following examples.

- | | |
|---|--|
| 1. $5ab^3, 7a^2b^4.$ | 2. $12x^2y^3, 54yZ^2.$ |
| 3. $12R^3, 45L^2.$ | 4. $72x^3y, 96y^2Z^4.$ |
| 5. $x^2 + x - 6, x^2 + 4x + 4.$ | 6. $R^2 + RL, RL - L^2.$ |
| 7. $Z^2 - 9, Z^2 + 10Z + 21.$ | 8. $ax - x^2, a^2 - x^2.$ |
| 9. $x^2 - y^2, x + y, x^2 + xy.$ | 10. $x^2 - ax - bx + ab, x^2 - 2ax + a^2.$ |
| 11. $12E - 36, E^2 - 9, E^2 - 5E + 6.$ | 12. $4a^2 - 2ab, 4ab + 2b^2, 4a^2 - b^2.$ |
| 13. $x^{12} + x^{11}, x^{14} + x^{13}.$ | 14. $a^2(x - y), b^2(x^2 - y^2).$ |
| 15. $ax - ay, bx^2 - by^2, cx + cy.$ | 16. $a^2 - 2ab + b^2, b^2 - a^2, a - b.$ |
| 17. $x(m + n)^2, y(m^2 - n^2), Z(m - n)^2.$ | 18. $a(b - c), b(c - a), c(a - b).$ |
| 19. $R^2 - 3R - 10, R^2 - 9R + 20.$ | 20. $E^2 + 2E - 3, E^2 + 10E + 21.$ |

2-10. Fractions: Addition and Subtraction. In arithmetic, fractions are added and subtracted after they have been reduced to the same denominator. In the case of numerical fractions, the common denominator is always taken to be the least common multiple (see Sec. 1-10) of the denominators of the given fractions and is called the **least common denominator**, abbreviated **L.C.D.** The same procedure holds for fractions involving algebraic expressions. *The L.C.D. of fractions involving algebraic expressions is the lowest common multiple of their denominators.* When dealing with numerical fractions we use the term **least**, whereas in algebraic expressions this is replaced by the term **lowest**. Let the student give the reason for this change.

Before finding the lowest common multiple of the denominators it is often convenient to reduce each fraction to its lowest terms. A fraction

is in its **lowest terms** when all factors common to numerator and denominator have been eliminated by dividing numerator and denominator by these factors.

The method for the addition or subtraction of fractions involving algebraic expressions will become apparent from the following illustrations.

Example 1.

$$\frac{2a}{(x+3)a} + \frac{3x^2}{(x+2)x^2} - \frac{13}{x^2+5x+6}.$$

By factoring the denominators of each fraction we obtain

$$\frac{2a}{(x+3)a} + \frac{3x^2}{(x+2)x^2} - \frac{13}{(x+3)(x+2)}.$$

Reducing each fraction to its lowest terms, the expression becomes

$$\frac{2}{x+3} + \frac{3}{x+2} - \frac{13}{(x+3)(x+2)}.$$

Finding the L.C.D. and expressing each fraction as one having the L.C.D., we have

$$\frac{2(x+2)}{(x+3)(x+2)} + \frac{3(x+3)}{(x+2)(x+3)} - \frac{13}{(x+3)(x+2)}.$$

Combining the new numerators and placing over the L.C.D., we obtain

$$\frac{2(x+2) + 3(x+3) - 13}{(x+2)(x+3)}.$$

Removing the parentheses in the numerator and simplifying, we have

$$\frac{2x+4+3x+9-13}{(x+2)(x+3)} = \frac{5x}{(x+2)(x+3)}.$$

This illustration suggests the following steps in the addition or subtraction of fractions.

1. Factor the numerator and denominator of each fraction.
2. Reduce each fraction to its lowest terms.
3. Find the lowest common denominator (L.C.D.).
4. Change each fraction to one having for its denominator the L.C.D. (obtained in Step 3) by multiplying numerator and denominator of each fraction by the quotient of the L.C.D. by the denominator of the fraction.
5. Combine the new numerators and place over the L.C.D.
6. Simplify the numerator and reduce the result to lowest terms (sometimes further simplification is possible if the numerator can be factored).

Example 2.

$$\begin{aligned}
& \frac{r+1}{r^2-4r+4} + \frac{r^2+5r+6}{r^2+r-6} - \frac{r^3}{r^2(r-1)} \\
&= \frac{r+1}{(r-2)(r-2)} + \frac{(r+3)(r+2)}{(r+3)(r-2)} - \frac{r^2 \cdot r}{r^2(r-1)} \quad \text{Step 1} \\
&= \frac{r+1}{(r-2)(r-2)} + \frac{r+2}{(r-2)} - \frac{r}{r-1} \quad \text{Step 2} \\
&= \frac{(r+1)(r-1)}{(r-2)(r-2)(r-1)} + \frac{(r+2)(r-2)(r-1)}{(r-2)(r-2)(r-1)} - \frac{r(r-2)(r-2)}{(r-2)(r-2)(r-1)} \quad \begin{array}{l} \text{Step 3} \\ \text{Step 4} \end{array} \\
&= \frac{(r+1)(r-1) + (r+2)(r-2)(r-1) - r(r-2)(r-2)}{(r-2)(r-2)(r-1)} \quad \text{Step 5} \\
&= \frac{(r^2-1) + (r^3-r^2-4r+4) - r(r^2-4r+4)}{(r-2)(r-2)(r-1)} \\
&= \frac{r^2-1+r^3-r^2-4r+4-r^3+4r^2-4r}{(r-2)(r-2)(r-1)} \\
&= \frac{4r^2-8r+3}{(r-2)(r-2)(r-1)} \\
&= \frac{(2r-1)(2r-3)}{(r-2)(r-2)(r-1)}. \quad \text{Step 6}
\end{aligned}$$

EXERCISES

Perform the indicated operations.

1. $\frac{x}{xy^2} + \frac{y}{x^2y} + \frac{z}{xyz}$.
2. $\frac{a}{a^2bc} - \frac{ab}{ab^2c} + \frac{ac}{bc^2}$.
3. $\frac{t-3}{2t} + \frac{3t-7}{2t}$.
4. $\frac{x}{y-1} - \frac{y}{y^2(y-1)} - \frac{xy}{(y-1)^2}$.
5. $\frac{x}{x-4} + \frac{4}{4-x}$.
6. $\frac{2}{x-2} - \frac{1}{2-x}$.
7. $\frac{r(r-1)}{(r+1)} - \frac{r(r-1)}{(r-1)}$.
8. $\frac{a}{a^2-1} - \frac{2-a}{1-a}$.
9. $\frac{1}{a-1} + \frac{1}{1-a}$.
10. $\frac{a-b}{a+b} - \frac{c-d}{c+d}$.
11. $\frac{a-c}{b-d} - \frac{a}{b}$.
12. $\frac{1}{c^n} + \frac{1}{c^{n-1}}$.
13. $\frac{1-y}{y^n} + \frac{1}{y^{n-1}}$.
14. $\frac{2}{3} + \frac{a}{a+b} + \frac{b}{a-b}$.
15. $\frac{x-(5x+y)}{5x+y} + 1$.
16. $\frac{m+n}{m-n} - \frac{m-n}{m+n} + \frac{m \cdot n}{m^2-n^2}$.

$$17. \frac{3}{2x-1} + \frac{7}{2x+1} - \frac{x}{4x^2-1}.$$

$$19. \frac{9y^n}{14b^6c^4} - \frac{5b^{n-3}}{21yc^2} - \frac{2c^{n-4}}{15yb^5}.$$

$$21. \frac{n}{a^n+1} - \frac{n}{a^n-1}.$$

$$23. \frac{r-1}{r+1} - \left(\frac{r+1}{1-r} + \frac{r^2+1}{r^2-1} \right).$$

$$25. \frac{x^m+y^m}{x^m-y^m} - \frac{x^m-y^m}{x^m+y^m}.$$

$$27. \frac{r+s}{(s-t)(t-r)} + \frac{s+t}{(t-r)(r-s)} + \frac{t+r}{(r-s)(s-t)}.$$

$$28. \frac{a^2y}{(a-b)(a-c)} + \frac{b^2y}{(b-a)(b-c)} + \frac{c^2y}{(c-a)(c-b)}.$$

$$29. \frac{x(x-3)}{x^2(x+5)} - \frac{x^2+2x-15}{x-3} + \frac{x^2-8x+7}{x-7}.$$

$$30. \frac{C+Q}{C-Q} - \frac{C^2+2QC+Q^2}{C+Q} - \frac{C^3-QC^2-2Q^2C}{C(C^2-Q^2)}.$$

$$31. \frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}.$$

$$32. \frac{1}{z+a} + \frac{3z}{(z+a)^2} - \frac{2z-3a}{z^2-2az-3a^2}.$$

$$33. \frac{4Z^2-2Z}{4Z^2+4Z+1} + \frac{Z}{2Z+1}.$$

$$35. \frac{2}{x} + \frac{2}{x^2+2x+1} + \frac{1}{x+1}.$$

$$37. 3x-4 - \frac{9x^2-16}{3x+4}.$$

$$39. \frac{1}{1-x} + \frac{x}{x^2-1} + \frac{x+1}{(x-1)^2}.$$

$$41. \frac{1}{4z^2-4z+1} + \frac{1-z}{1-2z}.$$

$$42. \frac{2}{x^2-11x+30} - \frac{1}{x^2-36} + \frac{1}{x^2-25}.$$

$$43. \frac{9}{x^2+7x-18} - \frac{8}{x^2+6x-16}.$$

$$44. \frac{1}{(a-b)(b-c)} - \frac{1}{(b-a)(c-d)} + \frac{1}{(b-c)(c-d)}.$$

$$45. \frac{3x-4}{x-9} - \frac{4x-1}{x+2} - \frac{x-1}{x^2-7x-18}.$$

$$46. \frac{4-x^2}{x^2-5x-14} - \frac{3x+6}{x+2} + \frac{2x+1}{x-7}.$$

$$18. \frac{R+X}{R-X} - \frac{R+2X}{R+X} + \frac{2R-X}{R^2-X^2}.$$

$$20. \frac{a-b}{a+b} - \left(\frac{a+b}{a-b} - \frac{a+b}{2a+2b} \right).$$

$$22. \frac{WL}{W^2-4y^2} - \frac{L^2}{WL+2yL}.$$

$$24. \frac{a(16-a)}{a^2-4} + \frac{3+2a}{2-a} - \frac{2-3a}{a+2}.$$

$$26. \frac{5}{y-2} + \frac{7}{y-1} - \frac{5}{y-2} - \frac{7}{y+1}.$$

$$34. \frac{R^2}{R^2-2RX+X^2} + \frac{R+X}{R-X}.$$

$$36. \frac{x+y}{x-y} + \frac{1}{x^2-2xy+y^2} - 2.$$

$$38. 1 + \frac{2x^2}{x^2+2ax+a^2} - \frac{x-a}{x+a}.$$

$$40. \frac{3}{x^2-y^2} - \frac{x}{y-x} + \frac{2}{(x+y)^2}.$$

$$47. \frac{20x^2 + 7x - 3}{9x^2 - 1} - \frac{3x + 1}{3x - 1}. \quad 48. c^2 - \frac{c^4 + d^4}{c^2 + cd + d^2} - cd + d^2.$$

$$49. \frac{c + 2d}{c^2 - 2cd + d^2} - \frac{1}{d - c} - \frac{2}{c + 2d}.$$

$$50. \frac{a + 2}{a^2 - 7a + 12} - \frac{a + 2}{6 + a^2 - 5a} - \frac{1}{6a - a^2 - 8}.$$

2-11. Fractions: Multiplication and Division. Multiplication and division of fractions involving algebraic expressions follow the same methods given for arithmetic fractions in Secs. 1-10 and 1-11.

To multiply fractions:

1. Reduce each fraction to its lowest terms.
2. Multiply the numerators to form the numerator of the product; multiply the denominators to form the denominator of the product.
3. Reduce the resulting fraction to lowest terms.

Example 1. Simplify:

$$\begin{aligned} & \frac{ax + x^2}{2b - cx} \cdot \frac{2bx - cx^2}{(a + x)^2} \cdot \frac{a^2 + 2ax + x^2}{ax^2 + x^3} \\ &= \frac{x(a + x)}{2b - cx} \cdot \frac{x(2b - cx)}{(a + x)(a + x)} \cdot \frac{(a + x)(a + x)}{x^2(a + x)} && \text{Step 1} \\ &= \frac{x(a + x)}{2b - cx} \cdot \frac{x(2b - cx)}{(a + x)(a + x)} \cdot \frac{a + x}{x^2} \\ &= \frac{x(a + x)x(2b - cx)(a + x)}{(2b - cx)(a + x)(a + x)x^2} && \text{Step 2} \\ &= 1. && \text{Step 3} \end{aligned}$$

Note that it is convenient in step 2 to keep the expressions in factored form.

To divide one fraction by another invert the divisor and multiply it by the dividend.

Example 2. Simplify:

$$\frac{x^2 + 7x + 12}{x^2 + 2x - 15} \div \frac{x + 4}{x + 5}$$

Factoring the dividend and inverting the divisor, we obtain

$$\frac{(x + 4)(x + 3)}{(x + 5)(x - 3)} \cdot \frac{x + 5}{x + 4}$$

Multiplying and reducing the result to lowest terms,

$$\frac{(x + 4)(x + 3)(x + 5)}{(x + 5)(x - 3)(x + 4)} = \frac{x + 3}{x - 3}.$$

EXERCISES

Perform the indicated operations and simplify:

1. $\frac{7y}{a^3} \cdot \frac{5ab^3}{14y^2}$.
2. $\frac{15r^3s^2}{22x^2y^5} \cdot \frac{14xy^2}{25r^2s}$.
3. $\frac{8a^2}{15b^3x^{n-1}} \cdot \frac{x^{m+n}}{a^{n+1}}$.
4. $\frac{5a}{15x - 10y} \cdot (3x - 2y)$.
5. $\frac{4L + 12}{7} \cdot \frac{14Z}{3L + 9}$.
6. $\frac{r - p}{q + s} \cdot \frac{r + s}{r - p}$.
7. $\frac{abc}{def} \cdot \frac{bcd}{efa} \cdot \frac{cde}{fab}$.
8. $\frac{mv^2g^3}{rs^2t^3} \cdot \frac{r^3s^2t}{m^3v^2g}$.
9. $\frac{ab^2xy^2}{a^3bx^3y} \cdot \frac{a^2bx^2y}{ab^3xy^3}$.
10. $\frac{e - i}{e + i} \cdot \frac{e + i}{e - i}$.
11. $\frac{8r^2}{r^2 - s^2} \cdot \frac{r + s}{2r}$.
12. $\frac{mn^2 - n^3}{m^2 + mn} \cdot \frac{m^3 - mn^2}{2n^2}$.
13. $\frac{P^2 - r^2}{Pq + rq} \cdot \frac{P^2 + r^2}{P - r}$.
14. $\frac{a^2b}{a + b} \cdot \frac{a^2 - b^2}{ab^2}$.
15. $\frac{lx + x^2}{2b - cx} \cdot \frac{2bx - cx^2}{(l + x)^2} \cdot \frac{l + x}{x^2}$.
16. $\frac{x^2 + 3x + 2}{x^2 + 4x + 3} \cdot \frac{x + 3}{x + 2}$.
17. $\frac{ac + bc + a^2 + ab}{ab + ac + b^2 + bc} \cdot \frac{b + c}{c + a}$.
18. $\frac{ei - er - i^2 + ir}{ir - ei - r^2 + er} \cdot \frac{r - e}{e - i}$.
19. $\frac{x^2 - (c + d)x + cd}{x^2 - (c + e)x + ce} \cdot \frac{x^2 - e^2}{x^2 - d^2}$.
20. $\frac{x^2 + 15x + 56}{x^2 + 11x + 30} \cdot \frac{x + 6}{x^2 + 12x + 35}$.
21. $\left(x + y - \frac{xy}{x + y}\right) \div \left(x + \frac{y^2}{x + y}\right)$.
22. $\frac{my^n}{4ax} \div \frac{my^n}{ax}$.
23. $\frac{x^5y^6}{r^7s^8} \div \frac{x^3y^4}{r^6s^6}$.
24. $\frac{u^{n+1}v^{n-1}}{w^{m+1}x^{m-1}} \div \frac{u^nv^n}{w^mx^m}$.
25. $\frac{E^2 - e^2}{E + 2e} \div \frac{E - e}{3E + 6e}$.
26. $\frac{x^2 - y^2}{x} \div \frac{x + y}{y}$.
27. $\frac{r^4 - s^4}{(r - s)^2} \div \frac{r^2 + rs}{r - s}$.
28. $\frac{t^4 - v^4}{t^2 - v^2} \cdot \frac{t^2 - v^2}{(t + v)^2}$.
29. $\frac{4(a^2 - ab)}{b(a + b)^2} \div \frac{6ab}{a^2 - b^2}$.
30. $\frac{m^2 - 1}{m^2 - 3m - 10} \div \frac{m^2 - 12m + 35}{m^2 + 3m + 2}$.
31. $\frac{y^2 + y - 6}{y^2 - 4} \cdot \frac{y^2 + y - 12}{y^2 + y - 6} \cdot \frac{y^2 - 3y - 10}{y^2 - y - 20}$.
32. $\frac{12X_c^5X_l^6}{35w^7\pi^3} \div \frac{24X_c^6X_l^5}{7w^4\pi^6}$.
33. $\left(r^2 - \frac{1}{r^2}\right) \div \left(r - \frac{1}{r}\right)$.
34. $(ax + bx) \div \frac{b^2 - a^2}{3x}$.
35. $\frac{1 + p - p^2 - p^3}{1 - a^2} \div \frac{p^2 - 1}{a^2 - 1}$.
36. $\frac{2y}{x - 4} \cdot (x^2 - 11x + 28)$.

37. $\frac{2x^3 - 2xy^2}{x + 2y} \div \frac{x^2 - y^2}{2x + 4y}$.
38. $\frac{i^2 - 6i + 8}{i^2 + 2i + 1} \div \frac{i - 4}{i + 1}$.
39. $\frac{24dx - 80dy}{14bx - 35by} \div \frac{6xd - 20dy}{28gx - 70gy}$.
40. $\frac{(g_m - 3)^2}{(g_m + 4)^2} \cdot \frac{g_m^2 - 16}{g_m^2 - 9}$.
41. $\frac{4z - 1}{z - 1} \cdot \frac{(z + 1)^2}{16z^2 - 1} \div \frac{z^2 - 1}{4z + 1}$.
42. $\frac{x^2 + 2x - 3}{x^2 + 5x + 6} \cdot \frac{4x^2 - 12x + 40}{9x^2 - 24x + 15}$.
43. $\frac{t^2 + 5t + 6}{t^2 + 7t + 12} \cdot \frac{t^2 + 9t + 20}{t^2 + 11t + 30}$.
44. $\frac{4R^2 - 6R + 2}{4R^2 - 8R + 3} \cdot \frac{4R^2 - 10R + 6}{4(R^2 - 2R + 1)}$.
45. $\frac{I^2 + I - 12}{I^2 - 13I + 40} \cdot \frac{I^2 + 2I - 35}{I^2 + 9I + 20}$.
46. $\frac{(a - b)^2 - 9}{(a + b)^2 - 9} \div \frac{a - b + 3}{a + b + 3}$.
47. $\frac{4a^2 + b^2 - c^2 + 4ab}{4a^2 - b^2 - c^2 - 2bc} \div \frac{2a + b + c}{2a - b - c}$.
48. $\frac{14x^2 - 7x}{12x^3 + 24x^2} \div \frac{2x - 1}{x^2 + 2x}$.
49. $\left(\frac{a + b}{2a + b}\right)^n \left(\frac{2a + b}{a + b}\right)^{n-1} \div \left(\frac{a + b}{2a + b}\right)^2$.
50. $\left(\frac{y - 1}{y + 1}\right)^4 \div \left(\frac{y - 1}{y + 1}\right)^2$.

2-12. Equations. *An equation is a statement of equality. Thus:*

- (1) $a^2 - x^2 = (a + x)(a - x),$
- (2) $4x + 2 = 10,$
- (3) $x + 1 = x$

are equations. In making a statement of equality, we make no assertion concerning whether there are any values of the letters for which the statement is true. The statement may be true for all values of the letters, as (1) above; the statement may limit the values of the letters for which it is true, as (2) above which is true only for $x = 2$; or the statement may not be true for any values of the letters, as (3) above. A more complete discussion of this matter will be found in Chapter 7.

In this section we shall consider only equations which are true for a single value of a single letter. *The value of the letter for which the equation is true is the root or solution of the equation.*

When a statement of equality of this kind is given, our interest is in finding the value of the letter for which it is true. The following rules will aid in finding the root.

1. *The roots of an equation remain the same if the same expression is added to or subtracted from both sides of the equation.*

2. *The roots of an equation remain the same if both sides of an equation are multiplied or divided by the same expression other than zero and not involving the letter whose value is in question.*

Thus the equation

$$(4) \qquad 2x = 4,$$

where x is the unknown, is true for $x = 2$. To illustrate the first of the above two rules, add $5x$ to both sides of (4) and get

$$2x + 5x = 4 + 5x$$

which, like equation (4), is true for only $x = 2$. To illustrate the importance of the restriction in the second of the above two laws, multiply both sides of equation (4) by x and get

$$(2x)x = (4)x$$

which is true not only for $x = 2$ but also for $x = 0$.

Example 1. Find the value of x which satisfies

$$7x + 13 = 3x + 41.$$

In finding the solution of this equation both sides may be decreased by 13 (Rule 1), yielding

$$7x + 13 - 13 = 3x + 41 - 13,$$

$$7x = 3x + 28.$$

Applying Rule 1 again, both sides may be decreased by $3x$, thus giving

$$4x = 28.$$

Dividing both sides by 4 (Rule 2) we get

$$\frac{4x}{4} = \frac{28}{4},$$

$$x = 7.$$

Although the above rules assure us that $x = 7$ satisfies the given equation, we verify the solution $x = 7$:

$$7 \cdot 7 + 13 = 3 \cdot 7 + 41,$$

$$49 + 13 = 21 + 41,$$

$$62 = 62.$$

It is always a good plan to check the accuracy of one's work by substituting the result in the **original** equation to see whether the equation is true for this value.

Rule 1 is applied very frequently. It is, therefore, desirable to state it in a way which mechanizes its application. Observe that the effect of decreasing both sides of the equation in Example 1 by 13 is equivalent to the deletion of the term $+13$ on the left side of the equation and the addition of the term -13 on the right side.

If the equation

$$4x = 28 - 3x$$

is given, then, applying Rule 1, $3x$ may be added to both sides of the equation, yielding

$$\begin{aligned} 4x + 3x &= 28 - 3x + 3x \\ &= 28. \end{aligned}$$

The result of the operation consists in omitting the term $-3x$ of the right side and adding the term $+3x$ to the left side. We call this operation **transposition** of the term $3x$. This operation is an application of Rule 1 and may be explained in the following way:

Any term of one side of an equation may be transposed to the other side if its sign is changed.

Example 2. Find the value of x which satisfies

$$(5) \quad 3x + 7(4 - x) + 6x = 15.$$

Clearing of parentheses and combining terms:

$$\begin{aligned} 3x + 28 - 7x + 6x &= 15, \\ 2x + 28 &= 15. \end{aligned}$$

Transposing $+28$ from the left to the right side:

$$\begin{aligned} 2x &= 15 - 28, \\ 2x &= -13. \end{aligned}$$

Dividing each side by 2, according to Rule 2:

$$\begin{aligned} \frac{2x}{2} &= -\frac{13}{2}, \\ x &= -\frac{13}{2}. \end{aligned}$$

To check, substitute this solution in (5)

$$\begin{aligned} 3(-\frac{13}{2}) + 7(4 + \frac{13}{2}) + 6(-\frac{13}{2}) &= 15, \\ -\frac{39}{2} + 28 + \frac{91}{2} - \frac{78}{2} &= 15, \\ -\frac{117}{2} + \frac{147}{2} &= 15, \\ \frac{30}{2} &= 15, \\ 15 &= 15. \end{aligned}$$

An equation which can be reduced to the form

$$ax + b = 0, \quad a \neq 0,$$

is called a **linear equation in x** . All equations considered in this section are of this form.

To solve an equation containing fractions, first reduce each fraction to its lowest terms. Then multiply each side of the equation by the least common denominator of all the denominators. This process is called **clearing of fractions**.

Example 3. Find the value of x which satisfies

$$\frac{5x}{6} - \frac{x-1}{2} = \frac{3x+1}{3} + \frac{3}{8}(x+1).$$

The common denominator is 24. Then

$$\frac{20x}{24} - \frac{12(x-1)}{24} = \frac{8(3x+1)}{24} + \frac{9(x+1)}{24},$$

whence

$$\frac{20x - 12(x-1)}{24} = \frac{8(3x+1) + 9(x+1)}{24}.$$

By Rule 2, we multiply both sides of the equation by 24, obtaining

$$20x - 12(x-1) = 8(3x+1) + 9(x+1).$$

Clearing parentheses

$$20x - 12x + 12 = 24x + 8 + 9x + 9.$$

By Rule 1,

$$20x - 12x - 24x - 9x = 8 + 9 - 12,$$

$$-25x = 5,$$

$$x = \frac{5}{-25},$$

$$x = -\frac{1}{5}.$$

The validity of this result may be verified by substituting it in the original equation.

EXERCISES

Solve the following equations (a , b , c are to be treated as known quantities).

1. $4x + 13 = 21.$

2. $7y + 14 = 2y - 6.$

3. $3Z + 2(Z - 4) = 6Z - 10(Z - 2).$

4. $4I + 7 = 3(I - 4) + \frac{6I}{4}.$

5. $\frac{5R}{4} + 2R = \frac{3+R}{3} - 7R.$

6. $ax + a = ab.$

7. $\frac{x}{a+2} + \frac{x}{a-2} = 2a.$

8. $3E + 2 = E + 30.$

9. $\frac{4X_L}{5} - 6X_L + 2 = \frac{X_L}{4}$.
10. $7s - 55 = 18 - 2s - 1$.
11. $2v + \frac{v}{3} = \frac{35}{6}$.
12. $5w - \frac{7}{2}w = 17 - w$.
13. $12x + 5x + 20 - 8x = 48 + 3x - 4$.
14. $3(k - 5) + 4k + 8 = 5(4k - 20)$.
15. $5(2q - 10) + 7q - 15 = 20q$.
16. $\frac{x+1}{3} + \frac{x+1}{7} = 4$.
17. $\frac{3s}{4} + 5 = 91 - 10s$.
18. $7y + 2 - y = 17$.
19. $8 + 2y + \frac{y}{4} = \frac{3}{4} + \frac{2y}{3}$.
20. $4K - 15 = 2K + 11$.
21. $2v + \frac{v}{2} - \frac{3v}{4} + 14 = 7v - \frac{v}{4} + \frac{2}{7}v - \frac{15}{2}$.
22. $14k - 20 + 7k - 2 = 6k + \frac{37}{2}$.
23. $x - 0.02x - 5 = 117.50$.
24. $2l + 5 = l + 20$.
25. $22w + 30 = 17w + 90$.
26. $250r - 20 = 20r + 440$.
27. $12.75i + 6.25 = 7.25i + 17.25$.
28. $\frac{2}{3}x + 4 = \frac{1}{3}x + 4\frac{2}{3}$.
29. $-2s + 4 = -12$.
30. $2 - (3 - 4 - x) = 3$.
31. $5(2n + 5) = 7(4n - 17)$.
32. $\frac{14 - 3q}{13} = \frac{4q - 5}{10}$.
33. $\frac{2}{3}(11 - 2l) = \frac{1}{4}(3l - 2) - \frac{1}{2}$.
34. $(a + y)(b + y) - (c - y)(a - y) = 0$.
35. $\frac{4x}{a-3} + \frac{2x}{a+3} - \frac{x}{a^2-9} = 6$.
36. $(t-1)(t-2) - (t-3)(t+1) - 17 = 0$.
37. $0.1x + 0.21x = -0.62$.
38. $x^2 + b^2 = (a - x)^2$.
39. $(x-7)(2x+1) - (3x-1)(2x-5) + (4x-3)(x+7) + 4 = 0$.
40. $q^3 + 1 = (q+1)(q^2 - 4q + 7) + 3q^2$.
41. $\frac{1}{2}(x-1) + \frac{1}{3}(x - \frac{1}{2}) + \frac{1}{4}(x - \frac{1}{3}) = \frac{1}{5}(x - \frac{1}{5})$.
42. $\frac{5E - 12}{0.6} = 0.06E + 5 - 0.3E - 0.083E$.
43. $\frac{x+2b}{a} + \frac{x-a}{2b} = 0$.
44. $\frac{7r}{10} - \frac{2}{5}(11r-5) = \frac{8}{15}(r+10) - 24\frac{1}{2}$.
45. $k - 3 + \frac{k}{12} = (k+3)^2 - \frac{(2k+1)^2}{4}$.
46. $0.125q - 0.7q + \frac{9.2}{160} = 0$.
47. $\frac{x-2}{5} - \frac{x-3}{4} = \frac{x+1}{2} - \frac{x+5}{3}$.

$$48. \frac{I-a}{2a} + \frac{I+6b}{3b} + \frac{I-2c}{4c} = 1.$$

$$49. \frac{e+3}{4} + \frac{7e-2}{5} = \frac{1}{2} \left(\frac{5e-1}{2} + \frac{10e+8}{9} \right).$$

$$50. \frac{7k+6}{13} + \frac{5k+4}{9} = 3 - \frac{4k-1}{3}.$$

2-13. Factorable Quadratic Equations. A quadratic equation is one which can be reduced to the form

$$(1) \quad ax^2 + bx + c = 0, \quad a \neq 0,$$

where a , b , and c are known and x is unknown.

Occasionally the left side of (1) can be factored by the type forms of Sec. 2-8. Then the left is the product of two factors. Now if $A \cdot B = 0$, either $A = 0$ or $B = 0$, or $A = 0$ and $B = 0$. Hence equation (1) is satisfied if at least one factor is zero. The numbers which satisfy (1) are the **roots** or solutions of the quadratic equation.

Two examples will illustrate the process.

Example 1. Find the values of x which satisfy the quadratic equation

$$(2) \quad x^2 - 4x + 3 = 0.$$

Factoring the left side we obtain

$$(x-3)(x-1) = 0.$$

This product is zero if

$$x-3 = 0 \quad \text{or} \quad x-1 = 0;$$

hence,

$$x = 3 \quad \text{or} \quad x = 1.$$

Substituting $x = 3$ in (2), we obtain

$$(3)^2 - 4(3) + 3 = 0,$$

$$9 - 12 + 3 = 0,$$

$$0 = 0.$$

Substituting $x = 1$, we obtain

$$(1)^2 - 4(1) + 3 = 0,$$

$$1 - 4 + 3 = 0,$$

$$0 = 0.$$

Therefore both solutions satisfy the original equation.

Example 2. Solve

$$(3) \quad -\frac{x}{4} + \frac{3}{2} = \left(\frac{x}{2}\right)^2.$$

Rewriting (3) in form (1) we obtain,

$$(4) \quad x^2 + x - 6 = 0.$$

Factoring (4)

$$(x + 3)(x - 2) = 0.$$

Equating each factor to 0, we obtain

$$x + 3 = 0,$$

First solution:

$$x = -3;$$

$$x - 2 = 0,$$

Second solution:

$$x = 2.$$

Substituting the first solution into (3), we obtain,

$$-\frac{-3}{4} + \frac{3}{2} = \left(\frac{-3}{2}\right)^2,$$

$$\frac{9}{4} = \frac{9}{4}.$$

Substituting the second solution into (3) we obtain,

$$-\frac{2}{4} + \frac{3}{2} = \left(\frac{2}{2}\right)^2,$$

$$1 = 1.$$

The solutions obtained are therefore correct.

EXERCISES

Solve the following by factoring and check the solutions obtained.

1. $x^2 + 4x + 4 = 0.$
2. $y^2 + 6y + 9 = 0.$
3. $t^2 + 5t + 6 = 0.$
4. $E^2 - 9 = 0.$
5. $4x^2 + 12x = -9.$
6. $I^2 = 3 - 2I.$
7. $12V^2 - 12V = -3.$
8. $4R^2 = 25.$
9. $2x^2 - 22x + 60 = 0.$
10. $14 = 9t - t^2.$
11. $3C^2 + \frac{8}{3}C + \frac{3}{25} = 0.$
12. $9y^2 - 4y + \frac{2}{9} = 0.$
13. $2R^2 - \frac{4}{3}R + \frac{2}{9} = 0.$
14. $5V^2 - 40V + 80 = 0.$
15. $4E^2 = \frac{1}{5}E - \frac{9}{25}.$
16. $R^2 - \frac{2}{3}R = -\frac{1}{9}.$
17. $\frac{V_0^2}{6} - \frac{5}{6}V_0 = 1.$
18. $3R^2 - 9R - 84 = 0.$
19. $9Z^2 - 16 = 0.$
20. $8E^2 - 50 = 0.$
21. $4C^2 - \frac{9}{49} = 0.$
22. $3Z^2 + 30Z + 75 = 0.$
23. $I^2 - 14I = -49.$
24. $9R^2 + 2R + \frac{1}{9} = 0.$
25. $5Z^2 = 5Z + 30.$
26. $2I^2 - 32 = 0.$
27. $E^2 - \frac{9}{4} = 0.$
28. $3I^2 + 3I + \frac{3}{4} = 0.$
29. $3x^2 - \frac{27}{4} = 0.$
30. $25C^2 - 81 = 0.$

2-14. Applications to Engineering Formulas. Many experimental or theoretical facts in science are stated by equations. For example, the heat in calories produced by a steady direct current is equal to 0.24 multiplied by the square of the current in amperes, by the resistance in ohms, and by the time in seconds during which the current flows. Symbolically, this law may be expressed by the formula

$$(1) \quad H = 0.24I^2RT,$$

where H is the heat in calories, I the current in amperes, R the resistance in ohms, and T the time in seconds. In the form given, (1) gives H when I , R , and T are known. Suppose that H , I , and R are given, and T is to be determined. We state this problem by saying that equation (1) has to be **solved for T** . In the light of the preceding section, then, we may regard T as unknown and the other letters as known, and by Rule 2

$$T = \frac{H}{0.24I^2R}.$$

In like manner, if H , I , and T are given and the equation has to be solved for R , we obtain from (1) the result

$$R = \frac{H}{0.24I^2T}.$$

Many formulas encountered in practical work, like the illustration above, have the form

$$(2) \quad ax + b = 0,$$

where a and b are given and x is to be determined. The methods of the previous section therefore apply.

EXERCISES

In the formulas below, consider all the quantities as given except the indicated quantity to be solved for. A hint as to the meaning of most of the formulas is given. If the reader is interested in a description of the exact physical meaning of the formula and the units involved, an appropriate textbook should be consulted.

GIVEN	SOLVE FOR	PHYSICAL APPLICATION
1. $R_T = R_1 + R_2 + R_3$.	R_3 .	Total resistance in a series circuit.
2. $L_T = L_1 + L_2 + 2M$.	M .	Total inductance of two coils with inductive coupling.
3. $L_T = L_1 + L_2 - 2M$.	L_1 .	
4. $L_T = L_1 + L_2$.	L_2 .	Total inductance of uncoupled coils.
5. $I_z = I_{zo} + Ad^2$.	I_{zo}, A .	Rotational inertia.
6. $M = D_o - pc + 3D_w$.	D_o, p .	
7. $I = \frac{E}{R}$.	E .	Ohm's law.

GIVEN	SOLVE FOR	PHYSICAL APPLICATION
8. $P = IE.$	$I.$	Electrical power.
9. $P = I^2R.$	$R.$	Electrical power.
10. $p = ei.$	$i.$	Instantaneous power in alternating current circuits.
11. $Fd = Wh.$	$d, W.$	Work done on a falling body.
12. $\frac{P_2V_2}{T_2} = \frac{P_1V_1}{T_1}.$	$P_1, \frac{V_1}{V_2}.$	Equation for perfect gas.
13. $d = \frac{wl^2}{8d}.$	$w.$	
14. $C = \frac{5}{9}(F - 32).$	$F.$	Centigrade-Fahrenheit relationship.
15. $v^2 = v_0^2 + 2gh.$	$g, v_0^2.$	Initial-final speed relation.
16. $H = \frac{tws}{33,000}.$	$w.$	Horsepower.
17. $F = \frac{w}{g}a.$	$a, w.$	One of Newton's laws.
18. $F = 2Wah.$	$h.$	
19. $d = vt.$	$t.$	Distance traveled.
20. $a = \frac{v - v_0}{t}.$	$v_0.$	Average acceleration.
21. $fd = \frac{1}{2}mv^2.$	$m, d.$	Work-kinetic energy relation.
22. $R = \frac{kl}{d^2}.$	$l.$	Resistance of wire.
23. $R = \frac{kl}{A}.$	$l.$	Reluctance ("magnetic resistance").
24. $Ft = mv_2 - mv_1.$	$m, F.$	Mechanical impulse.
25. $W = \frac{Kbd^2}{L}.$	$b.$	
26. $Q = CE.$	$E.$	Quantity of charge on a condenser.
27. $T = \frac{12(D - d)}{l}.$	$D.$	
28. $L = \pi(R + r) + 2d.$	$d, r.$	Length of a belt around two pulleys.
29. $B = \frac{\phi}{A}.$	$\phi.$	Magnetic field intensity.
30. $f = \frac{8.94B^2A}{10^8}.$	$A.$	Tractive force of electromagnets.
31. $H = \frac{D^2N}{2.534}.$	$N.$	
32. $P = \frac{KB_{\max}^{1.6}}{10^7}.$	$B_{\max}^{1.6}.$	The Steinmetz equation for hysteresis loss.
33. $\frac{r_1}{r_2} = \frac{r_3}{r_4}.$	$r_1.$	Wheatstone bridge formula.
34. $E = \frac{N\phi}{10^8t}.$	$\phi, N.$	Average electromotive force generated by cutting magnetic lines.

GIVEN	SOLVE FOR	PHYSICAL APPLICATION
35. $F = \frac{22.5BI}{10^8}$.	I, B .	Force of magnetic field on a coil carrying a current.
36. $S = \frac{nW \cdot 33 \cdot 10^8}{r^4}$.	W .	
37. $F = \frac{F_1 d}{2\pi l}$.	F_1 .	
38. $L_{av.} = \frac{1.26N^2 A \mu}{10^8 l}$.	A .	Self-inductance of long coils.
39. $M = \frac{-1.26N_1 N_2 A \mu}{10^8 l}$.	N_2 .	Mutual inductance.
40. $S = \frac{d^2 w}{8p}$.	w .	Stretch of trolley wires.
41. $i = \frac{wh}{R}$.	h .	
42. $y = \frac{1}{4}x(R - x)$.	R .	
43. $C = \frac{8.84KA}{10^8 d}$.	A .	Calculation of capacitance.
44. $F = \frac{W}{e^a}$.	W .	
45. $X_L = 2\pi fL$.	f, L .	Inductive reactance.
46. $f = \frac{v}{\lambda}$.	v .	Frequency of radio wave in terms of velocity and wavelength.
47. $L = 0.0251d^2 n^2 l$.	d^2, l .	Inductance of a single-layer solenoid.
48. $D_o = \frac{(N + 2)P_c}{\pi}$.	N, P_c .	
49. $i_{av.} = C \frac{e_2 - e_1}{t_2 - t_1}$.	e_2, C .	Current flowing into a condenser.
50. $e_{av.} = L \frac{i_2 - i_1}{t_2 - t_1}$.	L, i_2 .	Average induced voltage.

2-15. Substitution of Particular Values in Formulas. The formulas given in the preceding section show how to solve for certain quantities if other quantities connected with them are given. By the statement, **a quantity is given**, it is understood that the following is true:

1. A unit is known by which the particular quantity is measured. It must be explained, for instance, whether a certain length is measured in feet or in inches.

2. The number of units of measurement in the given quantity is known. This number of units is often called the numerical value of the quantity.

For instance, if an electromotive force of a certain battery is to be given, a unit of measurement has to be chosen first. In this case the

unit is the volt. Thus, if an experiment shows that the electromotive force of this battery is 6.2 volts, then 6.2 is the numerical value of the electromotive force when measured in volts.

In almost all practical applications it is necessary to compute the numerical value of quantities which are given by mathematical formulas. For example, the heat produced in an electric circuit is computed by the formula

$$(1) \quad H = 0.24I^2RT.$$

In every formula we suppose that the quantities involved in it are measured in certain definite units. As a rule the formula will change if the units of measurements on which it is based are replaced by other units. For example, in formula (1), it is supposed that H is measured in calories, I in amperes, R in ohms, and T in seconds.

The numerical value of H can be computed if the numerical values of I , R , and T are known. To find H it is necessary only to replace the letter symbols I , R , and T by their numerical values and to perform the operations indicated by the formula. Thus, if it is known that $I = 4$ amperes, $R = 50$ ohms and $T = 5$ minutes = 300 seconds, then

$$H = 0.24 \times 4^2 \times 50 \times 300 \text{ calories} = 57,600 \text{ calories.}$$

In this case we say that *the numerical values* $I = 4$, $R = 50$, $T = 300$, have been substituted in the formula (1) or that H has been computed from formula (1) for the particular values $I = 4$, $R = 50$, $T = 300$.

It should be observed that the particular values which are substituted in a formula are, as a rule, results of measurements and are, therefore, numbers of limited accuracy, given only with a certain number of significant digits. It is therefore important to observe the rules developed in Chapter 1 for dealing with inaccurate numbers. The slide rule is often very useful for the evaluation of formulas and should be used whenever its precision is sufficient.

Example 1. The resistance R of a wire, measured in ohms, is given by the formula

$$R = \frac{kl}{d^2},$$

where l is the length of the wire measured in feet, d its diameter measured in mils (1 mil = 0.001 in.), and k the resistance in ohms per mil-foot of a wire of the same material, that is, a wire of length 1 ft. and diameter 1 mil. Compute the resistance of a wire of length 5 miles and diameter 80.8 mils when $k = 10.4$ ohms per mil-foot.

Since the formula requires that the length be expressed in feet, the value $l = 5$ miles has to be changed to feet, giving

$$l = 5 \times 5280 = 26,400 \text{ ft.}$$

The substitution of the given data in the formula gives

$$R = \frac{10.4 \times 26,400}{80.8^2} = \frac{10.4 \times 26,400}{6530} = 42.0 \text{ ohms,}$$

the computations being performed on a slide rule.

Example 2. The resonance frequency in cycles per second of a circuit is given by the formula

$$f = \frac{1}{2\pi\sqrt{LC}},$$

where L is the inductance in henries and C the capacitance in farads. Find the value of f for

$$L = 4 \times 10^{-6} \quad \text{and} \quad C = 3 \times 10^{-10}.$$

Substitution of the given values in the formula for f gives

$$f = \frac{1}{2\pi\sqrt{4 \times 10^{-6} \times 3 \times 10^{-10}}} = \frac{1}{2\pi\sqrt{12 \times 10^{-16}}} = \frac{1}{2\pi\sqrt{12} \cdot 10^{-8}} \\ = 0.0459 \times 10^8 = 4.59 \times 10^6,$$

where the computations were performed on a slide rule.

EXERCISES

In the following problems observe the rules about significant figures given in Sec. 1-13.

1. Given $P = I^2R$, find the value of P in watts when $I = 15.4$ amperes and $R = 25.7$ ohms.

2. Given $H = 0.24I^2R$, find the value of H in calories when $I = 0.56$ amperes and $R = 195$ ohms.

3. Given $P = \frac{E^2}{R}$, find the value of P in watts when $E = 115$ volts and $R = 75$ ohms.

4. Given $v = \sqrt{v_0^2 + 2gh}$, find v in feet per second, when $v_0 = 20$ ft. per sec., $g = 32$ ft. per sec. per sec., and $h = 500$ ft.

5. Given $s = v_0t + \frac{g}{2}t^2$, find s in feet when $v_0 = 850$ ft. per sec., $t = 23$ sec., and $g = 32$ ft. per sec. per sec.

6. Given $a = \frac{v - v_0}{t}$, find a in feet per second per second when $v = 85$ ft. per sec., $v_0 = 17$ ft. per sec., and $t = 7.2$ sec.

7. The watts of power P dissipated in a resistor R ohms is given by the relation $P = \frac{E^2}{R}$, where E is the voltage drop across the resistor in volts. What power is dissipated in a 1.37 megohm resistor, across which the voltage is 3.1 microvolts?

8. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, where $\pi = 3.1416$. What is the volume of a sphere 2 ft. in diameter?

9. How many cubic inches of rubber are required to make a hollow rubber ball having an outer radius of $2\frac{1}{4}$ in. and an inner radius of $1\frac{3}{4}$ in.?

10. The distance s in feet through which an object falls owing to the action of gravity is $s = \frac{1}{2}gt^2$, where g is the acceleration due to gravity and t is the time of fall in seconds. If the acceleration due to gravity is 32 ft. per sec. per sec., how far will an object fall in 8 sec.?

11. If two point electric charges of magnitude q and q' are separated by a distance r centimeters, Coulomb's law states that the force F in dynes which one charge exerts on the other is given by

$$F = \frac{qq'}{kr^2},$$

where k is a constant. Given two point charges of 5 and 10 electrostatic units respectively, separated by a distance of 4 cm., what is the force acting between them, if it is assumed that $k = 1$?

12. The tractive force f in pounds of an electromagnet is given by

$$f = \frac{8.94B^2A}{10^8},$$

where B is measured in gausses and A in square centimeters. Compute f when $B = 1.87 \times 10^4$, $A = 21.3$ sq. cm.

13. The impedance Z , measured in ohms, of a circuit is given by

$$Z = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}.$$

Compute Z when $R = 145$ ohms, $f = 5.4 \times 10^6$ cycles per sec., $L = 3.2 \times 10^{-8}$ henry, and $C = 2.5 \times 10^{-10}$ farad.

14. The self-inductance L in henries of a certain type of coil is given by

$$L = \frac{1.26N^2A\mu}{10^9l}.$$

Compute L when $N = 450$, $A = 320$ sq. cm., $l = 210$ cm., and $\mu = 3200$.

15. The acceleration a in feet per second per second which the gravitational attraction of the earth would produce at the distance of the moon is given by

$$a = \frac{4\pi^2R}{T^2},$$

where R is measured in miles and T in seconds. Find the value of a when $R = 2.4 \times 10^5$ miles, $T = 27.3$ days.

16. The reactance X_c in ohms of a condenser is given by

$$X_c = \frac{1}{2\pi fC},$$

where f is the frequency of the impressed voltage in cycles per second and C is the capacity of the condenser in farads. What is the reactance of a 115-micromicrofarad condenser at a frequency of 3.289 megacycles?

17. The resonant frequency f_r in cycles per second of a condenser and coil in series is given by

$$f_r = \frac{1}{2\pi\sqrt{LC}},$$

where L is the inductance of the coil in henries and C is the capacitance of the condenser in farads. What is the resonant frequency of a 2.1374 millihenry coil and a 12-micromicrofarad condenser in series? What value of π should be used in this problem? Why?

2-16. Systems of Two Linear Equations in Two Unknowns. Consider the equation

$$(1) \quad x - 2y = 5,$$

where x and y are both unknown. This equation is satisfied when $x = 7$ and $y = 1$, also when $x = 5$ and $y = 0$. There are many such pairs of values which satisfy (1). To find pairs other than those given, choose a value of one letter, say y , arbitrarily, and then from (1) find the corresponding value of x . For example, let $y = 3$. Then from (1)

$$(2) \quad x = 5 + 2y,$$

whence

$$x = 5 + 2 \cdot 3,$$

$$x = 5 + 6 = 11,$$

and the pair of values $x = 11$, $y = 3$ satisfies the equation (1).

Consider now a second equation in two unknowns.

$$(3) \quad x + y = 2.$$

As before, this equation is satisfied by many pairs of values.

Is there a pair of values, one for x and one for y , which satisfies both (1) and (3)? The x -value satisfying (3) must also satisfy (2). Substituting (2) in (3),

$$(5 + 2y) + y = 2,$$

$$5 + 3y = 2,$$

$$3y = -3,$$

$$y = -1.$$

Using this value in (2),

$$x = 5 + 2(-1),$$

$$x = 5 - 2,$$

$$x = 3.$$

The reader may verify that $x = 3$, $y = -1$ satisfies both (1) and (3).

The method for finding the pair of values satisfying both equations indicated above usually applies to pairs of equations of the form

$$(4) \quad \begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are known, and x and y are unknown. A more complete discussion is given in Chapter 13. The equations (4) are termed **linear** because the unknowns x and y enter to the first power only.

To solve a system of two linear equations in two unknowns, solve for one unknown in one equation and substitute this result in the other equation, thus obtaining one equation in one unknown.

An alternative way of solving a system of two linear equations, which is usually more convenient, is given by the following rule.

Multiply the two equations with numerical factors which are chosen so that the coefficients of one of the two unknowns have the same numerical values in both equations. By adding or subtracting the two equations, a new equation with only one unknown quantity is obtained. Solve this equation. In order to find the second unknown quantity, substitute the value which has been found and solve for the remaining unknown quantity. An alternative method for finding the second unknown is to repeat the above process of forming equal coefficients for the other unknown.

From the following example the reader will see that this method works in a very convenient way.

Example. Find the values of I_1 and I_2 which satisfy

$$\begin{aligned} 7I_1 - 3I_2 &= 26, \\ 2I_1 + 11I_2 &= 43. \end{aligned}$$

Multiply the first equation by 2, the second by 7:

$$\begin{aligned} 14I_1 - 6I_2 &= 52, \\ 14I_1 + 77I_2 &= 301. \end{aligned}$$

Subtract the first of these equations from the second:

$$\begin{aligned} 83I_2 &= 249, \\ I_2 &= 3. \end{aligned}$$

Substituting this value for I_2 in the first equation, we obtain

$$\begin{aligned} 7I_1 - 9 &= 26, \\ 7I_1 &= 35, \\ I_1 &= 5. \end{aligned}$$

The two values give the desired solution.

Check: Substituting 5 for I_1 and 3 for I_2 in the given equations, we obtain

$$35 - 9 = 26,$$

$$26 = 26.$$

$$10 + 33 = 43,$$

$$43 = 43.$$

EXERCISES

Solve the following systems of equations by either of the methods described above and check your answers.

1. $x + y = 3,$
 $3x + 2y = 1.$
3. $I_1 - 3I_2 = 0,$
 $5I_1 - 4I_2 = 11.$
5. $I_1 + I_2 = 7,$
 $2I_1 + 5I_2 = 20.$
7. $7Z_1 - 5Z_2 = 1,$
 $5Z_1 + Z_2 = 19.$
9. $6Z_1 - 5Z_2 = -3,$
 $4Z_1 + 2Z_2 = 14.$
11. $4X_1 - 2X_2 = 2,$
 $3X_1 + X_2 = 14.$
13. $3X_1 + 4X_2 = 17,$
 $X_1 + 2X_2 = 7.$
15. $3R_1 - 2R_2 = 28,$
 $R_1 - 3R_2 = 7.$
17. $7E_1 + 3E_2 = 1,$
 $4E_1 - E_2 = 13.$
19. $3I_1 - 9I_2 = 15,$
 $6I_1 - 7I_2 = 41.$
21. $2E_1 - 3E_2 = 6,$
 $2E_1 + 4E_2 = -8.$
23. $4Z_1 + 7Z_2 = 26,$
 $3Z_1 - 4Z_2 = 1.$
25. $5L - 6C = 2,$
 $4L - 7C = -5.$
27. $Z_1 - 2Z_2 = 3,$
 $2Z_1 + Z_2 = 4.$
29. $2E_1 + 3E_2 = 13,$
 $5E_1 - 2E_2 = 7.$
31. $2L + 5C = 5,$
 $5L + 2C = 20.$
33. $15I_1 - 17I_2 = 100,$
 $15I_1 - 7I_2 = 100.$
35. $3R_1 + 2R_2 = 19,$
 $2R_1 - 6R_2 = 20.$
2. $I_1 + I_2 = 0,$
 $3I_1 + 5I_2 = 50.$
4. $8E_1 - 4E_2 = 0,$
 $3E_1 - E_2 = 2.$
6. $4s + 3t = 10,$
 $s + 5t = 11.$
8. $3L + 9C = 15,$
 $5L - 4C = 6.$
10. $Z_1 + 3Z_2 = 14,$
 $3Z_1 - 4Z_2 = 3.$
12. $3x - 4y = 15,$
 $x + 2y = 5.$
14. $4R_1 - 3R_2 = 3,$
 $2R_1 - R_2 = 3.$
16. $6Z_1 + Z_2 = 8,$
 $Z_1 - 2Z_2 = -3.$
18. $-2x + y = 3,$
 $x + 2y = 4.$
20. $x + 6y = 4,$
 $2x - y = -3.$
22. $2R_1 - 3R_2 = 5,$
 $3R_1 - 4R_2 = 6.$
24. $3R_1 - R_2 = 9,$
 $5R_1 - 6R_2 = 2.$
26. $4E_1 + 5E_2 = 6,$
 $2E_1 + 3E_2 = 4.$
28. $2x + 4y = 3,$
 $4x - 2y = -9.$
30. $5x + 4y - 6 = 0,$
 $3x + 2y - 4 = 0.$
32. $4I_1 - 10I_2 = 0,$
 $7I_1 - 11I_2 = 2.$
34. $6x - 4y + 42 = 0,$
 $7x - 5y + 50 = 0.$
36. $3x + 2y = 60,$
 $2x - 3y = -24.$

$$37. E_1 - E_2 = 1,$$

$$\frac{2E_1}{5} + \frac{3E_2}{4} = 5.$$

$$39. \frac{L+C}{8} + \frac{L-C}{6} = 5$$

$$\frac{L+C}{4} - \frac{L-C}{3} = 10.$$

$$38. \frac{R_1}{3} - \frac{R_2}{4} = 5,$$

$$\frac{R_1}{8} + \frac{R_2}{3} = 7.$$

$$40. 2s + \frac{t-2}{5} = 21$$

$$4s + \frac{t-4}{6} = 29.$$

2-17. Applications of Linear Equations in One Unknown to Solution of Problems. In arithmetic, problems are solved by direct computation with the given numbers. In algebra, the method of solving a problem is quite different.

The fundamental idea of algebra is to denote numbers by letter symbols. We can extend this and also denote by letter symbols quantities which are not known and have to be computed in a problem. Thus in solving a problem by the algebraic method, a letter is chosen to represent the unknown quantity. Using this letter the statements in the problem are translated into algebraic expressions and equations to which the various processes of algebra can be applied.

The algebraic method of solving problems is much more powerful than the arithmetic method. To appreciate this statement, the reader need only try to solve arithmetically almost any problem in this section.

Example 1. A city consumes 28,000,000 kilowatts of electric power which is supplied by four sources. The first source supplies twice as much as the second, and each of the third and fourth sources supplies 1,000,000 kilowatts less than the second. How much power comes from each source?

To form an algebraic solution, we let x represent an unknown quantity.

x = Number of kilowatts supplied by the second source.

Then according to the statement of the problem, the first source supplies twice as much as the second source, whence

$2x$ = Number of kilowatts supplied by the first source.

Since each of the third and fourth sources supplies 1,000,000 kilowatts less than the second, we have

$x - 1,000,000$ = Number of kilowatts supplied by the third source,

and

$x - 1,000,000$ = Number of kilowatts supplied by the fourth source.

The sum of all these is 28,000,000:

$$x + 2x + (x - 1,000,000) + (x - 1,000,000) = 28,000,000.$$

We may now dismiss the requirements of the problem and apply the formal processes of algebra to the equation which has been obtained. Then it becomes

$$x + 2x + x - 1,000,000 + x - 1,000,000 = 28,000,000,$$

$$5x - 2,000,000 = 28,000,000,$$

$$5x = 30,000,000,$$

$$x = 6,000,000.$$

Recalling the question asked in the problem and using the expressions previously stated, we have the following:

$$6,000,000 = \text{Number of kilowatts supplied by the second source,}$$

$$12,000,000 = \text{Number of kilowatts supplied by the first source,}$$

$$5,000,000 = \text{Number of kilowatts supplied by the third source,}$$

$$5,000,000 = \text{Number of kilowatts supplied by the fourth source.}$$

Checking the correctness of the result,

$$6,000,000 + 12,000,000 + 5,000,000 + 5,000,000 = 28,000,000.$$

The importance of a clear statement of the relation between the conditions of the problem and the algebraic expressions used in its solution cannot be overemphasized. Such a statement, which may be regarded as the *English-Algebra dictionary* for the problem under consideration, is essential for a clear understanding of the solution and will aid very materially in the construction of this solution.

In connection with the algebraic solution of problems the student should note the following two facts.

1. Any one of the unknown quantities may be denoted by the letter.
2. Any letter symbol may be used to denote the unknown.

Example 2. Find a number such that the sum of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ of the number is equal to 47.

Let x be the number sought. Therefore

$$\frac{x}{3} = \text{Third part of the number,}$$

$$\frac{x}{4} = \text{Fourth part of the number,}$$

$$\frac{x}{5} = \text{Fifth part of the number.}$$

Thus the sum of the third, fourth, and fifth parts of the number is equal to

$$\frac{x}{3} + \frac{x}{4} + \frac{x}{5},$$

and we have the equation

$$\frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 47,$$

which when solved gives us

$$20x + 15x + 12x = 2820,$$

$$47x = 2820,$$

$$x = 60,$$

the number sought.

To check the correctness of the result we see that the third, fourth, and fifth parts of 60 are 20, 15, and 12, which when added give 47.

In applying algebra to the solution of problems the student should proceed as follows.

1. *Denote the unknown quantity by any symbol. Usually x or y is used. If there are several quantities whose values are not known, denote any one of them by a symbol. If the equation can be solved by the methods of this paragraph, the other unknown quantities can be expressed in terms of this symbol.*

2. *Write down in algebraic language every statement of the given problem or any inference which can possibly be made from it. This should be done by short steps requiring only a few words at a time.*

3. *Using the results of the preceding step, frame an equation by expressing the conditions of the problem in algebraic language.*

4. *Solve the equation resulting from the preceding step.*

5. *Check whether the values found are the true solutions of the problem.*

Example 3. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 36 greater than the first number. Find the number.

Both digits of the number are unknown. We decide to denote one of them, for instance the digit in the ten's place, by the symbol x . Then

$$6 - x = \text{Digit in unit's place.}$$

Hence the number is

$$(6 - x) + 10x.$$

Reversing the order of the digits, we have for the new number,

$$10(6 - x) + x.$$

Hence we have the equation

$$10(6 - x) + x = (6 - x) + 10x + 36,$$

$$60 - 10x + x = 6 - x + 10x + 36,$$

$$x = 1,$$

$$6 - x = 5.$$

The number is 15.

Checking, we have

$$51 - 15 = 36.$$

Example 4. A milling machine M can face a certain number of castings in 8 days. Machine N can face them in 10 days. In how many days can the castings be faced by both machines working together?

Let

x = Number of days required if M and N work together.

We note that

$$\frac{1}{8} = \text{Part of work done by } M \text{ in one day.}$$

Hence

$$x \cdot \frac{1}{8} = \text{Part of the work done by } M \text{ in } x \text{ days.}$$

Likewise

$$\frac{1}{10} = \text{Part of the work done by } N \text{ in one day,}$$

and

$$x \cdot \frac{1}{10} = \text{Part of the work done by } N \text{ in } x \text{ days.}$$

Therefore

$$x \cdot \frac{1}{8} + x \cdot \frac{1}{10} = \text{Part of the work done by } M \text{ and } N \text{ in } x \text{ days.}$$

But the work is done in x days, hence

$$x \cdot \frac{1}{8} + x \cdot \frac{1}{10} = 1.$$

The number 1 on the right refers to the whole piece of work. Solving the equation we get

$$x = 4\frac{4}{13}.$$

Therefore M and N working together will do the piece of work in $4\frac{4}{13}$ days. The student should check this solution by substitution.

EXERCISES

1. The sum of two numbers is 30 and their difference is 2. Find the numbers.
2. The sum of the seventh and eighth parts of a number is 15. Find the number.
3. A tungsten steel drill will last 3 times as long as a carbon steel drill, and a high-speed steel drill will last 5 times as long as a tungsten steel drill. The average number of hours of operation for the three drills together is 665 hours. What is the life of each?
4. If a number is multiplied by 3, the product is equal to twice the sum of the number and 2. What is the number?
5. Manganin, an alloy which has an extremely low temperature coefficient of resistance and which is used in the manufacture of standard resistance boxes, contains 5 times as much manganese as nickel, and $27\frac{1}{3}$ times as much copper as nickel. How much of each element is required to make 1 kg. of manganin wire?
6. Find the numbers whose sum is 21 if 4 times the smaller number is equal to twice the larger.
7. Three machine guns fire 43.5 kg. of bullets in 3 minutes. Machine gun B fires twice the weight of bullets that machine gun A fires, and 5 times the weight of bullets that C fires. What is the weight of bullets fired by each machine gun? What is the rate per minute for each gun?
8. A , who is 40 years older than B , is 6 times as old as B . Find their respective ages.

9. One hundred grams of phosphor bronze, a metal used in bearings, contains 3.7 grams more of copper than 8 times the amount of antimony. It contains 0.5 gram more tin than antimony and 0.003 gram less of phosphorus than $\frac{1}{100}$ of the amount of copper. How much of each metal is present?

10. A is 3 times as old as B , and in 12 years A will be twice as old as B will then be. How old is A ?

11. A truss is loaded 5 times as much as another; together they carry a load of 216 lb.. How much load is there on each truss?

12. The current from a battery is divided among three circuits. Circuit 1 draws 20 milliamperes more than circuit 2, and circuit 2 draws 20 milliamperes more than circuit 3. If the total current drawn is 240 milliamperes, what is the current in each circuit?

13. A and B have equal amounts of money. A gives B \$15 and then has one-fourth as much as B . How much did each have at first?

14. Find three consecutive integers whose sum is 27.

15. A tank is filled by two pipes; the first can fill it in 4 hours and the second in 3 hours. How long will it take both together to fill it?

16. Of four pipes, the first fills a tank in 1 day, the second in 2 days, the third in 3 days, and the fourth in 4 days. How long will it take all the pipes together to fill the tank?

17. A battery is charged by three chargers; the first can charge it in 1 hour and 20 minutes, the second in 200 minutes, and the third in 5 hours. How long will it take all three together to charge it?

18. A code operator can send 1000 messages in 8 days. A machine can send 1500 messages in 2 days. How long will it take both working together to send 300 messages?

19. A can do a piece of work in 18 days which B can do in 21 days. In how many days can it be done by both working together?

20. A does an amount of work in $4\frac{1}{3}$ hours; B does the same task in $4\frac{3}{4}$ hours; and C does the same amount in $4\frac{5}{6}$ hours. In how many hours can it be done by all working together?

21. A can do a piece of work in 15 hours which B can do in 25 hours. After A has worked a certain time, B completes the job, working 9 hours longer than A . How many hours did A work?

22. A number is composed of two digits whose sum is 9. Reversing the order of the digits, we obtain a number which is 27 greater than the number in its original form. What is the number?

23. A number is composed of two digits, the digit in the unit's place being 4 less than the digit in the ten's place. If the number is increased by two and the result multiplied by two, a number equal to fifteen times the sum of the digits is obtained. What is the number?

24. A number is composed of two digits, the digit in the ten's place being 3 greater than the digit in the unit's place. The number is 3 more than 7 times the sum of the digits. What is the number?

25. A number is composed of two digits whose sum is 9. The number formed by reversing the digits is 2 greater than 5 times the remainder when the unit's digit is taken from the ten's digit. What is the number?

26. The digit in the unit's place is 6 less than 3 times the digit in the ten's place. The number has the same value if the order of the digits is reversed. What is the number?

27. A number is composed of three digits. The digit in the unit's place is 1 greater than the digit in the ten's place, which in turn is 3 greater than the digit in the hundred's place. The number is 9 greater than 19 times the sum of the digits. What is the number?

28. How many gallons of a mixture containing 70% alcohol should be added to 6 gallons of a 20% solution to give a 30% solution?

29. What percentage of a 30% solution of hydrochloric acid should be drawn off and replaced by water to give a 15% solution?

30. A washer is to be made with the radius of the hole equal to r . What will the radius of the washer be if the area of the metal part is equal to the area of the hole?

31. A square court has the same area as a rectangular court whose length is 8 yd. greater and whose width is 6 yd. less than the side of the square. What is the area of the court?

2-18. Applications of Linear Equations to the Solution of Engineering Problems. Very often the relationships in a problem which are to be translated into algebraic notation are those of physical laws. A few examples of such physical situations are given below.

The speed v of a body which travels a distance d in a time t is given by

$$(1) \quad v = \frac{d}{t},$$

and is expressed in units of distance per unit time. Thus a man driving 60 miles in 2 hours has a speed of $\frac{60}{2} = 30$ miles per hour. Also from (1), the distance traveled is the speed multiplied by the time of travel, or

$$(2) \quad d = vt.$$

The speed of a body is **uniform** if v is the same regardless of the interval of time during which the distance traveled is measured.

Example 1. A freight and passenger train travel toward each other from towns 390 miles apart, the freight at 30 miles an hour, the passenger train at 45 miles an hour. If the freight train starts half an hour sooner than the passenger train, how many hours will the passenger train travel until it meets the freight train?

Let

x = Hours the passenger train travels to meet the freight train,

$x + \frac{1}{2}$ = Hours the freight train travels to meet the passenger train,

$45x$ = Miles the passenger train travels,

$30(x + \frac{1}{2})$ = Miles the freight train travels.

Hence, since together they traverse the entire 390 miles,

$$45x + 30(x + \frac{1}{2}) = 390,$$

$$45x + 30x + 15 = 390,$$

$$75x = 390 - 15,$$

$$75x = 375,$$

$$x = 5 \text{ hours the passenger train travels.}$$

The lever is a bar upon which force may be applied to overcome a resistance. The bar is pivoted at some point (called the fulcrum) around which it turns. The teeter-totter, the crowbar, the wheelbarrow, and pliers are examples of levers.

When a lever, which is assumed to be weightless, is in equilibrium (or balanced) on a fulcrum by weights W_1 and W_2 at distances a_1 and a_2 from the fulcrum, respectively (see Fig. 2-1), it is known that

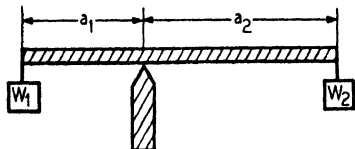


FIG. 2-1.

$$(3) \quad W_1 a_1 = W_2 a_2.$$

The product of a force and the distance of a point from the line in which the force is acting is called the **moment** or **torque** of the force with respect to the point. Equation (3) then states that the moments with respect to the fulcrum are equal. More generally, if several weights are placed on either side of the fulcrum, *the lever is in equilibrium if the sum of the moments with respect to the fulcrum on one side of the fulcrum is equal to the sum of the moments on the other.*

Example 2. A weight of 180 lb. is placed 6 ft. from a fulcrum, and a weight of 73 lb. 5 ft. from the fulcrum on the other side. Considering the lever to be weightless, where should a weight of 55 lb. be added to balance the lever?

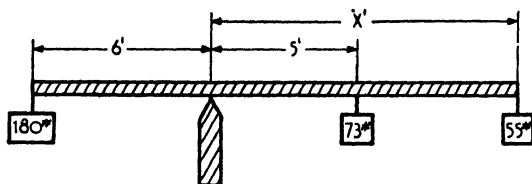


FIG. 2-2.

The moment on the left in Fig. 2-2 is greater than that on the right, so that the 55-lb. weight must be placed at an unknown distance to the right, which we will call x .

Then

$$180 \cdot 6 = 73 \cdot 5 + 55x,$$

$$1080 = 365 + 55x,$$

$$55x = 1080 - 365,$$

$$55x = 715,$$

$$x = 13 \text{ ft.}$$

Thus the weight should be placed 13 ft. to the right of the fulcrum and 8 ft. to the right of the 73-lb. weight.

If E is the voltage across an electrical device D of resistance R ohms, the following formulas are valid when a direct current of I amperes is flowing.

$$(4) \quad E = IR \quad (\text{Ohm's law})$$

$$(5) \quad P = IE = I^2R = \frac{E^2}{R}$$

$$(6) \quad H = 0.24I^2RT = 0.24PT$$

where

E = voltage (potential drop) across D ,

I = current in amperes,

H = heat in calories developed in D ,

R = resistance in ohms,

P = power in watts developed in D ,

T = time in seconds,

and 0.24 is accurate to two significant figures.

If these formulas are to be applied to a closed circuit like Fig. 2-3 the total resistance of the circuit consists of the external resistance R_e and

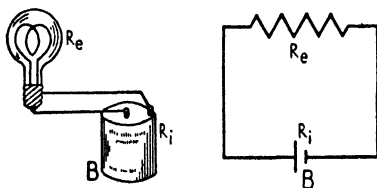


FIG. 2-3.

the internal resistance R_i of the battery B . If E is the electromotive force developed by the battery the following formulas are valid

$$E = I(R_i + R_e),$$

$$P = I^2R_e,$$

$$H = 0.24I^2R_eT = 0.24PT,$$

where P is the power developed in R_e and H is the heat developed in R_e .

Example 3. A 6.0-volt battery supplies power to a 30-watt lamp. What is the resistance of the lamp, the current flowing, and how much heat is generated in 10 minutes, if the internal resistance of the battery is neglected?

From (5),

$$30 = 6 \cdot I,$$

$$I = 5 \text{ amperes.}$$

Then from (4),

$$6 = 5R,$$

$$R = 1.2 \text{ ohms.}$$

Finally from (6),

$$H = 0.24(30)60 \times 10 = 0.24(18,000) = 4.3 \times 10^3 \text{ calories.}$$

EXERCISES

1. Two columns of troops 90 miles apart, setting out at the same time, travel toward each other at the rates of 5 and 4 miles an hour, respectively. After how many hours will they meet? How far will each have traveled when they meet?

2. The speed of a freight train is two-thirds that of an express train. The express train covers 180 miles in 2 hours less time than the slower train. Find the speed of each train.

3. An airplane leaves the deck of a carrier and travels north at the rate of 128 miles an hour. The carrier travels north at the rate of 28 miles an hour. When will the airplane pass out of range of wireless communication with the ship if the plane's radio has a range of 500 miles?

4. A freight train with an average speed of 40 miles per hour leaves a certain point 5 hours before an express train traveling in the same direction with a speed of 65 miles per hour. How long will it be before the express train overtakes the freight train?

5. An airplane can cover a distance in 4 hours with the wind, but can travel only two-thirds of the way back in the same time. If the airplane travels 110 miles per hour in still air, what is the wind velocity?

6. From a place *A* a messenger goes to a place *B*, 21 miles distant from *A*, and immediately returns, going at a rate of 4 miles an hour. Simultaneously with the messenger's departure from *A*, another messenger starts from *B* at the rate of 3 miles an hour, goes to *A*, and immediately returns. Find the distance between the two points at which they meet.

7. Jones drove 3 hours and 20 minutes at a uniform speed. Engine trouble forced him to drive at two-fifths of this former speed for an hour. Upon reaching a garage at the end of this time he found that he had driven a distance of 168 miles. What was Jones' original speed?

8. A traveler sets out to walk at the rate of 4 miles an hour. Twenty hours later a second traveler sets out on a bicycle at 18 miles an hour. How long has the marathon walker been walking when the bicyclist passes him if they travel the same route?

9. What voltage is required for a current of 0.6 ampere to flow through a resistance of 10 ohms?

10. An electric heater takes 6 amperes when the voltage across its terminals is 120 volts. What is the resistance?

11. The voltage across a street car heater is 220 volts. If the resistance is 5 ohms, what current will flow through the heater?

12. A flashlight operates from a battery of two cells in series, each having an electromotive force of 1.5 volts. If a current of 1.5 amperes flows through the bulb, what is the resistance of the entire circuit? Neglect internal resistance of batteries. (Note: If batteries are in series, the electromotive forces are additive.)

13. What is the resistance of an electric iron which takes a current of 8 amperes when used on a 120-volt circuit?

14. What is the internal resistance of a battery of electromotive force 6.4 volts which sends a current of 8 amperes through a resistance of 0.7 ohm?

15. What is the resistance in ohms of a bulb which uses 60 watts if operated at 115 volts?

16. What is the amount of heat developed in 1 minute in the street car heater described in Exercise 11?

17. What is the current in amperes in an electric power line which transmits 100,000 kilowatts at 50,000 volts?

18. An electric heater has to be constructed which uses 600 watts if 120 volts are applied. What is the resistance of this heater?

19. A battery of an electromotive force of 32 volts has an internal resistance of 0.3 ohm. What is the external resistance required for a current of 20 amperes?

20. If the current through a lamp is 0.75 ampere and the fall of potential between its terminals is 120 volts, at what rate is heat produced per second?

21. If weights of 160 lb., 100 lb., and 130 lb. are placed 8 ft., 6 ft., and 4 ft., respectively, from the fulcrum and on the same side, where must a weight of 400 lb. be placed to balance them?

22. How much force can be exerted on a body 3 ft. from the fulcrum by a force of 300 lb. acting 6 ft. from the fulcrum on the opposite side?

23. *A* and *B* together weigh 299 lb. They balance when seated 5 ft. and 8 ft. respectively from the fulcrum on opposite sides. What is the weight of each?

24. If weights of 400 lb. and 160 lb. are placed on a lever 14 ft. apart, where would the fulcrum have to be in order to balance them?

25. Two weights of 250 lb. and 150 lb. are placed 8 ft. and 5 ft. respectively from the fulcrum of a lever and on the same side; a third weight of 200 lb. is placed 4 ft. at the other side of the fulcrum, and a fourth weight 6 ft. from the fulcrum on the same side as the third weight. What is the amount of the fourth weight if the lever is in equilibrium?

26. A load is placed 8 ft. from the fulcrum of a lever and a second load, weighing 600 lb. more, is placed 5 ft. of the fulcrum on the same side. What is the weight of the two loads if they are balanced by a weight of 6000 lb., placed 7 ft. from the fulcrum on the other side of the lever?

PROGRESS REPORT

While Chapter 1 was primarily devoted to computations with numbers, the chapter just concluded is the first one to deal with algebra.

At the beginning of the chapter, letter symbols were introduced to denote numbers. The introduction of letter symbols served the purpose of giving general rules for operating with numbers and also solving problems which are difficult to handle if particular numbers are used.

The ideas and rules involved in operating with numbers were all expressed in letter symbols. Thus the student has learned how to add, subtract, multiply, and divide various algebraic forms. Certain special products which occur frequently were introduced. The extension of

resolving integers into the products of prime factors in arithmetic led to the algebraic operation of factoring. Rules for factoring certain simple forms were given. This, in turn led to an extensive discussion of methods for handling fractional forms of algebra.

These powerful methods were then applied first to various formulas and then to some problems arising in the engineering sciences.

CHAPTER 3

FUNCTIONS AND THEIR GRAPHS

An old Chinese proverb states, "A picture is worth a thousand words." Numerical data taken from an experiment or calculations derived from a formula require interpretation. Frequently this can best be done by means of a picture, more commonly referred to in engineering work as a graph or a curve. Such a graph provides the engineer with a pictorial résumé of the interrelationship of the data and points out at a glance in a graphical way how one factor depends on another. Various relationships can be plotted on the same graph, showing how two or more factors may depend on any one. Since such graphs and curves are universally used by engineers, it is, of course, important that the student follow a general scheme of plotting the data so that these graphs may be read as easily by other engineers as tables of data.

3-1. Constants and Variables. In the discussions and formulas of mathematics, physics, engineering, and other branches of science, some of the symbols are intended to represent fixed numbers, while others may be assigned various values in the course of a single discussion.

A symbol which represents a single number throughout a discussion is called a constant.

A symbol which represents more than one number in the course of a discussion is called a variable. These values may be assigned to the symbol arbitrarily, or they may be determined by some law.

Example. In Sec. 2-18 it was stated that the heat H in calories generated in a light bulb of resistance R ohms, by a current I amperes, during a time T seconds is $H = 0.24I^2RT$. Suppose we ask how much heat is generated in a bulb having a resistance of 1.2 ohms in 10, 20, and 30 seconds for 5, 10, and 15 amperes of current. During this discussion 0.24 and R do not change and therefore are constants. I and T change arbitrarily, and therefore I and T are variables. H also changes as I and T vary, and therefore H is also a variable.

An **absolute constant** is a symbol which represents the same number in all discussions. The number 0.24 in the example is an absolute constant; 3, π , $6\frac{1}{2}$ are absolute constants.

An **arbitrary constant** is a symbol which represents the same number throughout a given discussion, but which may represent another number in another discussion. R is an arbitrary constant in the example; its

value may differ in the next discussion, for instance, if a lamp with a different resistance is substituted.

In mathematics the first letters of the alphabet are used more often to represent constants, whereas the letters at the end of the alphabet are used to represent variables, especially the letters x , y , and z . However, this practice is not uniform; in the other sciences and occasionally in mathematics, the symbol is often the first letter of the name of the quantity, making it easier to remember the meaning of the symbol.

3-2. Functions. The formula for the volume V of a sphere in terms of its radius r is $V = \frac{4}{3}\pi r^3$. In this formula r and V are variables, since each may assume different values. Further, the formula gives a definite relationship between V and r , since, for any specified value of r , the value of V is determined. For instance, if $r = 3$ in., then $V = 36\pi$ cu. in. Thus the value of V may be thought of as depending on r . To express this idea, we say that V is a *function* of r .

If two variables are so related that for each value which may be assigned to one of the variables, there correspond one or more values of the other variable, then the second variable is said to be a function of the first.

The first variable to which values are assigned arbitrarily is called the **independent variable**; the second variable, whose values are determined, is called the **dependent variable**.

If a specific formula is known which expresses the relation between one variable and another, the formula is spoken of as the function. However, a functional relation need not be given by means of an algebraic formula; it may be given by a table of corresponding values or by other information. The set of values which may be assigned to the independent variable, called its **range** of values, may include all numbers, or it may be limited in some way by physical or mathematical considerations. The following examples will illustrate.

Example 1. Given $y = x^2 - 5x + 6$. Here x may be construed as the independent variable, and y is then a function of x , making y the dependent variable. We speak of $x^2 - 5x + 6$ as the function of x . In this example we may assign any number to x , and the corresponding value of y is determined by the given function.

Example 2. Given $y = \frac{x^2 - 2}{(x + 1)(x - 1)}$. Construing x as the independent variable, the function of x is $\frac{x^2 - 2}{(x + 1)(x - 1)}$. The value of the dependent variable y

is specified by this function for any number x except $x = 1$ and $x = -1$. In these two cases the denominator is zero, leaving the value of the function undefined for these two values of x .

Example 3. A grid bias of -2.8 volts is placed on a 6J5 triode radio tube, and while this bias is held fixed the plate voltage E_p is changed, and the resulting changes in plate current I_p are observed. The following data are obtained.

E_p (in volts)	2.5	65	70	90	100	118	127	136	142	155
I_p (in milliamperes)	0.2	1	2	3	4	6	7	8	9	10

For each value of E_p a value of I_p can be found, and hence I_p may be considered a function of E_p . However, the knowledge of the values of I_p is limited to the ten values of E_p given in the table. Hence this function is defined for only ten values of the independent variable E_p . Some knowledge of the behavior of the triode may enable us to predict the approximate values that I_p may have for values of E_p other than those given, but from a mathematical point of view I_p simply remains undefined for values of E_p other than those given in the table. We may, on the other hand, consider I_p as the independent variable, for the table may be construed as giving the values of E_p for ten values of I_p .

Example 4. Given $y = x^2$. Here for any value of x , which we consider the independent variable, the value of the dependent variable y is determined. However, it is possible to consider y as the independent variable and x as the dependent variable. Then if y is given, the values of x are given by the formula $x = \pm\sqrt{y}$. We note that two values of x are possible, $+\sqrt{y}$ and $-\sqrt{y}$. This function is thus called a **double-valued function**. If the reader will reexamine the definition of function, he will find this case included in the definition. The range of values of y is limited, for there is no real value of x for negative values of y since there is no real square root of a negative number.

The preceding example shows a double-valued function. In general, functions which have three, four, and more values of the dependent variable for each value of the independent variable are possible, but we shall deal most frequently with single-valued functions in this book.

The last two examples show also that, if only the formula or table of values relating the two variables is given, we may often choose the dependent and independent variables at our pleasure, keeping in mind only the requirements of the definitions.

3-3. Functional Notation. Since the volume V of a sphere in terms of its radius is $\frac{4}{3}\pi r^3$, we may regard V as a function of r . This statement is conveniently abbreviated by the **functional notation** $V = f(r)$. This equation is read *V equals the f -function of r* , or more briefly, *V equals f of r* . It states simply that V depends upon r . The reader should note carefully that $f(r)$ does *not* mean f times r .

Thus, in general, *the statement that y is a function of x is abbreviated by $y = f(x)$* .

When, in a given discussion, an algebraic formula for the dependent variable in terms of the independent variable is known, the functional notation is used to represent this formula. Thus, in the example above, since $V = f(r)$ and $V = \frac{4}{3}\pi r^3$, we may use $f(r)$ interchangeably with

$\frac{4}{3}\pi r^3$. In discussions where complicated expressions are involved this convention is very convenient.

The introduction of the functional notation is an extension of the idea of algebra; not only an arbitrary number is denoted by a letter but also an arbitrary function.

The notation $y = f(x)$ can be used conveniently to indicate that the values of y are to be considered which correspond to a particular value of x . For example, suppose that $y = x^2 - 5x + 6$. Then we may write

$$y = f(x) = x^2 - 5x + 6.$$

If we wish to find the value of y corresponding to $x = 2$, we may replace x by 2. Then we have

$$y = f(2) = 4 - 10 + 6 = 0.$$

In the same way, we obtain

$$f(6) = 12,$$

$$f(0) = 6,$$

$$f\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^2 - 5 \cdot \frac{a}{2} + 6 = \frac{a^2}{4} - \frac{5a}{2} + 6,$$

$$f(t^2) = (t^2)^2 - 5(t^2) + 6 = t^4 - 5t^2 + 6.$$

In a single discussion, a functional symbol such as $f(x)$ refers to the same function or law of dependence throughout the discussion. If other functional relationships occur in the same discussion, different prefixed letters are used to distinguish the functions. Other letters often used in this way are: g , F , G , ϕ (phi), θ (theta), ψ (psi), μ (mu). However, any letter may be used. In different discussions $f(x)$ may represent different functions of x . Further, the functional notation is used whether an algebraic formula is known for the law of dependence or not. Thus in Example 3 of the preceding section, we may write $I_p = f(E_p)$, or $E_p = g(I_p)$, even though no algebraic formula is known for the law of dependence. This functional notation is used for multiple-valued as well as single-valued functions. Its use is as general as the definition of the functional relationship itself.

Example 1. The area A of the surface of a sphere as a function of its radius r is $A = g(r) = 4\pi r^2$.

Example 2. Given $f(x) = x^3 - 8x + 10$.

Then

$$f(2) = 8 - 8 \cdot 2 + 10 = 8 - 16 + 10 = 2,$$

$$f(-3) = (-3)^3 - 8(-3) + 10 = -27 + 24 + 10 = 7,$$

$$f(0) = 0 - 8 \cdot 0 + 10 = 10,$$

$$f(a + b) = (a + b)^3 - 8(a + b) + 10,$$

$$f(ab) = (ab)^3 - 8(ab) + 10 = a^3b^3 - 8ab + 10.$$

3-4. Functions of Several Variables. A dependent variable may be a function of more than one independent variable. Thus in the formula for the volume V of a right circular cylinder, $V = \pi r^2 h$, where h is the altitude, and r is the radius of the base. The volume V is a function of the two independent variables r and h . In functional notation we write $V = f(r, h)$, which is read, V equals the f -function of r and h , or V equals f of r and h . The formula for the area A of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the lengths of the bases. Here A is a function of three independent variables, h , b_1 , and b_2 . We may state this fact by writing $A = g(h, b_1, b_2)$.

Example. Given $f(x, y) = xy^2 - x^2 + y$.

Then

$$f(0, 1) = 0 \cdot 1^2 - 0^2 + 1 = 1,$$

$$\begin{aligned} f(3, -2) &= 3(-2)^2 - 3^2 + (-2) \\ &= 12 - 9 - 2 = 1, \end{aligned}$$

$$f(a, b) = ab^2 - a^2 + b,$$

$$\begin{aligned} f(0, a + b) &= 0(a + b)^2 - 0^2 + (a + b) \\ &= a + b. \end{aligned}$$

EXERCISES

Express each of the following statements in functional notation, and then give the exact formula for the function.

1. The circumference C of a circle is a function of its radius r .
2. The area A of a circle is a function of its radius r .
3. The area A of a square is a function of the length d of one side.
4. The volume V of a cube is a function of the length e of one edge.
5. The volume V of a right circular cone is a function of its altitude h and the radius r of its base.
6. The area A of a rectangle is a function of its width w and length L .
7. The volume V of a rectangular parallelepiped is a function of its width w , height h , and thickness t .
8. The area A of the surface of a cube is a function of the length e of one edge.

9. The area A of a triangle is a function of its base b and altitude h .
10. The area A of the surface of a right circular cylinder is a function of its altitude h and the radius r of its base.
11. The volume V of a pyramid is a function of the area A of its base and its altitude h .
12. The lateral area of a regular pyramid is a function of the perimeter p of its base and its slant height s .
13. The total area A of the surface of a right circular cone is a function of the slant height s and the radius r of the base.
14. The area A of an equilateral triangle is a function of the length s of one side.
15. The simple interest I on \$50 at 3 per cent per year is a function of the time t in years.
16. The cost C of x oranges at 20 cents a dozen is a function of x .
17. The cost C of riding q miles in a cab at 20 cents for the first half mile and 10 cents for each additional one-third mile is a function of q .
18. The power P in a direct current circuit is a function of the current I and resistance R .
19. The heat H generated in a circuit in a time T is a function of T , the current I , and the resistance R .
20. The distance d traveled in a time t at a speed v is a function of t and v .
21. Given $f(x) = x - 8$, find $f(2)$, $f(0)$, and $f(6)$.
22. Given $f(x) = x^3 - 8x + 3$, find $f(0)$, $f(-2)$, and $f(1)$.
23. Given $f(x) = \frac{x+1}{x-3}$, find $f(2)$, $f(-3)$, and $f(4)$.
24. Given $G(Z) = \frac{Z^2+1}{1-Z}$, find $G(16)$, $G(\sqrt{2})$, and $G(8)$.
25. Given $F(y) = y^3 - 8y + 10$, find $F(0)$, $F(-3)$, and $F(2)$.
26. Given $\theta(x) = x(x-1)(x-2)$, find $\theta(0)$, $\theta(-1)$, and $\theta(2)$.
27. Given $\phi(y) = (y-a)(y-b)$, find $\phi(a)$, $\phi(b)$, and $\phi(c)$.
28. Given $f(y) = \frac{1-y}{y^3-8}$, find $f(\frac{1}{2})$, $f(\frac{3}{4})$, and $f(0)$.
29. Given $\psi(x) = (x-a)(x-b)$, find $\psi(1)$, $\psi(a)$, $\psi(a^2)$.
30. Given $f(x) = x - 5$, find $f(1) \cdot f(6)$.
31. Given $f(x) = \left(\frac{1}{x} + x\right)\left(x^2 - \frac{1}{x}\right)$, find $f\left(\frac{1}{y}\right)$, and $f(2)$.
32. Given $Q(x) = \frac{1+x^2}{x^3}$, find $\frac{Q(a) - Q(a+1)}{Q(3)}$.
33. Given $\theta(t) = \frac{1-t^2}{1+t^2}$, find $\frac{\theta(t) \cdot \theta(t^2)}{\theta(t^4)}$.
34. Given $g(y) = \frac{1+y}{1-y}$, find $g(2)$, $g(y^2)$, $[g(y)]^2$, $g\left(\frac{1}{y}\right)$, $\frac{1}{g(y)} \cdot g[g(y)]$.
35. Given $f(x) = x^2 + x + 1$, find $\frac{f(a) - f(b)}{f(a-b)}$.
36. Given $\theta(x) = x^2$, find $\frac{\theta(4) - \theta(1)}{\theta(3)}$.
37. Given $f(x) = x^2$, find $f(x+h) - f(x)$.

38. Given $f(x) = x^2$, find $\frac{f(x+h) - f(x)}{h}$.
39. Given $f(x) = x^3$, find $\frac{f(x+h) - f(x)}{h}$.
40. Given $f(x) = x^2 - 2x$, find $\frac{f(x+h) - f(x)}{h}$.
41. Given $f(x) = x^2 + x$, find $f[f(3)]$.
42. Given $g(x) = 3x + 5$, find $g[g(4)]$.
43. Given $f(x) = x^2 + x$, find $f[f(x)]$.
44. Given $g(x) = 3x + 5$, find $g[g(x)]$.
45. Given $h(x) = 4x^2 - 8$, find $\frac{h(a+k) - h(a)}{k}$.
46. Given $f(x, y) = x + y$, find $f(0, 1)$, $f(0, 0)$, $f(1, 1)$ and $f(3, 4)$.
47. Given $f(x, y) = x^2y$, find $f(1, -2)$, $f(0, -3)$, $f(1, 5)$, and $f(1, 4)$.
48. Given $f(x, z) = x^2 + z^2$, find $f(1, -5)$, $f(3, 2)$, $f(0, 6)$, and $f(1, 2)$.
49. Given $g(p, q) = pq$, find $g(3, -1)$, $g(0, 16)$, $g(1, -1)$, and $g(a, b)$.
50. Given $f(x, y) = x^2 + 2xy + y^2$, find $f(0, 0)$, $f(0, 1)$, $f(3, 4)$, and $f(1, 2)$.
51. Given $f(x, y) = x^2 - 4y^2 + 20$, find $f(0, 1)$, $f(3, 4)$, $f(5, 6)$, and $f(a, b)$.
52. Given $f(x, y) = \frac{x+y}{xy}$, find $f(1, -2)$, $f(2, 2)$, $f(5, -6)$, and $f(0, 1)$.
53. Given $f(r, h) = r^2h$, find $f(8, -1)$, $f(-2, 2)$, $f(2, 3)$, and $f(1, 5)$.
54. Given $f(h, b) = \frac{1}{2}hb$, find $f(1, -3)$, $f(2, 3\frac{1}{2})$, and $f(9, 5.2)$.
55. Given $f(x, y) = xy$, find $f(1, -1)$, $f(3, -4)$, $f(\pi, -1)$, and $f(\pi, \pi)$.
56. Given $f(x, y, z) = xyz$, find $f(1, 1, 1)$, $f(1, 2, 3)$, $f(-1, -5, 3)$, $f(-1, -2, 3)$, and $f(0, 1, 2)$.
57. Given $f(x, y, z) = x^2 + y^2 + z^2$, find $f(1, 1, 1)$, $f(-1, -1, -1)$, $f(2, 3, -2)$, and $f(-1, -2, -3)$.
58. Given $f(x, y, z) = x^2 + 2xyz - z^2$, find $f(1, 2, 3)$, $f(0, 0, 1)$, $f(1, 0, 4)$, and $f(5, 0, 7)$.
59. Given $g(p, q, r) = \frac{pq}{r}$, find $g(1, 1, 1)$, $g(1, -1, 2)$, $g(4, -3, 8)$, and $g(-5, 2, 3)$.
60. The altitude h of a triangle is half the length of its base. Find the area A as a function of h .
61. Find the perimeter p of a square as a function of its area A .
62. An isosceles triangle whose equal sides are 5 in. long has a third side of x units. Find the area A of the triangle as a function of x .
63. A right triangle with hypotenuse of 10 in. has one leg of length x . Find the area A of the triangle as a function of x .
64. An open box is to be made from a square of cardboard 12 in. on a side by cutting out equal squares of side x inches in length from each of the four corners and bending up the sides. Express the volume V of the box and its surface S as functions of x .
65. A rectangle has an area of 10 sq. in., and one side is of length x . Express the perimeter p as a function of x .
66. A rectangular garden of 400 sq. rods is to be laid out so that one side of the rectangle, of length x , is along a neighbor's lot. Fencing costs \$5 a rod, and the neighbor offers to pay half the cost of the fence along his lot. Express the cost C of fencing in the garden plot as a function of x .

67. A piece of wire 40 in. long is cut into two pieces, one of which is x inches long. The piece of length x inches is bent into the form of a circle, and the other into an equilateral triangle. Express the sum A of the areas of the two figures as a function of x .

68. A theater is reserved for 100 persons at 35 cents per person. For each additional person the management agrees to charge 5 cents less per person. If x additional persons attend the show, how much does the management receive?

69. John and James leave a given point at the same time on foot, John walking west and James east. If John walks at 3 miles per hour and James at 4 miles per hour, find their distance apart d at the end of t hours as a function of t . (Assume that the surface of the earth where they are is level, and that this surface is a plane.)

70. A coast guard patrol plane flies over a buoy at 285 m.p.h. at an altitude of 500 ft. above the buoy. If the plane is flying in a straight line, find the distance d of the plane from the buoy t seconds after being directly above it as a function of t .

71. A rectangle of sides x and y inches is inscribed in a circle having a diameter of 10 in. Express y as a function of x . Express the area A of the rectangle as a function of x . Express the perimeter P of the rectangle as a function of x .

72. A rectangular box having a volume of 50 cu. in. is open at the top and stands on a square base with sides of x inches. Find the total area A of the four sides and the base as a function of x .

73. A tin can which is a right circular cylinder has a volume of 100 cu. in. If its base has a radius r , find its height h as a function of r . State the number of square inches of metal in the can as a function of r .

74. A right circular cylinder is inscribed in a sphere of radius 10 in. If the altitude of the cylinder is y and its radius is x , express y as a function of x . Express the volume V of the cylinder as a function of x . Express the total area S of the cylinder as a function of x .

75. A right circular cone, with a base radius r inches and altitude h inches, is inscribed in a sphere 36 cu. in. in volume. Express r and the volume V of the cone as functions of h .

76. A rectangular sheet of galvanized iron of width 5 ft. and length 20 ft. is bent to form a trough along lines parallel to its sides. If the bottom of the trough is 2 ft. wide and each side is $1\frac{1}{2}$ ft. wide, express the cross-sectional area A of the trough as a function of its depth d . Express the volume V of the trough as a function of its depth d .

3-5. Inverse Functions. As we have seen, the functional relationship expressed in the equation $y = x^2$ can also be expressed in the equation $x = \pm\sqrt{y}$. In general, when an equation of the form $y = f(x)$, which defines y as a function of x , is solved for x in terms of y , taking the form $x = g(y)$, the function $g(y)$ is called the inverse function of $f(x)$. Thus in the inverse function $g(y)$, the independent variable y was formerly the dependent variable.

Example 1. Given $y = f(x) = 3x - 5$. To find the inverse function:

$$3x = y + 5,$$

$$x = g(y) = \frac{y + 5}{3}.$$

Example 2. Express the volume V of a sphere as a function of its surface area S .

We know that $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ where r is the radius of the sphere. From the formula $S = 4\pi r^2$ we obtain $r^2 = \frac{S}{4\pi}$ and $r = \pm\sqrt{\frac{S}{4\pi}}$. Since the radius of a sphere is considered to be positive, we choose $r = \sqrt{\frac{S}{4\pi}}$. Substituting this value in the formula for volume, we have

$$\begin{aligned} V &= \frac{4}{3}\pi \left(\sqrt{\frac{S}{4\pi}}\right)^3 = \frac{4}{3}\pi \left(\frac{S}{4\pi}\right)\sqrt{\frac{S}{4\pi}} \\ &= \frac{S}{3}\sqrt{\frac{S}{4\pi}} = \frac{S}{6}\sqrt{\frac{S}{\pi}}. \end{aligned}$$

EXERCISES

1. Given $y = f(x) = 3x - 18$. Find the formula for the inverse function $x = g(y)$.
2. Given $y = f(x) = 6x - 12$. Find the formula for the inverse function $x = g(y)$.
3. Given $y = f(x) = x^2 - 9$. Find the formula for the inverse function $x = g(y)$.
4. Given $y = F(x) = 18 - x^2$. Find the formula for the inverse function $x = G(y)$.
5. Given $y = f(x) = x^3$. Find the formula for the inverse function $x = \phi(y)$.
6. Given $y = \theta(x) = x^3 + 8$. Find the formula for the inverse function $x = \phi(y)$.
7. Given $w = H(Z) = 48 + Z^3$. Find the formula for the inverse function $Z = G(w)$.
8. Given $Z = q(x) = 1 - x^2$. Find the formula for the inverse function $x = r(Z)$.
9. Express the radius r of a circle as a function of its circumference C .
10. Express the radius r of a circle as a function of its area A .
11. Express the diameter d of a circle as a function of its area A .
12. Express the radius r of a sphere as a function of its volume V .
13. Express the diameter d of a sphere as a function of its volume V .
14. Express the diameter d of a sphere as a function of the area A of its surface.
15. Express the edge e of a cube as a function of its volume V .
16. Express the edge e of a cube as a function of the area of its surface A .
17. Express the current I in a simple direct-current circuit of resistance 10 ohms as a function of the electromotive force E .
18. Express the current I in a simple direct-current circuit of resistance 10 ohms as a function of the power P .
19. Express the speed v of a car which travels for 3 hours as a function of the distance d which the car travels.
20. Express the base b of a triangle of height 10 in. as a function of the area of the triangle.

3-6. The Number Line. In Sec. 1-2 we indicated how to represent numbers on a straight line. On a straight line XX' (see Fig. 3-1) of

unlimited length choose a point O , called the origin, from which to measure distances. Choose a positive direction on the line. The usual choice on a horizontal line is shown by the arrow in Fig. 3-1. Also choose a unit of measurement. Now consider a number x as corresponding to the point a distance of x units from O , in the direction of the arrow from O if

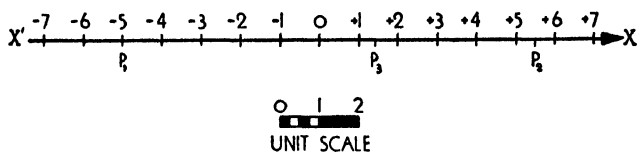


FIG. 3-1.

x is positive, in the opposite direction from O if x is negative. It is customary to mark and label the points corresponding to the positive and negative integers as shown in Fig. 3-1.

Example. On the number line in Fig. 3-1, P_1 corresponds to -5 , P_2 corresponds to $5\frac{1}{2}$, O corresponds to zero, and P_3 corresponds to $\sqrt{2}$. P_3 can be found exactly by geometrical methods by constructing the diagonal of a square whose side length is the unit used on the number line. Then the length of the diagonal of this square is the distance of P_3 from O .

The relative size of two numbers is indicated graphically by the relative positions of the points corresponding to these numbers. If $a > b$, then the point corresponding to a is to the right of the point corresponding to b ; if $a < b$, the point corresponding to a is to the left of the point corresponding to b .

The **absolute value** of a number can be thought of as the number of units that the point representing it is from the origin, regardless of direction.

EXERCISES

Locate the numbers given on a number line and insert the proper inequality sign, $>$ or $<$, between the pairs.

- | | | |
|--------------------|-------------------------------------|---------------------|
| 1. 3 and 5. | 2. 6 and 2. | 3. -5 and 2. |
| 4. -3 and -5 . | 5. 8 and -3 . | 6. 1.2 and -3.6 . |
| 7. 8 and -1.2 . | 8. $\sqrt{\pi}$ and $-\sqrt{\pi}$. | 9. 18 and -5 . |
| 10. -6 and 14. | | |

Locate the numbers in each exercise on a number line and arrange the numbers in each set in the order of increasing magnitude.

- | | |
|--|---|
| 11. 4, -8 , $\frac{2}{3}$, $\sqrt{2}$. | 12. -8 , 6, $4\frac{2}{3}$, $-\sqrt{5}$, $\sqrt{\pi}$. |
| 13. 7, 5, $-3\frac{1}{2}$, 6.7, -5 . | 14. 8, 10, -5 , 0, -4 . |
| 15. 4, -5 , 6, 7, -8 . | 16. 1.6, -8.5 , -10 , -3.7 , -18 . |
| 17. 18, 12, -5 , 1.6, 4.7. | 18. 1, 4, 3, -5 , 6.7, -18 . |

3-7. Rectangular Coordinates. In the preceding section a correspondence was found between numbers and points on a straight line. In this section it will be shown how a correspondence can be set up between points on a plane and pairs of numbers.

Let XX' be a number line drawn horizontally with its positive direction to the right. Let YY' be another number line drawn perpendicular to XX' so that the point of intersection is the origin on both lines. As a rule, we use the same unit on both lines, but sometimes the choice of

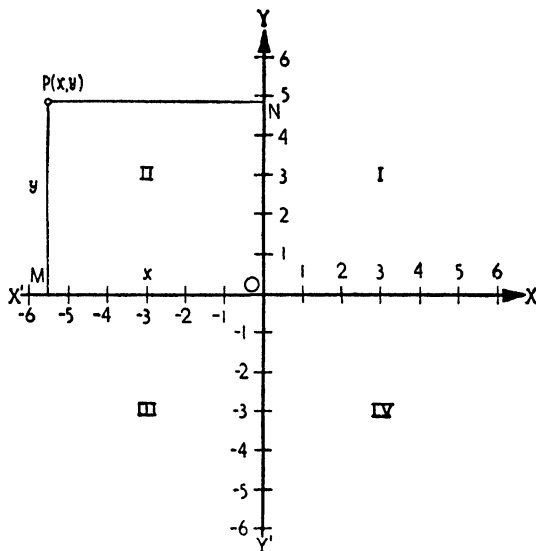


FIG. 3-2.

different units is advantageous. Let the positive direction of YY' be upward (see Fig. 3-2). The line XX' is called the **x -axis** and the line YY' is called the **y -axis**; the two lines together are called the **coordinate axes**. The point of intersection O of the two lines is called the **origin**.

Let P be any point in the plane. From P drop a perpendicular to the x -axis and another to the y -axis. The perpendicular to the x -axis cuts this axis in the point M which corresponds to the number x , called the **abscissa** of P . The perpendicular to the y -axis cuts this axis in a point N which corresponds to the number y , called the **ordinate** of P . With P we associate the numbers x and y , called the **rectangular coordinates** of P . Then to indicate that P has coordinates x and y , we write $P(x, y)$, and we refer to the point P as the **point $P(x, y)$** or, briefly, the **point (x, y)** .

Given a pair of numbers (x, y) , the point corresponding to this pair of numbers can be found as follows. Erect a perpendicular to the x -axis

at the point corresponding to the number x . Erect a perpendicular to the y -axis at the point corresponding to y . The point of intersection of these two perpendiculars is $P(x, y)$. The process of locating the point with given coordinates is called **plotting the point**.

Instead of erecting the two perpendiculars as just described, it is often more convenient to erect only the perpendicular to the x -axis and to measure the distance y directly on this line instead of on the y -axis.

The coordinate axes divide the plane into four parts called **quadrants**, numbered I, II, III, and IV as shown in Fig. 3-2.

The system of coordinates with equal units on both axes, as described above, is called the **Cartesian system of rectangular coordinates**, after the distinguished French mathematician and philosopher Descartes (1596-1650), who was the first man to make extensive use of this system of coordinates. Although the rectangular system is not the only coordinate system in the plane, it is the most convenient one for most purposes in this book and the one most frequently used in practical work. Other systems may be constructed, for instance, by reversing the positive direction on the axes, by using different units on the two axes, or by allowing them to intersect at an angle not a right angle. Later we shall make use of a system of polar coordinates.

Example. Given $P_1(2, 4)$, $P_2(-3, 5)$, $P_3(-4, -6)$, $P_4(3, -2)$, $P_5(5, 0)$.

These points are plotted in Fig. 3-3. Points are customarily labeled as shown in Fig. 3-3.

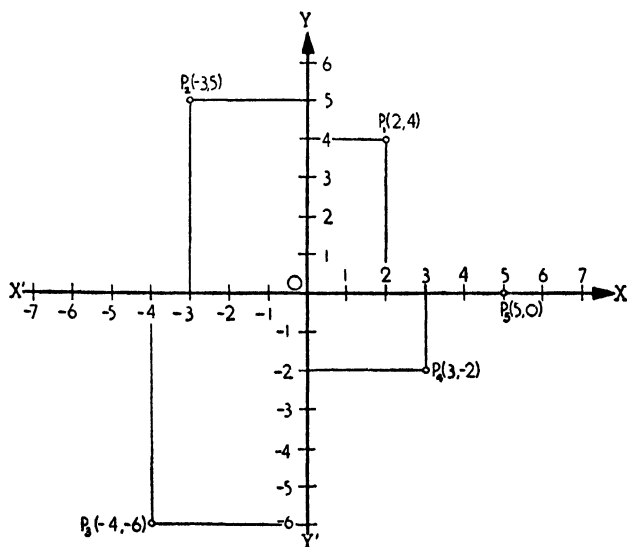


FIG. 3-3.

EXERCISES

Plot each of the following points on rectangular Cartesian coordinates, and state the quadrant, if any, in which each lies.

- | | | |
|--------------------------------------|------------------|------------------|
| 1. (6, 3). | 2. (5, -4). | 3. (-3, 6). |
| 4. (-2, -4). | 5. (6, -2). | 6. (0, 5). |
| 7. (3, 0). | 8. (-5, 0). | 9. (0, -4). |
| 10. $(3\frac{1}{2}, 4\frac{2}{3})$. | 11. (5.6, -6.5). | 12. (-1.4, -27). |

13. In what two quadrants do the points have positive abscissas? negative ordinates?

14. In what two quadrants do the points have positive ordinates? negative abscissas?

15. In what quadrant are the abscissas and ordinates both positive?

16. In what quadrant are the abscissas and ordinates both negative?

17. In what two quadrants do the abscissas and ordinates have opposite signs?

18. What is the abscissa of all points on the y -axis?

19. What is the ordinate of all points on the x -axis?

20. Where are all the points for which $x = 5$?

21. Where are all the points for which $x = 15$?

22. Where are all the points for which $x = -5$?

23. Where are all the points for which $x = 0$?

24. Where are all the points for which $y = 0$?

25. Where are all the points for which $y = -8$?

26. Where are all the points for which $x = 2.5$?

27. In what quadrants is the ratio $\frac{y}{x}$ positive? negative?

28. Draw the triangle whose vertices are (3, 1), (-4, 4), and (-3, -5).

29. Draw the triangle whose vertices are (-2, 6), (3, 2), and (0, -3).

30. Draw the quadrilateral whose vertices, connected in the order given, are (-3, -4), (-1, -8), (3, 0), and (0, 5).

31. Draw the quadrilateral whose vertices, connected in the order given, are (1, 3), (-3, 4), (-2, -5), and (3, -2).

32. Plot the points in the following table and connect them by straight segments in the order of increasing values of x .

x	-3	-2	-1	0	1	2	3	4
y	18	8	2	0	2	8	18	32

33. Plot the points in the table of Exercise 32 and sketch a smooth curve passing through them in the order of increasing values of x .

34. Plot the points in the following table and sketch a smooth curve passing through them in the order of increasing values of x .

x	-3	-2	-1	0	1	2	3
y	-37	-8	5	8	7	7	17

3-8. Graphs of Functions Given by Tables of Data. The single-valued function $y = f(x)$ gives a number y for each number x in a given range of values for x . The function $f(x)$ may therefore be regarded as specifying a set of number pairs (x, y) , one for each x in the given range. The Cartesian coordinate system provides a geometrical interpretation of a number pair as a point. The set of points in the plane corresponding to the number pairs given by the function $y = f(x)$ is called the **graph of the function** $y = f(x)$. We thus arrive at a very useful geometrical interpretation of the idea of a function.

Example 1. Plot the graph of the function defined by the following table.

x	-5	-3	-1	0	1	3	4	6
y	3	1	0	1	2	0	-2	-6

The graph is shown in Fig. 3-4. As an aid to the eye, the points are sometimes connected by straight segments as shown in Fig. 3-5 or by

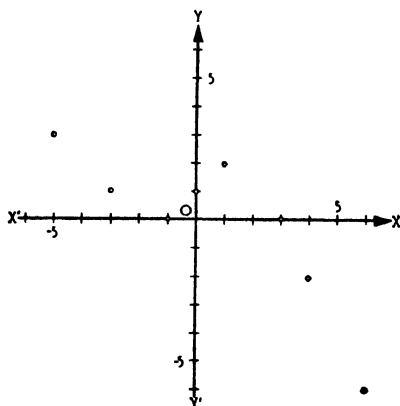


FIG. 3-4.

some convenient smooth curve as shown in Fig. 3-6. A **smooth curve** is one without corners, drawn in such a way that its curving is as even as possible. Most of the quantities which are observed in physics and engineering change in such a manner that the corresponding graph is a smooth curve.

This procedure does *not* imply that the function is defined for other values of x , nor is it intended to extend the definition; it simply serves as a helpful, artistic aid in reading the graph.

Many practical considerations in physical experiments, engineering work, and business involve relations between two variables which fall under the definition of a function given in Sec. 3-2. One of the most

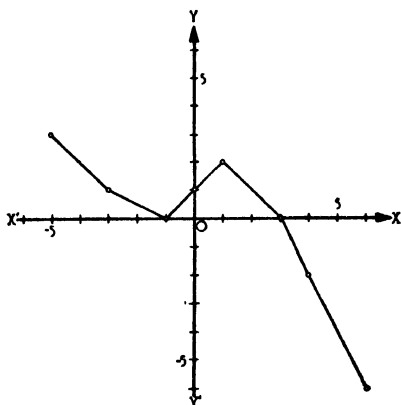


FIG. 3-5.

forceful ways to present such data is to plot them on a Cartesian coordinate system. *The x -axis is replaced by one on which the units of the independent variable are measured; the y -axis is replaced by an axis on which the units of the dependent variable are measured.* It is often convenient

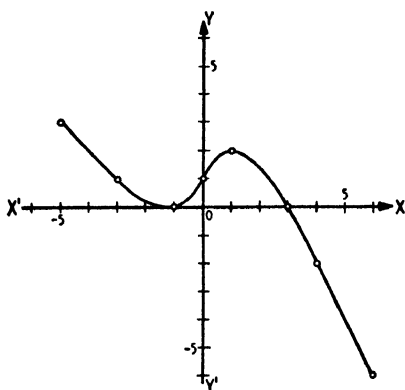


FIG. 3-6.

to use a different scale on the horizontal axis from that used on the vertical axis. Sometimes data obtained in connection with practical work involve only positive numbers; it is then sufficient to use only the first quadrant in plotting such data.

Example 2. A body falling in a vacuum attracted by gravity will fall distance s in feet in time t in seconds as given by the table.

t	0	0.5	1.0	1.5	2.0	2.5	3.0
s	0	4.0	16.1	36.2	64.4	100.6	144.9

We may interpret s as a function of t and plot the function on Cartesian coordinates, replacing the x -axis by a t -axis, the y -axis by an s -axis. This procedure is often called, *plotting the curve of the distance s against the time t* , or simply, *plotting s against t* . It is also convenient to use a much smaller unit on the s -axis than on

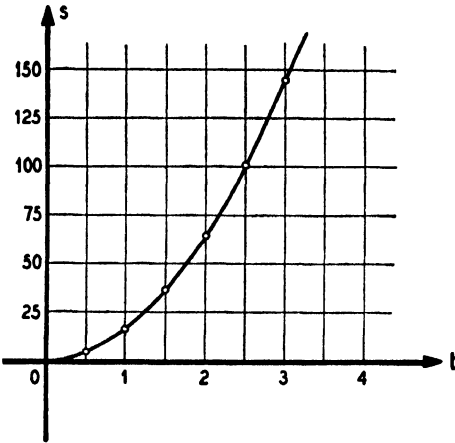


FIG. 3-7.

the t -axis. In this example, our intuition suggests that the value of s could be found by experiment for any positive value of t not given in the tables, and we may surmise that the proper values would be approximately represented by the smooth curve drawn in Fig. 3-7 through the points given by the table.

Example 3. The hourly temperatures, in degrees Fahrenheit, in Chicago, Illinois, on a certain date are given in the table.

	A.M.							P.M.		
H	5	6	7	8	9	10	11	12 Noon	1	2
T	3.5	3.5	3.5	4.0	5.0	7.5	12.5	17.5	22.0	25.0

P.M.

<i>H</i>	3	4	5	6	7	8	9	10	11	12 Mid- night
<i>T</i>	27.0	28.0	27.0	23.5	22.5	24.0	27.5	30.0	30.5	30.5

The function is plotted in Fig. 3-8. As in the preceding example, we may surmise that the temperatures for times not given in the table can be approximated fairly closely by drawing a smooth curve through the points given in the table.

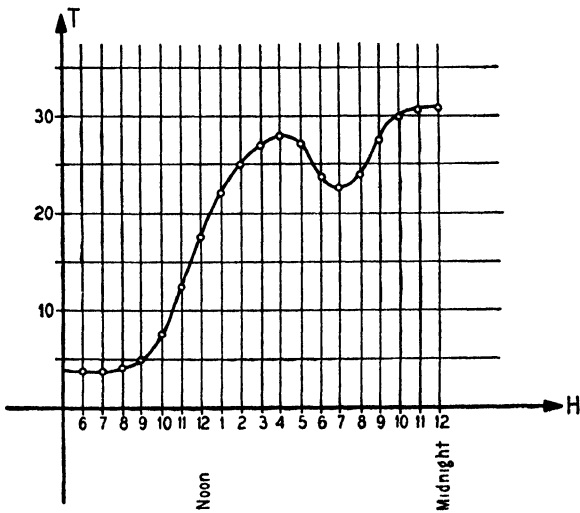


FIG. 3-8.

Example 4. Mr. Sam Jones had in his bank account during 1942 the amounts designated in the table.

Date	Jan. 15	Feb. 15	Mar. 15	Apr. 15	May 15	June 15
Amount A	\$187	\$125	\$100	\$205	\$186	\$142

Date	July 15	Aug. 15	Sept. 15	Oct. 15	Nov. 15	Dec. 15
Amount A	\$217	\$108	\$146	\$150	\$50	\$19

A graph of this function is given in Fig. 3-9. The x -axis is replaced by one on which the dates are given, and the y -axis by one on which the amount A is given. The table of values given does not indicate in any way what Mr. Jones had in his account on other days than the fifteenth of each month. Since this is so, we connect the points plotted on the graph by straight lines, to aid the eye in reading the graph. However, we note carefully that we cannot infer from them what Mr. Jones' bank account was on days other than those given.

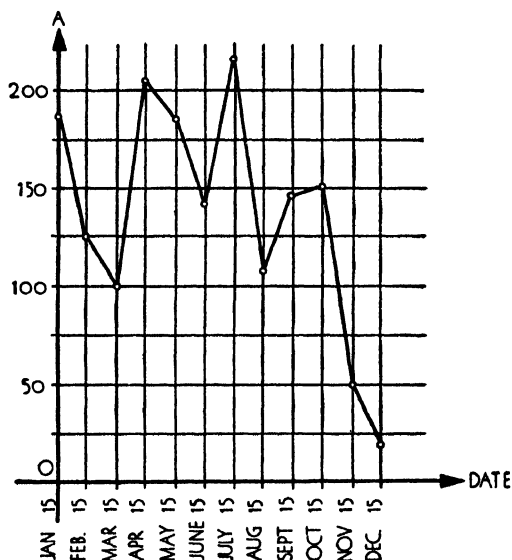


FIG. 3-9.

Strictly speaking, the functions in the preceding examples are defined only for the values of the independent variable given in the tables. However, Example 4 is different from Examples 2 and 3 in one important respect: in Example 4 the given data yield no information about what values the function might have if its definition were extended, whereas in Examples 2 and 3 we can guess rather accurately what the other values of the function might be in an extended definition. In Example 4 we join the points with straight lines merely as an aid in reading the graph; in Examples 2 and 3 we draw a smooth curve through the points as an indication of our conviction that the points plotted are indications of a general trend which might be discovered if the function were defined more fully.

Certain suggestions about the details of plotting a graph will be useful.

1. Use graph paper (cross-section paper) whenever possible.
2. The coordinate axes should be made heavier than the other lines of the network, especially if cross-section paper is used.

3. *The axes should be properly labeled by the quantities measured on them.* If these labels are omitted, the graph is meaningless.

4. *The scales on the axes should be as large as possible while at the same time keeping the graph, or the parts in which we are interested, within the space available.* It may be convenient to choose a unit length for the ordinates differing from the unit length for the abscissas. Before choosing the units on the axes, it is well to examine the table for the maximum and minimum values of the variables and then to choose the units on the axes so as to fit these values into the space available for the graph. The scales should be indicated by numbering points at uniform intervals along the entire length of each axis. Without these labels the graph is meaningless.

5. *In joining the plotted points, proceed from one point to another in order of increasing (or decreasing) values of x .* When a smooth curve through the plotted points is called for by the nature of the problem, first sketch a curve through the points lightly, and then trace it heavily when it appears satisfactory.

3-9. Reading a Graph. Graphs are widely used in engineering and science to indicate the behavior of related physical quantities. Many properties of these functions can then be inferred directly from the graph. The process of finding properties of a function by inspection of the graph representing it is called **reading the graph**.

Example 1. An object is thrown upwards with an initial speed of 100 ft. per sec. It is known that its distance S from the starting point after t seconds is approximately given by this table.

t	0	1	2	3	4	5	6	7
S	0	84	136	156	144	105	34	-74

The graph corresponding to this table is shown in Fig. 3-10. From this graph, we can read the following information. The object moves up for approximately 3 seconds and then begins to move down. The highest point reached by the object is about 156 ft. The object returns to its starting point after approximately 6.25 seconds.

Studying the graph in Fig. 3-10, we can now answer some questions about the motion of this object.

(a) What height is reached by the object after 4.50 seconds? To answer this question, draw a perpendicular to the t -axis at the point $t = 4.50$. Its length, measured with the unit of the s -axis, is approximately 130. Thus, 130 ft. is roughly the height of the object after 4.50 seconds.

(b) After how many seconds is the object 100 ft. above the starting point? To answer this question, draw a line perpendicular to the s -axis through the point $s =$

100. It intersects the graph in two points, whose abscissas are measured to be 1.30 and 5.05. We can state, therefore, that the object is 100 ft. above the starting point after 1.30 seconds and again after 5.05 seconds.

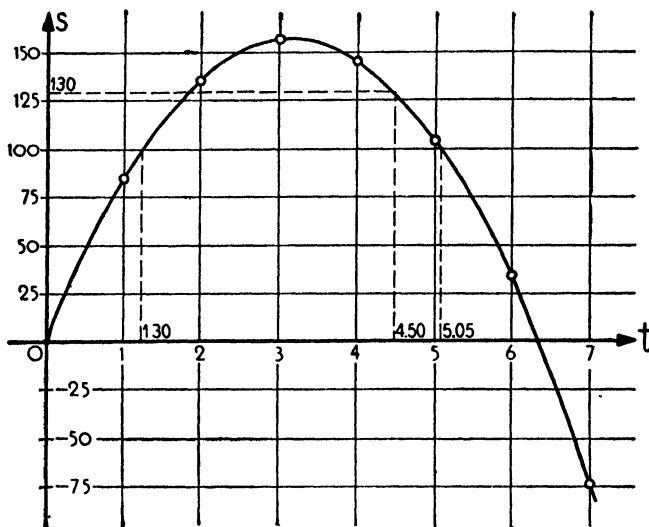


FIG. 3-10.

Example 2. When two coils are arranged so that a change in current in one coil causes a voltage to be induced in the other, the two coils are said to possess mutual

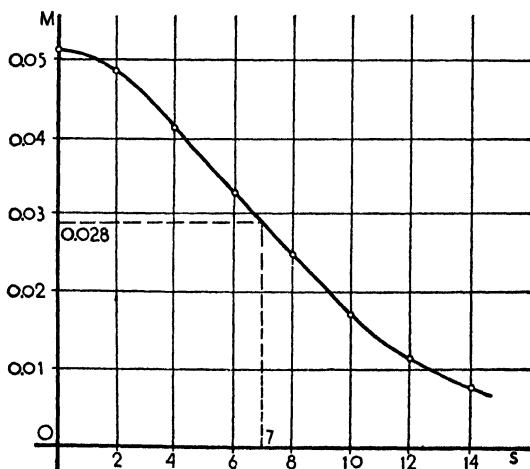


FIG. 3-11.

inductance. The table gives the mutual inductance M in henries of two coils for several distances of separation s in centimeters. Plot the curve of the mutual induct-

ance against the distance of separation between the coils. Determine the mutual inductance when the two coils are separated a distance of 7 cm.

s	0	2	4	6	8	10	12	14
M	0.051	0.049	0.041	0.033	0.025	0.017	0.011	0.007

The corresponding graph is plotted in Fig. 3-11. If the mutual inductance for the separation of 7 cm. is to be found, draw a vertical line from the point 7 to the curve. We note that the corresponding mutual inductance is 0.028 henry.

From the curve we can also read the following information. The mutual inductance does not change rapidly for the first 2 cm. of separation. From 2 cm. to 10 cm. the mutual inductance decreases more rapidly with separation, after which the mutual inductance changes more gradually as the distance is increased.

Example 3. The following table shows the values of current I in milliamperes obtained by applying E volts to a selenium rectifier plate. Plot I against E and determine the current when the voltage is 0.8 volt.

E	1.5	1.3	1.1	0.9	0.7	0	-2	-4	-6	-8	-10
I	100	80	60	40	20	0	-0.05	-1	-2	-3	-4

The curve is plotted in Fig. 3-12. To find the value of current corresponding to 0.8 volt, we draw a vertical line at 0.8 on the E axis and find that the corresponding

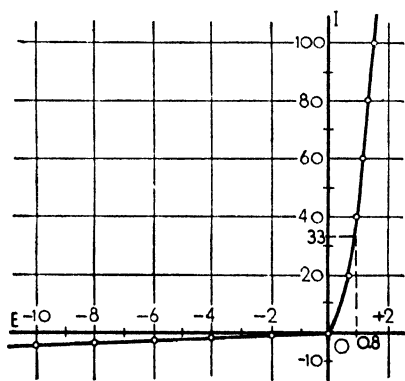


FIG. 3-12.

value of current is 33 milliamperes. From this curve we note that the current rises abruptly when positive voltages between 0 and 1.5 volts are applied. When negative voltages between zero and -2 are applied, the negative or reverse current is extremely small; when voltages between -2 and -10 are applied, the negative current remains comparatively small.

Example 4. From electrical engineering, it is known that, when a condenser is charged through a resistance, the current I rises to a maximum value in an extremely short time after the circuit is closed and then decreases toward zero as the time T increases. An experiment was set up to prove this, and the data given below were obtained, I being measured in amperes and T in seconds. Plot a graph of I against T . After how many seconds does the current reach a value which is 36.8 per cent of its maximum value as indicated in the table?

T	0	0.01	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
I	0	0.80	0.65	0.53	0.42	0.34	0.28	0.23	0.19	0.16	0.13

The curve is plotted in Fig. 3-13. The maximum value of current as given in the data is 0.80 ampere. Of this 36.8 per cent is 0.29 ampere. By drawing a horizontal line at the point 0.29 ampere to the curve, we note that the corresponding time value is 0.48 second.

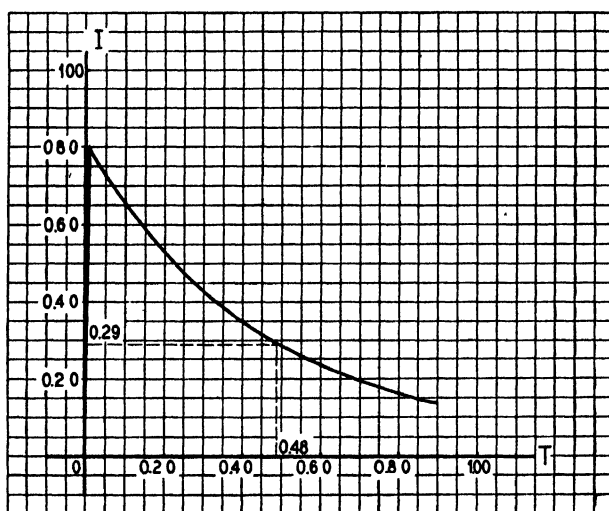


FIG. 3-13.

From the developments in the preceding paragraphs it may be seen that a function $y = f(x)$ can be given in three different ways.

1. *By a formula*, from which the values of y corresponding to given values of x may be computed. For example, $y = x^2 - 3x + 5$.

2. *By a table*, which gives values of y for certain values of x . The functions of Sec. 3-8 are examples of this kind.

3. *By a graph*. If in Example 1 only the graph in Fig. 3-10 were given, it would describe the distance s traveled by the object as a func-

tion of time t , for we can find from the graph the value of s corresponding to a given value of t .

EXERCISES

Plot the following functions on Cartesian coordinate systems, connecting the successive points by straight segments.

1.

x	-4	-3	-2	-1	0	1	2	3	4	5
y	-12	-10	-8	-6	-4	-2	0	2	4	6

2.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
y	24	21	18	15	12	9	6	3	0	-3	-6

3.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

4.

x	-10	-8	-6	-4	-2	0	2	4	6	8	10
y	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25

5.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	15	6	-1	-6	-9	-10	-9	-6	-1	6	15

6.

x	-3	-2	-1	0	1	2	3	4	5	6
y	21	12	5	0	-3	-4	-3	0	5	12

7.

x	-6	-4	-2	0	2	4	6
y	-30	-11	-4	-3	-2	5	24

8.

x	0	2	3	4	5	7	8	9
y	5	-1	2	10	6	-2	0	5

9.

x	-3	-1	0	1	2	3	5	7	8
y	6	14	8	6	-2	-4	-6	0	6

10.

x	-4	-2	0	1	4	5	6	7	8
y	-4	0	5	2	-8	-10	-6	0	10

Plot the functions given in Exercises 11–23 on Cartesian coordinate systems, connecting the successive points by a smooth curve.

11.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2

12. The function of Exercise 1.

13. The function of Exercise 2.

14.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-20	-11	-4	1	4	5	4	1	-4	-11	-20

15. The function of Exercise 3.

16. The function of Exercise 4.

17. The function of Exercise 5.

18. The function of Exercise 6.

19. The function of Exercise 7.

20.

x	-3	-2	-1	0	1	2	3	4
y	16	-36	-28	-8	0	-4	4	72

21.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-42	0	20	24	18	8	0	0	14	38

22.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	-7	-1.6	1	1.6	1	-0.1	-1	-0.9	1

23.

x	-1	0	1	2	3
y	-13	-5	-3	-1	7

The data in the exercises below were obtained by various experiments. In each case plot the graph of the data on a Cartesian coordinate system, connecting the points by a smooth curve.

24. The hourly temperatures in Chicago, Illinois, on a certain day in January are given in the following table. Plot T against H .

A.M.								P.M.		
Hour H	5	6	7	8	9	10	11	Noon	1	2
Temp. T	6	6	6	5	5	7	10	14	15	17

P.M.										
Hour H	3	4	5	6	7	8	9	10	11	Mid- night
Temp. T	20	21	21	20	19	19	18	19	19	19

25. With a grid bias of -6 volts on a 6J5 triode tube, the values of plate current I_p in milliamperes for various values of plate voltage E_p in volts are found by experiment to be:

E_p	120	140	158	170	180	190	200	210	218	225	235
I_p	0.5	1	2	3	4	5	6	7	8	9	10

Plot I_p against E_p . Find by reading the graph approximately what plate voltage is required if the plate current is to be (a) 1.5 milliamperes, (b) 8.6 milliamperes.

26. The experiment of Exercise 25 is performed again with a grid bias of -10 volts, and the following data are obtained.

E_p	215	250	275	295	310	320	325
I_p	0.5	1	2	3	4	5	5.5

Plot I_p against E_p on the same sheet used for the graph of Exercise 25. Compare the two graphs. What plate voltage is required if the plate current is to be (a) 1.5 milliamperes, (b) 4.5 milliamperes?

27. With the plate voltage fixed at 47 volts on a 6J5 triode tube, the values of plate current I_p in milliamperes for various values of grid voltage E_g are found by experiment to be:

E_g	-9.0	-6.0	-3.0	-2.5	-2.0	-1.5
I_p	0.4	0.4	0.4	0.6	1.0	1.9

E_g	-1.0	0.0	0.5	1.0	1.5	2.0
I_p	2.8	4.8	6.0	7.3	8.4	9.8

Plot I_p against E_g . What is the plate current when the grid voltage is -1.2 volts? What grid voltage is required for a plate current of 0.5 milliamperes?

28. The experiment of Exercise 27 is performed again with a plate voltage of 100 volts, and the following data are obtained.

E_g	-6	-5	-4	-3	-2	-1
I_p	0.7	0.8	1.7	3.4	5.8	8.7

Plot I_p against E_g on the same sheet used for the graph of Exercise 27 and compare the two graphs.

29. When testing the brake horsepower P of an engine, a number of readings are taken at various speeds. One set of experimental data is given below, the speed S being given in revolutions per minute. Plot P against S . What is the range of speed for which the horsepower is above 55? Describe in general terms the results of this experiment which you can read from the graph.

S	400	600	800	1000	1200	1400	1600	1800	2000
P	14	22	31	39	47	52	56	57	54

30. The subject of grounding is very important in an electrical distribution system. The following data concerning the variation of the resistance R in ohms of the ground connection against the number of rods N in parallel were obtained. Plot the curve.

N	1	2	3	4
R	100	55	35	28

31. As the output current I_o in milliamperes was varied for a full-wave rectifier voltage quadrupler, the output voltage E_o in volts changed in accordance with the following data. Plot the curve and determine the current at a voltage of 380 volts.

I_o	45.5	42.0	39.5	36.0	32.5	28.5	24.0	19.5	14.0	8.0	4.0
E_o	292	305	317	330	350	370	390	415	448	488	515

32. In determining the volt-ampere characteristic of a tungsten lamp, the following data were obtained, the current I being measured in milliamperes and the voltage E being measured in volts. Plot I against E . What is the change in current in milliamperes as the voltage varies between 45 and 75 volts?

E	10	20	30	40	50	60	70	80	90	100	110
I	150	195	235	270	305	340	365	410	425	450	485

33. The deflection on the screen of a cathode-ray tube is a function of the voltage applied to the plates. From the following data, plot a curve of deflection d in inches against the applied voltage E . When the voltage changes from 15 to 25 volts, what is the corresponding change in deflection?

E	52	40	26	13	0	-13	-26	-40	-52
d	0.8	0.6	0.4	0.2	0	-0.2	-0.4	-0.6	-0.8

34. In determining the dynamic characteristic of 6J5 triode tube, the following relations were found to exist between the grid voltage e_c in volts and the plate current i_b in milliamperes with a fixed plate supply voltage. Plot i_b against e_c .

e_c	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
i_b	13.5	11.0	7.4	5.6	4.9	2.3	1.0	0.5	0.1	0

35. The rectified current I in microamperes of a crystal detector is a function of the impressed signal voltage E_s . From the following experimental data, plot the curve. Determine the change in current when the voltage changes from 0.25 to 0.35 volt; also when the voltage changes from 0.75 to 0.85 volt.

E_s	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
I	0	9	13	18	21	24	26	28	30	31	31

36. The output voltage e_o in volts of a superheterodyne radio receiver varies with the frequency setting F in kilocycles per second as shown in the following table. Plot the curve, using F as the independent variable. The curve is called a **selectivity curve**.

F	1410	1408	1406	1404	1402	1400	1398	1396	1394	1392	1390
e_o	1.2	1.2	1.6	2.5	4.5	5.5	4.5	2.5	1.6	1.2	1.2

37. Upon setting the grid bias of a 884 Thyratron tube at zero, the following data were obtained for the plate voltage e_b in volts and the plate current i_b in milliamperes. Plot i_b against e_b .

e_b	36	56	74	94	113	132	149	167
i_b	2	4	6	8	10	12	14	16

38. The following data were obtained in testing a 1000-kilowatt 500-volt alternator, where L is the percentage of full load, and E is the percentage of efficiency. L is the independent variable. Plot the curve and determine the range, within the limits of the test data, of L , where the efficiency will be 90 per cent or greater.

L	0	20	40	60	80	100	120
E	0	80	91	94	95	96	96

39. In the design of power lines, the vertical sag S in the cable depends on the temperature T . The following set of data was obtained for a 400-ft. span, S being measured in feet and T in degrees Fahrenheit. Plot S against T . If the sag is not to exceed 8.1 ft., what is the maximum permissible temperature?

T	-40	-20	0	20	40	60	80	100
S	6.8	7.0	7.2	7.4	7.6	7.8	8.0	8.2

40. In studying certain magnetic properties for Armco iron, the following data were obtained, the magnetization H being measured in gilberts per centimeter and the flux density B being measured in lines per square centimeter. Plot B against H . The curve is known as a magnetization, or B - H curve.

H	0	0.9	1.0	1.05	1.15	1.4	2.2	4.5
B	0	2000	4000	6000	8000	10,000	12,000	14,000

41. The antenna resistance R in ohms varies as the height H in wavelengths of a horizontal half-wave antenna according to the values given in the table below. Plot the curve of antenna resistance against the antenna height. At what antenna height is the resistance a maximum?

H	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
R	0	30	87	95	68	60	77	85	68

42. The mutual conductance g_m in micromhos for a variable- μ pentode tube in radio-frequency amplifiers varies with the grid bias E_c in volts. From the given data plot the curve.

E_c	-2	-5	-10	-15	-20	-25	-30
g_m	2000	1000	300	125	90	50	25

43. As the setting of a field rheostat was varied on a shunt motor, the following set of data was obtained, the rheostat resistance R_f being measured in ohms and the speed S of the armature being measured in revolutions per minute. Plot S against R_f . At what resistance should the rheostat be set so that the motor will run at a rate of 1700 revolutions per minute? of 1400 revolutions per minute?

R_f	53.2	58.0	68.0	76.5	82.7	93.0	96.0
S	1300	1340	1420	1500	1590	1680	1730

44. As zinc is added to aluminum, the tensile strength of the alloy is increased. For various percentages P of zinc alloyed with aluminum, the corresponding tensile strengths S in pounds per square inch were found to be as follows.

P	0	1	2	3	4	5	6	7	8
S	12,600	13,000	13,500	14,000	14,800	15,700	16,600	18,000	19,200

Plot S against P .

45. A venturi meter is an apparatus designed to measure the flow of water. For various heights H of water in a certain tank, the rates of discharge D were measured by a venturi meter. For H in feet and D in cubic feet per second, the following data were obtained. Plot D against H . What is the effect of the height of the water on the discharge?

H	0	0.3	1.0	2.0	3.5	5.5	8.0	11.0
D	0	5	10	15	20	25	30	35

46. When a load is placed on a timber beam, the beam is compressed. The following data were obtained by applying various loads L to a beam and measuring the compression d parallel to the grain of the timber, L being measured in thousands of pounds and d in inches. Plot d against L . What will be the compression for a load of 50 tons? This curve is called a stress-deformation diagram.

L	0	30	67	90	114	124	126
d	0	0.01	0.02	0.03	0.04	0.05	0.055

47. In determining the relative merit of timber, it is necessary to dry or season the wood. The weights W in pounds of a red tie were measured after various times T in months during which the tie was being seasoned, and the following data were obtained. Plot W against T . After what length of seasoning will the loss in weight be 5 per cent of the original weight?

T	0	2	4	6	8	10	12	14	16
W	215	199	184	174	170	168	168	168	167

48. The valve lift L of a tangential cam was found to vary with the rotation θ of the shaft as shown in the table, L being measured in thousandths of an inch and θ in degrees. Plot L against θ . At what values of θ is the valve lift one-half of the maximum value?

θ	0	10	20	30	40	50	60	70	80	90	100	110
L	0	20	70	170	260	295	295	260	170	70	20	0

49. In the expansion of a gas at constant temperature, the pressure p times the volume v is equal to a constant. To verify this relationship known as Boyle's law,

the following experimental data were obtained, p being measured in pounds per square inch and v in cubic inches. Plot v against p .

p	14.7	16	17.5	21	25	30	37	50	76
v	10	9	8	7	6	5	4	3	2

50. An important factor in the treatment of carbon steel is the rate at which the furnace heats and cools. In a typical case, the temperatures T for various times t were found to be as follows.

t	1	2	3	4	5	6	7	8	9	10
T	550	900	1280	1460	1300	1250	1150	1000	880	780

The time t is measured in minutes, and the temperature T in degrees Fahrenheit. Plot T against t . For how long is the temperature of the sample above 1000° ?

51. The load voltage variations in a trolley system were studied by measuring the voltage drops V from the generator to various points on the track where a car was connected. Measuring V in volts and the distances D from the generator in miles, the following data were obtained. Plot V against D .

D	1	2	3	4	5	6	7	8
V	0.10	0.22	0.31	0.40	0.51	0.60	0.69	0.80

52. An experiment was made to determine the elongation e of machine steel due to a tensile stress S . Measuring S in thousands of pounds per square inch and e in 0.001 in. per in., the following data were obtained. Plot e against S . The graph is called a stress-strain diagram.

S	0	30	35	35	36	37	37.5
e	0	1	2	3	4	5	6

3-10. Graphs of Functions Given by Formulas. When a function $y = f(x)$ is given by an algebraic formula for $f(x)$, we must first construct from the given formula a table of number pairs from which the points on the plane may be plotted. However, most functions defined by formulas are defined for many more values of the independent variable than it would be feasible to use in finding number pairs from which to plot points. In fact, most of the functions considered in this chapter which are given by formulas are defined for all values of the independent

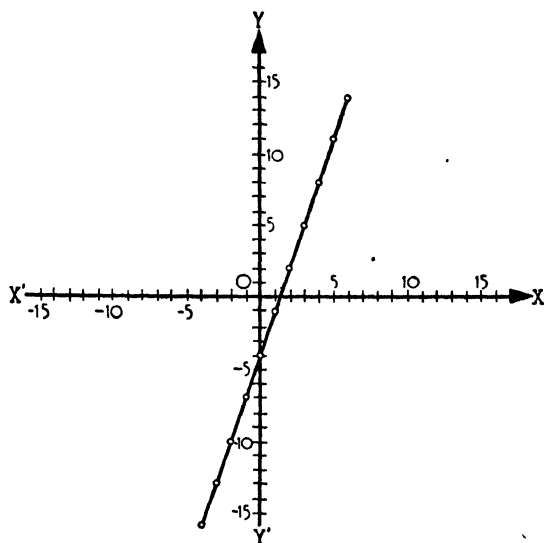
variable. What we do, therefore, is to select a few convenient number pairs and plot the corresponding points. Then the smooth curve through these points is approximately the graph of the function. This curve is sketched by proceeding from one point to another so that the values of x are taken in order of increasing or decreasing magnitude. How accurately this curve represents the function depends on how closely together the points are chosen: the closer together the points are chosen, the more accurately the curve can be drawn.

There is an essential difference between the graphs of the functions of this section and those of the functions in the preceding section: in the preceding section, the curve is drawn simply for convenience in reading the graph, whereas here all the points on the curve are points on the graph of the function.

Since most of the functions we consider in this section are defined for all values of the independent variable, it is convenient to plot only that portion of the graph which is required for the particular problem to be investigated by use of the graph. If nothing is known about the range within which the graph will be used, usually a portion of the graph near the origin of the coordinate system is plotted. It is then understood that the graph extends much further than is shown.

Example 1. Plot the graph of $3x - 4$.

We construct a table of sample values of the function as follows: Let $y = f(x) = 3x - 4$. Then $f(0) = -4$, $f(3) = 5$, etc. Values found in this way, by giving x



x	y
-4	-16
-3	-13
-2	-10
-1	-7
0	-4
1	-1
2	2
3	5
4	8
5	11
6	14

FIG. 3-14.

values and finding the corresponding values of y , fill out the table from which the graph of Fig. 3-14 is plotted.

Joining the points by a smooth curve in order of increasing or decreasing values of x , we see that the graph is a straight line. Although the line extends indefinitely in both directions, only a portion of it is shown in Fig. 3-14.

Example 2. Plot the graph of $2x^2 - 5x + 7$.

Setting $y = f(x) = 2x^2 - 5x + 7$, we find for various values of x the corresponding values of the function as given in the table.

x	y
-2	25
-1	14
0	7
1	4
2	5
3	10
4	19
5	32

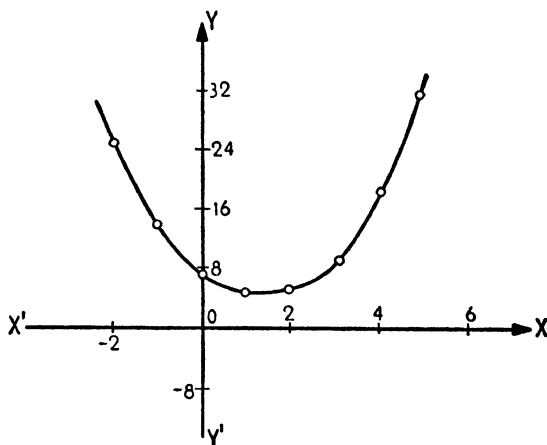


FIG. 3-15.

We plot as many of these points as can be located in the space available. Connecting them in the order of increasing values of x by a smooth curve, we have the graph of Fig. 3-15.

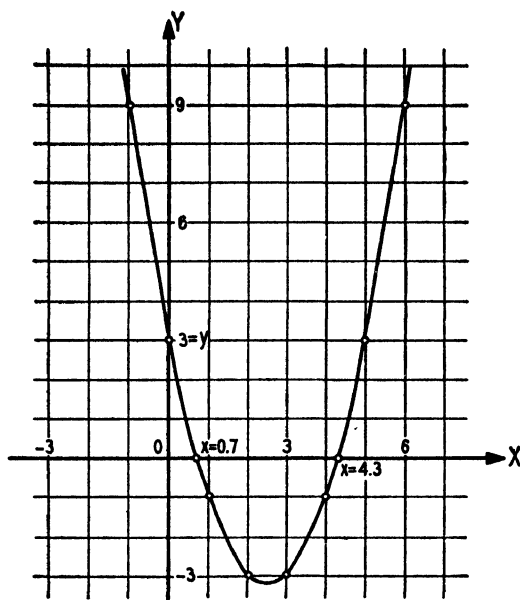
If the graph of the function $y = f(x)$ is plotted, its intersections with the y -axis give the values of $f(x)$ for $x = 0$. These values, the lengths of the segments from the origin to the points where the graph meets the y -axis, are often called the **y -intercepts**.

The intersections of the graph with the x -axis define x -values for which the corresponding y -value is zero. These values are sometimes called the **x -intercepts**. In order to find x -values so that $y = f(x) = 0$, look for the points where the graph of $f(x)$ intersects the x -axis.

The values of x for which an equation $f(x) = 0$ is satisfied are called the roots of the equation $f(x) = 0$. The inspection of the graph of the function $y = f(x)$ gives a very convenient method for estimating the roots of $f(x) = 0$.

Example 3. Estimate the roots of $x^2 - 5x + 3 = 0$ by plotting the corresponding graph.

Setting $y = x^2 - 5x + 3$ we obtain the following table of values.



x	y
-2	17
-1	9
0	3
1	-1
2	-3
3	-3
4	-1
5	3
6	9
7	17

FIG. 3-16.

The function is plotted in Fig. 3-16, and from where the graph crosses the x -axis we infer that the roots of the function are approximately 4.3 and 0.7. We also observe that the y -intercept is 3.

In addition to the suggestions about the details of plotting functions given in Sec. 3-8, the following suggestions will be useful in plotting the graphs of functions given by algebraic formulas.

1. *Intercepts.* The intercepts on the axes are usually of special interest and should be found and plotted whenever possible.

2. *Choosing points to plot.* We can usually obtain a satisfactory graph of the function by choosing the integral values of x near the origin from which to construct the table of values. Any sharp turn in the curve is usually important and may require that points be plotted more thickly. When there is doubt about the nature of a curve between two points, other points in the neighborhood should be found until its behavior is well established. In these matters, however, experience is the best teacher.

EXERCISES

Plot the graph of each of the following functions:

1. $3x$.
2. $-2x$.
3. $x + 5$.
4. $x - 3$.
5. $2x - 3$.
6. $4x + 6$.

- | | |
|---------------------------------------|---------------------------------------|
| 7. $8 - 3x$ | 8. x^2 . |
| 9. $x^2 - 6$. | 10. $3 - 2x^2$. |
| 11. $1 + x^2$. | 12. $1 + x + x^2$. |
| 13. $3 - 2x + x^2$. | 14. $\frac{-x^2}{2}$. |
| 15. x^3 . | 16. $-\frac{1}{8}x^3$. |
| 17. $x^2 - 5x$. | 18. $3x^2 + 2x$. |
| 19. $x^2 + x$. | 20. $x^2 - 5x + 6$. |
| 21. $3x^2 + 4x + 1$. | 22. $x^2 - 4x + 4$. |
| 23. $4x - 4 - x^2$. | 24. $2x^2 + x - 5$. |
| 25. $x^2 + x - 10$. | 26. $3 - 2x - 2x^2$. |
| 27. $2 - x + 3x^2$. | 28. $18 - 3x - 2x^2$. |
| 29. $3x^2 - 5x - 2$. | 30. $14 - 7x + x^2$. |
| 31. $x^3 + x$. | 32. $8x - x^3$. |
| 33. $x^2 + x^3$. | 34. $3x^3 - 2x^2$. |
| 35. $2x - x^3$. | 36. $4x^2 - 3x^3$. |
| 37. $x^3 + x + 1$. | 38. $x^3 - x^2 + 1$. |
| 39. $x^3 - 6x + 6$. | 40. $8 + x - x^3$. |
| 41. $x^3 - 3x^2 + 3x - 1$. | 42. $x^3 - x^2 + x - 1$. |
| 43. $\pm x$. | 44. $\pm 4\sqrt{x}$. |
| 45. $\pm\sqrt{25 - x^2}$. | 46. $\pm\sqrt{9 - x^2}$. |
| 47. $\pm\sqrt{16 - x^2}$. | 48. $\pm\frac{1}{3}\sqrt{25 - x^2}$. |
| 49. $\pm\frac{1}{4}\sqrt{x^2 + 25}$. | 50. $\pm\sqrt{x^2 + 4}$. |
| 51. $\pm\sqrt{2x + 4}$. | 52. $\pm\sqrt{x - 6}$. |
| 53. $\pm\sqrt{x + 12}$. | |

Find the roots of the following equations to the nearest tenth of a unit by plotting the corresponding graphs.

- | | |
|----------------------------|---------------------------|
| 54. $x^2 + x - 4 = 0$. | 55. $x^2 + x - 8 = 0$. |
| 56. $x^2 - 5x + 1 = 0$. | 57. $x^2 - 4x + 2 = 0$. |
| 58. $1 + x - x^2 = 0$. | 59. $1 - 2x - 2x^2 = 0$. |
| 60. $2x^2 - x - 5 = 0$. | 61. $x^2 - x - 10 = 0$. |
| 62. $2x^2 - 10x + 3 = 0$. | 63. $10 - 3x - x^2 = 0$. |

By plotting the corresponding graphs show that the following equations have no roots.

- | | |
|---------------------------|---------------------------|
| 64. $x^2 + x + 1 = 0$. | 65. $x^2 - x + 1 = 0$. |
| 66. $2x^2 - 2x + 1 = 0$. | 67. $x^2 + 2x + 2 = 0$. |
| 68. $6x + x^2 + 10 = 0$. | 69. $6x - x^2 - 10 = 0$. |
| 70. $x - x^2 - 1 = 0$. | 71. $2x - 2x^2 - 1 = 0$. |

3-11. Variation. A variable y is said to vary as x or to be proportional to x if

$$(1) \quad y = kx.$$

The constant number k is called the factor or constant of proportionality. The quotient or ratio y/x has the constant value k when x and y vary

so as to satisfy (1). The word **ratio** is used as a synonym for quotient. The statement that two ratios are equal is sometimes called a **proportion**.

From (1) it follows that $x = \frac{1}{k}y$. Since $\frac{1}{k}$ is also a constant, it follows that x is proportional to y if y is proportional to x . To emphasize this fact, it is often said that the variables x and y are proportional to one another.

Example 1. If E is an electromotive force, measured in volts, applied across a resistance of 5 ohms, the current measured in amperes in this resistance is $I = \frac{E}{5}$. I varies as E or is proportional to E , the factor of proportionality being $\frac{1}{5}$. If E changes, I varies so that the ratio $\frac{E}{I}$ has always the same value 5.

A variable y is said to vary inversely as another variable x or to be inversely proportional to x if

$$(2) \quad y = \frac{k}{x}.$$

As before the constant number k is called the factor of proportionality. From (2) it follows that

$$xy = k.$$

Thus, the statement that y varies inversely as x , or x varies inversely as y , is equivalent to the statement that the variables x and y vary in such a way that their product is constant.

Example 2. If a constant electromotive force of 6 volts is working across a resistance R_1 , the current produced is inversely proportional to the resistance and is given by the formula $I = \frac{6}{R}$. I varies inversely as R ; the product IR is constant, equal to 6.

A variable u is said to vary jointly as the variables x_1, x_2, x_3 and to vary inversely as the variables y_1, y_2, y_3 , if

$$(3) \quad u = k \frac{x_1 x_2 x_3}{y_1 y_2 y_3},$$

k being the factor of proportionality.

Example 3. The force F between two small conductors varies jointly as their electrical charges Q_1 and Q_2 and inversely as the square of their distance r . This statement is equivalent to the formula

$$F = k \frac{Q_1 Q_2}{r^2},$$

where k is a constant quantity.

In applications, it sometimes happens that quantities are connected by a relation like (1), (2), or (3), but the factor of proportionality is unknown. In order to find this unknown value, it is sufficient to know the value of the dependent variable for one set of values of the independent variables. Such a value, if not known, can often be found by an experiment.

Example 4. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter.

This statement is equivalent to the formula

$$R = k \frac{l}{d^2}.$$

If k were known, this formula could be used to find the resistance R for given values of l and d . The value of k depends on the units which are used to measure the quantities R , l , d . We suppose that R is measured in ohms, l in feet, and d in mils (1 mil = 0.001 in.). In order to find k , the formula $R = k \frac{l}{d^2}$ is applied to a particular case in which the values of R , l , d are known. Suppose that an experiment with a copper wire of length $l = 500$ ft. and diameter $d = 40$ mils yields a resistance of $R = 3.3$ ohms. By substituting these values, the following equation for k is obtained

$$3.3 = k \frac{500}{1600},$$

$$k = \frac{3.3 \times 1600}{500} = 10.56.$$

Using this value we obtain the result that the resistance of a copper wire can be found by the formula

$$R = 10.56 \frac{l}{d^2}.$$

This formula can be used to find the resistance for any given values of l and d .

Example 5. The time t required for an elevator to lift a weight varies jointly as the weight W and the distance d through which it is to be lifted and inversely as the power P of the motor. If it requires 20 seconds for a 5-horsepower motor to lift 500 lb. through 40 ft., what power is necessary to lift 1200 lb. 120 ft. in 30 seconds?

The first part of this problem states that

$$t = k \frac{Wd}{P}.$$

In order to find k , the data is used that $t = 20$ when $W = 500$, $d = 40$, and $P = 5$.

$$20 = k \frac{500 \cdot 40}{5} = 4000k,$$

$$k = \frac{20}{4000} = \frac{1}{200}.$$

The complete formula, therefore, is

$$t = \frac{1}{200} \cdot \frac{Wd}{P}.$$

In order to answer the question of the problem, the substitution $W = 1200$, $d = 120$, $t = 30$ is made.

$$30 = \frac{1}{200} \cdot \frac{1200 \times 120}{P},$$

$$P = \frac{1200 \times 120}{200 \times 30} = 24 \text{ hp.}$$

Example 6. If the rate of flow of water through a pipe varies as the square of the radius of the pipe, by how much would the rate of flow be increased if the diameter of the pipe were multiplied by $\frac{3}{2}$?

If the rate of flow is denoted by V and the radius of the pipe by r , then

$$V = kr^2.$$

If the new radius $r_1 = \frac{3}{2}r$, then the new rate of flow is

$$V_1 = kr_1^2 = k \cdot \frac{9}{4}r^2 = \frac{9}{4}kr^2 = \frac{9}{4}V.$$

The rate of flow, therefore, is multiplied by $\frac{9}{4}$.

EXERCISES

In the following exercises make use of the conventions about significant figures treated in Sec. 1-13.

1. The elongation E of a spring balance varies as the applied weight W . If $E = 3$ when $W = 40$, find E when $W = 65$.

2. The deflection D of a beam varies as the cube of the length L . If $D = 0.003$ when $L = 12$, find the formula giving D for any L . Find D when $L = 18$.

3. The acceleration g due to gravity in feet per second per second varies inversely as the square of the distance d from the center of the earth in miles. Write the formula for g if it is known that g is 32 when d is 4000 miles. Find the value of g when d is 6000 miles.

4. The distance a body falls, starting from a position of rest in a vacuum near the earth's surface, is proportional to the square of the number of seconds of fall. If a body falls 400 ft. in 5 seconds, how far will it fall in 9 seconds?

5. If the illumination from a source of light varies inversely as the square of the distance, how much farther from a candle must a book which is now 20 in. off be removed so as to receive just one-third as much light?

6. The number of revolutions per minute of a ball governor required to keep the balls h inches below the point of suspension varies inversely as the square root of h . If the balls are 8 in. below for 30 revolutions per minute, at what speed will they be $1\frac{1}{2}$ in. below the point of suspension?

7. The rate of vibration of a string under constant tension varies inversely as the length of the string. If a string 36 in. long vibrates 256 times per second, what must be the length of a string which vibrates 768 times per second?

8. The absolute temperature of a certain quantity of gas varies jointly as the pressure and the volume. The temperature of some gas in a cylinder is 300° and

its pressure is 20. What will the temperature become if the volume is doubled and the pressure is changed to 8?

9. The force of water p on the bottom of a containing vessel varies jointly as the area A of the bottom and depth d of the water. When the water is 1 ft. deep, the force on 1 sq. ft. of the bottom is 62.4 lb. Find the force, correct to the nearest pound, on the bottom of a circular tank which is 10 ft. in diameter and has water in it to a depth of 12 ft.

10. The approximate velocity of a stream of water necessary to move a round object is proportional to the product of the square roots of the object's diameter and its specific gravity. If a velocity of 12 ft. per sec. is needed to move a stone with a diameter of 1 ft. and specific gravity of 4, how large a stone with specific gravity 3 can be moved by a stream whose velocity is 20 ft. per sec.?

11. The force p of wind on a plane surface varies jointly as the area A of the surface and the square of the wind velocity v . If the force on 5 sq. ft. is 11 lb. when the wind velocity is 22 miles per hour, find (a) an equation connecting p , A , and v ; (b) the force on a surface 9 ft. by 15 ft. when the wind velocity is 50 miles per hour.

12. The horsepower that a shaft can safely transmit varies as its speed and the cube of its diameter. If a 2-in. shaft making 100 r.p.m. can transmit 30 horsepower, what size shaft of the same material is needed to transmit 200 horsepower at 150 r.p.m.?

13. The quantity Q of electricity that will flow into a radio condenser varies jointly as the capacity C and the voltage E . The factor of proportionality is 10^{-6} if Q is in coulombs, C in microfarads, and E in volts. If $C = 20$ microfarads and $E = 110$ volts, find Q .

14. The stress in the material of a pipe subject to internal pressure varies jointly as the pressure and the internal diameter of the pipe and inversely as the thickness. If the stress is 150 lb. per sq. in. when the diameter is 5 in., the thickness $\frac{3}{4}$ in., and the internal pressure 20 lb. per sq. in., find the stress when the pressure is 40 lb. per sq. in. if the diameter is increased to 6 in. and the thickness is reduced to $\frac{1}{2}$ in.

15. The weight W which a beam of uniform rectangular cross section, supported at both ends, will sustain varies jointly as the breadth b of the cross section and the square of the depth d and inversely as the distance between supports. If a beam 18 ft. long, 10 in. deep, and 6 in. in breadth will support a weight of 1200 lb., find the breadth of a beam of the same material, of the same depth, and 15 ft. long, that will support a weight of 1500 lb.

16. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. If a wire 450 ft. long and 4 mm. in diameter has a resistance of 1.06 ohms, find the length of a wire of the same material whose resistance is 0.71 ohm and diameter is 3 mm.

17. The amount of energy in joules stored by a condenser varies jointly as the capacity and the square of the voltage. The constant of variation is $\frac{1}{2}(10)^{-6}$ if C is in microfarads and E in volts. What is the energy stored in a 2-microfarad condenser with 1200 volts applied?

18. The weight of a body above the surface of the earth varies inversely as the square of the distance from the center of the earth. If a certain body weighs 25 lb. when it is 4000 miles from the center of the earth, how much will it weigh at a distance of 4100 miles from the center?

19. The distance of a body falling freely from rest varies as the square of the number of seconds required in falling. If a body falls 257.6 ft. in 4 seconds, how far will it fall in 15 seconds?

20. The electrical resistance of a substance varies directly as the length L and inversely as the area A of the cross section. If the resistance of a bar of annealed aluminum 1 in. long and 1 sq. in. in cross section is 0.000,001,144 ohm, find the resistance of a wire of the same material $1\frac{1}{2}$ ft. long and 0.003 in. in diameter.

21. The illumination received from a source of light varies inversely as the square of the distance from the source and directly as its candlepower. At what distance from a 60-candlepower light would the illumination be one-half that received at 25 ft. from a 50-candlepower light?

22. An electric power company figures that the cost C per kilowatt hour for supplying electric current to small consumers is equal to a fixed charge a plus an amount that varies inversely as the number m of kilowatt hours supplied. (a) Find the law connecting C , a , and m , if the company charges 4 cents per kilowatt hour at $m = 50$, and 3 cents per kilowatt hour at $m = 150$. (b) On this basis, what should be the charge at $m = 400$? (c) Find the value of m for which $C = 2$ cents.

23. The electrical resistance of a cable varies directly as its length and inversely as the square of its diameter. If a cable 4250 ft. long and $\frac{1}{3}$ in. in diameter has a resistance of 0.2 ohm, what resistance has a cable 7500 ft. long and $\frac{2}{3}$ in. in diameter?

24. Newton's law of gravitation states that the force of attraction between two masses of m_1 and m_2 pounds varies directly as the product of the masses and inversely as the square of the distance between the masses. Find the ratio of the force of attraction when two masses are 8000 miles apart to the force when they are 2000 miles apart.

25. The current in an electric circuit varies directly as the electromotive force and inversely as the resistance. In a certain circuit the electromotive force is E volts, the resistance is R ohms, and the current is I amperes. If the resistance is increased by 25 per cent, what per cent of increase must occur in the voltage to increase the current by 40 per cent?

3-12. The Straight Line. Consider the equation

$$(1) \quad Ax + By + C = 0$$

where A , B , and C are constants, and A and B are not both zero, and x and y are variables. Any equation which can be reduced by ordinary algebraic operations (those given in Chapter 2) to the form (1) is said to be **linear**.

Example 1. Given $x - 8y + 32 = 18 - (2x + y)$.

From this equation we obtain:

$$x - 8y + 32 = 18 - 2x - y,$$

$$x + 2x + y - 8y + 32 - 18 = 0,$$

$$3x - 7y + 14 = 0.$$

This last equation is of the form (1) where $A = 3$, $B = -7$, and $C = 14$. Thus the given equation is linear.

If the equation

$$(2) \quad 3x - 7y + 14 = 0$$

is solved for y , the result is the formula

$$(3) \quad y = \frac{3}{7}x + 2$$

by which y is defined as a function of x . The equations (2) and (3) are equivalent; that is, each pair of values (x, y) satisfying (3) also satisfies (2). If for instance $x = 7$, then we have from (3) $y = 5$. These two values also satisfy equation 2, for $3 \cdot 7 - 7 \cdot 5 + 14 = 21 - 35 + 14 = 0$.

The graph corresponding to $y = \frac{3}{7}x + 2$ consists of all points whose coordinates satisfy the relation $y = \frac{3}{7}x + 2$. According to the former statement, the coordinates of all those points also satisfy the equation $3x - 7y + 14 = 0$, and the graph belonging to $y = \frac{3}{7}x + 2$ consists of all points whose coordinates satisfy the equation $3x - 7y + 14 = 0$.

We shall, therefore, say that the graph corresponding to the function $y = \frac{3}{7}x + 2$ corresponds also to the equation $3x - 7y + 14 = 0$.

In this way we are able to construct a graph of each equation of the form (1). The graph consists of all points whose coordinates satisfy the equation $Ax + By + C = 0$, and we call it the graph of this equation.

It will be shown in Chapter 15 that the following statements are true.

The graph of every equation of the form (1) is a straight line. Conversely, every straight line has an equation of the form (1).

Equation (1) is therefore called the **general linear equation**. Since the variables enter only to the first power, it is said to be of **first degree**.

Since two points determine a straight line, we need plot only two points to plot a linear equation. Practically, it is a good idea to plot one or two more to insure accuracy. When the line does not pass through the origin or is not parallel to either axis, the two simplest points to plot are the intercepts on the axes.

Example 2. Plot the line $3x - 2y = 10$.

Setting $y = 0$, we see that the x -intercept is $\frac{10}{3}$, and setting $x = 0$, we see that the y -intercept is -5 . When $y = 1$, $3x - 2 = 10$, or $3x = 12$, and $x = 4$.

The line plotted in Fig. 3-17 by drawing the straight line through the intercepts is seen to pass through $(4, 1)$, and we are insured against error.

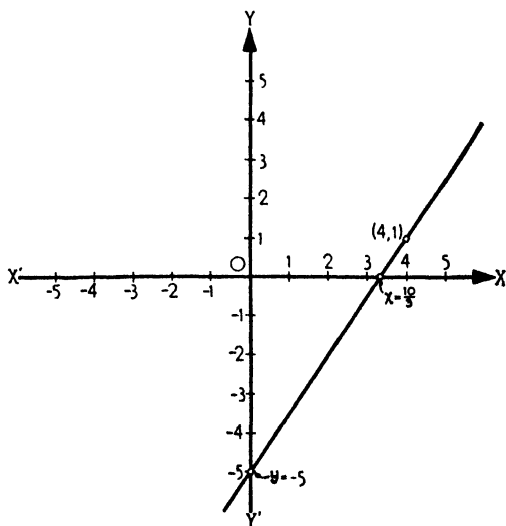


FIG. 3-17.

EXERCISES

Find the intercepts of the following straight lines and plot the graphs of the lines.

- | | |
|---------------------------|---------------------------|
| 1. $x + y - 5 = 0$. | 2. $x + 2y - 7 = 0$. |
| 3. $4x - 2y + 5 = 0$. | 4. $2x + 5y = 3$. |
| 5. $3x + y - 1 = 0$. | 6. $3x - 2y = 1$. |
| 7. $x + y - 2 = 0$. | 8. $2x - 3y = 0$. |
| 9. $5x + 12y = 39$. | 10. $12x - 5y + 65 = 0$. |
| 11. $4x + 3y - 15 = 0$. | 12. $x + 4y = 5$. |
| 13. $2x + y + 4 = 0$. | 14. $4x - 3y - 8 = 0$. |
| 15. $8x + y - 6 = 0$. | 16. $3x - 4y + 5 = 0$. |
| 17. $6x - 15y + 28 = 0$. | 18. $2x - 3y - 6 = 0$. |
| 19. $x + 2y - 3 = 0$. | 20. $6x + 10y + 43 = 0$. |

3-13. Graphical Solution of Simultaneous Linear Equations in Two Unknowns. In Sec. 2-16 we considered pairs of simultaneous linear equations of the form

$$(1) \quad \begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2. \end{aligned}$$

The graph of each of these equations is a straight line. For these two straight lines there are three possibilities.

1. *They may intersect in one point.* The coordinates (x, y) of the point of intersection give the numbers which satisfy the two equations.

2. *They may be parallel.* Since parallel lines do not have a point in common, there is no pair of numbers (x, y) which satisfies both equations.

3. *They may coincide.* In this case every number pair satisfying one equation satisfies the other.

The algebraic method of Sec. 2-16 yields the number pair in the first case. What happens in the other two cases will be indicated in the examples below.

It is possible to find the pair of numbers satisfying (1) approximately by plotting the two lines and reading the coordinates of their point of intersection from the graph.

Example 1. Find algebraically and graphically the values of x and y satisfying

$$2x - y = 3,$$

$$x + y = 3.$$

The graph is shown in Fig. 3-18. From the graph it appears that the solution is (2, 1).

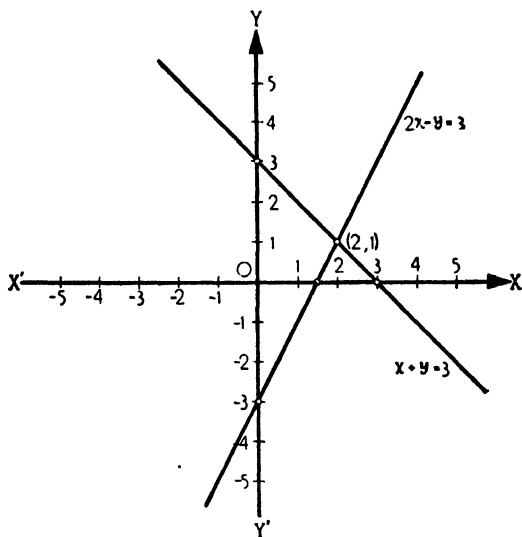


FIG. 3-18.

To verify this result algebraically, substituting $y = 3 - x$ from the second equation into the first, we obtain

$$2x - (3 - x) = 3,$$

$$2x - 3 + x = 3,$$

$$3x = 6,$$

$$x = 2,$$

and

$$y = 3 - 2 = 1.$$

Thus the correct solution is (2, 1).

Example 2. Given:

$$2x + y = 4,$$

$$2x + y = 8.$$

From Fig. 3-19 on which these two lines are plotted, we see that they are parallel, and hence there is no pair of numbers satisfying both.

Subtracting the first equation from the second we obtain $0 = 4$, which is absurd. When this algebraic process is performed, it is assumed that there is a solution, and since this assumption leads to a contradiction, the assumption is incorrect, and there is no solution.

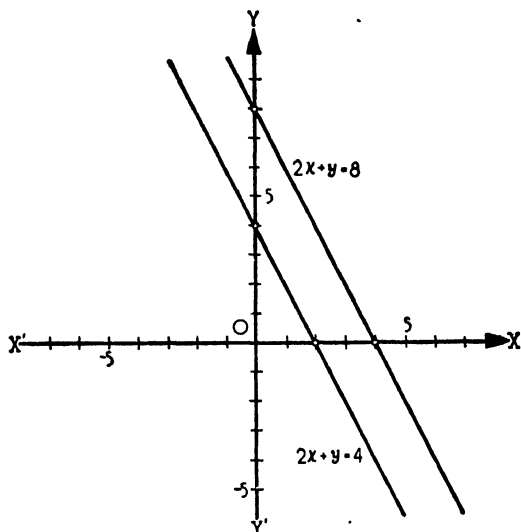


FIG. 3-19.

Example 3. Given:

$$x + 2y = 6,$$

$$2x + 4y = 12.$$

The student may verify that these two equations give the same line; consequently every pair of values satisfying one satisfies the other.

EXERCISES

Plot the lines in the following pairs of equations and from the graph state whether the lines intersect, are parallel, or coincide. If they intersect, estimate the coordinates of the point of intersection from the graph, and then verify your estimate by solving the equations algebraically.

- | | |
|-----------------------|------------------------|
| 1. $4x - 2y = 2,$ | 2. $4x - 2y = 2,$ |
| $2x + y = 7.$ | $2x - y = -3.$ |
| 3. $4x - 2y = 2,$ | 4. $2x + 3y = 13,$ |
| $2x - y = 1.$ | $5x - 2y = 4.$ |
| 5. $3x + 2y = 15,$ | 6. $5x + 3y - 11 = 0,$ |
| $3x + 2y = -2.$ | $y = 2x - 11.$ |
| 7. $3x - 5y - 1 = 0,$ | 8. $2x - y = 1,$ |
| $8x + 6y = 9.$ | $3x + 2y = 12.$ |
| 9. $4x + 3y = 2,$ | 10. $3x + 2y = 12,$ |
| $3x - 2y = -7.$ | $3x + 2y = 24.$ |
| 11. $x + y = 3,$ | 12. $x + y + 1 = 0,$ |
| $3x + 3y = 9.$ | $2x - y = 10.$ |
| 13. $x + y + 1 = 0,$ | 14. $2x - y = 3,$ |
| $3x = -3y - 3.$ | $4x - 2y = 11.$ |

$$\begin{aligned} 15. \quad & 2x + y = 1, \\ & x - y + 7 = 0. \end{aligned}$$

$$\begin{aligned} 17. \quad & 3x + y = -2, \\ & -x + 5y = 22. \end{aligned}$$

$$\begin{aligned} 19. \quad & x - 2y = 1, \\ & y = 1 + x. \end{aligned}$$

$$\begin{aligned} 16. \quad & x + y = 6, \\ & 3x - 5y + 10 = 0. \end{aligned}$$

$$\begin{aligned} 18. \quad & 3x + y = -2, \\ & 3x + y = 8. \end{aligned}$$

$$\begin{aligned} 20. \quad & 2x + y = 1, \\ & 3x - 4y = 29. \end{aligned}$$

PROGRESS REPORT

The numbers and symbols involved in the discussions and problems of the first two chapters were constants. But the quantities which appear in very many engineering problems are variables, and it is therefore of fundamental importance to investigate such quantities.

Mathematics, therefore, has to develop special methods in order to deal with these problems, methods for which a basis was laid by introducing the concept of a function to describe the relation between an independent and a dependent variable. Then methods were developed by which a function may be defined and examined. Tables, graphs, and mathematical formulas were used for this purpose. A large part of the chapter was devoted to the study of graphs; further it was shown how a number of problems concerning functions could be solved by graphs.

It should be pointed out that the corresponding values of any two variables can be used in order to construct a graph, even though there is no direct connection between the variables. Such graphs are frequently used in statistics. In engineering problems, however, there is a connection, described by some physical law, between the quantities which are plotted or given in a table.

Particular functions were studied in the sections about variation and about linear functions. Simultaneous systems of linear equations were studied, and the graphical method was used in their discussion.

CHAPTER 4

TRIGONOMETRIC FUNCTIONS

Trigonometry is the set of methods and procedures required to solve problems concerning triangles when angles of the triangles are involved. Since triangle problems occur in almost all branches of engineering and in much practical work, the scientist or engineer should have a working knowledge of trigonometry. The carpenter with his steel square makes use of trigonometric relations to find the length of one side of a hip roof. The modern structural engineer uses trigonometry to determine the stresses which will be applied to floor beams and to determine the dimensions of steel girders in a skyscraper. The civil engineer depends on it in measuring areas, in finding distances, and in other surveys. The mechanical engineer uses trigonometry in the design of crankshafts and engine sections; the electrical engineer uses it in the design of electric generators and alternators. Many problems about alternating current can be solved by using trigonometry. In gunnery and aerial bombing, we find trigonometry used in the location of targets and the design of gunsights. In meteorology, the height of a balloon above the surface of the earth is determined by the use of trigonometry.

4-1. Angles. We introduce the following definition of an angle.

If a line starting from a position OX is revolved, in one plane, about the point O to the position OA , we say that the line **generates an angle**

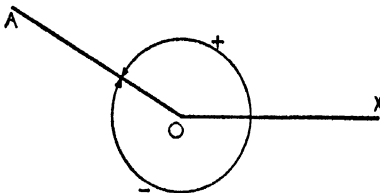


FIG. 4-1.

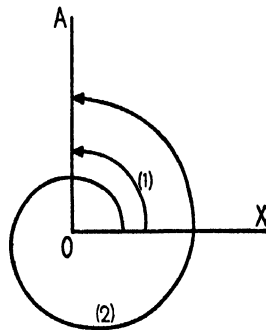


FIG. 4-2.

XOA (see Fig. 4-1). The original position OX of the generating line is called the **initial side** of the angle, the final position OA is called the

terminal side of the angle. If the generating line is rotated in a **counterclockwise** direction to position OA , the resulting angle XOA is called **positive**, and if the generating line is rotated in a **clockwise** direction to position OA the angle XOA is called **negative**. The positive angle in Fig. 4-1 is indicated by a $+$ sign, the negative angle by a $-$ sign.

From this definition the size of an angle is determined by the amount and direction of the generation used in constructing the angle. The definition can be used to compare any two angles with respect to their size. For example, the angle XOA , or (1) of Fig. 4-2 formed by revolving a line from position OX through one-quarter of a complete revolution about O in a counterclockwise direction to position OA is a different and smaller angle than the angle XOA or (2) formed by revolving a line from the position OX through one and one-quarter revolutions about O in a counterclockwise direction to position OA .

4-2. The Measurement of Angles: Degrees. There are two common systems used for the measurement of angles. In one the **degree** is the unit of measure; in the other, the **radian** is the unit of measure.

The angle of one degree is the angle which requires $\frac{1}{360}$ of the rotation needed to obtain one complete revolution. Thus a complete revolution is divided into 360 equal parts called degrees. Each degree is divided into 60 equal parts called **minutes**, and each minute into sixty equal parts called **seconds**. The symbols, $^{\circ}$, $'$, $''$ are used to denote degrees, minutes, and seconds respectively. Thus an angle of 31 degrees, 15 minutes, and 10 seconds may be written $31^{\circ} 15' 10''$. A frequent practice in engineering, especially in electrical engineering, is to use decimal parts of degrees rather than minutes and seconds, replacing, for example, $17^{\circ} 30'$ by 17.5° .

The computations in this book will be performed with the decimal parts of degrees almost exclusively.

Occasionally it is necessary to change from minutes and seconds to decimal parts of a degree, or vice versa. *To express an angle in decimal parts of a degree when it is given in minutes and seconds, first convert the seconds into a decimal part of a minute, and then the minutes into a decimal part of a degree.*

Example 1.

$$\begin{aligned}
 31^{\circ} 45' 54'' &= 31^{\circ} 45' + \frac{54'}{60} \\
 &= 31^{\circ} 45' + (0.9)' \\
 &= 31^{\circ} 45.9' \\
 &= 31^{\circ} + \left(\frac{45.9}{60}\right)^{\circ} = 31.765^{\circ}.
 \end{aligned}$$

To perform the inverse operation, multiply the decimal part of the given angle by 60 to obtain the number of minutes, and multiply the decimal part of the minutes so obtained by 60 to get the seconds.

Example 2.

$$\begin{aligned} 31.765^\circ &= 31^\circ + 0.765^\circ \\ &= 31^\circ 45.9' \text{ (since } 0.765^\circ = 0.765 \times 60' = 45.9') \\ &= 31^\circ 45' + 0.9' \\ &= 31^\circ 45' 54'' \text{ (since } 0.9' = 0.9 \times 60'' = 54''). \end{aligned}$$

At times it may be desirable to examine the accuracy of these conversions. It can be shown that:

(1) *Angles measured to the nearest minute are accurate, in decimal notation, to two decimal places.*

(2) *Angles measured to the nearest second are accurate to three decimal places.*

Example 3. Express $11^\circ 10' 13''$ in decimal parts of a degree to the number of decimal places permitted by the given data.

The angle $11^\circ 10' 13''$ has evidently been read to the nearest second. According to the preceding paragraph, the result will be significant to three decimal places. Hence, the computations are carried out to four places and rounded off to three places:

$$\begin{aligned} 11^\circ 10' 13'' &= 11^\circ 10' + \frac{13'}{60} \\ &= 11^\circ 10.2167' \\ &= 11^\circ + \frac{10.2167^\circ}{60} \\ &= 11.1703^\circ = 11.170^\circ. \end{aligned}$$

A positive angle is measured by a positive number of degrees; a negative angle is measured by a negative number of degrees. Thus the positive angle in Fig. 4-1 is 150° , and the negative angle in Fig. 4-1 is -210° .

EXERCISES

Convert to decimal parts of a degree, carrying out computations to the number of places permitted by the data given.

- | | | |
|---------------------------|----------------------------|----------------------------|
| 1. $21^\circ 12'$. | 2. $21^\circ 18'$. | 3. $173^\circ 42'$. |
| 4. $763^\circ 61'$. | 5. $173^\circ 85'$. | 6. $-71^\circ 2''$. |
| 7. $22'$. | 8. $28''$. | 9. $13^\circ 11' 47''$. |
| 10. $92^\circ 58' 19''$. | 11. $127^\circ 0' 14''$. | 12. $845^\circ 37' 14''$. |
| 13. $1^\circ 12''$. | 14. $215^\circ 20' 18''$. | 15. $-52^\circ 16' 17''$. |

16. $-763^{\circ} 11' 51''$.

17. $21^{\circ} 13'$.

18. $21^{\circ} 13' 0''$.

19. $32^{\circ} 0'$.

20. $32^{\circ} 1'$.

21. $142^{\circ} 17' 24''$.

22. $-52^{\circ} 16' 16''$.

23. $41^{\circ} 58' 3''$.

24. $12^{\circ} 0' 1''$.

Change to degrees, minutes, and seconds, writing the results as accurately as the given data will permit.

25. 13.1° .

26. 13.11° .

27. 75.79° .

28. 8.2° .

29. 465.273° .

30. 38.712° .

31. 16.50° .

32. 288.898° .

33. -15.555° .

34. -91.01° .

35. 42.10° .

36. 42.10° .

37. 42.13° .

38. 118.796° .

39. -0.01° .

40. 7.6296° .

Using a protractor, construct the following angles and indicate the corresponding rotation with an arrow (as in Sec. 4-1).

41. 30° .

42. 75° .

43. 90° .

44. 120° .

45. 190° .

46. 275° .

47. 345° .

48. -70° .

49. -120° .

50. -240° .

4-3. The Measurement of Angles: Radians. In the second system used for the measurement of angles, the radian is the unit of measure.

A **radian** is the measure of an angle which, placed with its vertex at the center of any circle, subtends on the circumference an arc equal in length to the radius of the circle.

Thus, if we take a circle with center at O and radius r (Fig. 4-3), and from a point A on the circumference measure an arc AB of length r , the angle AOB is by definition an angle of 1 radian.

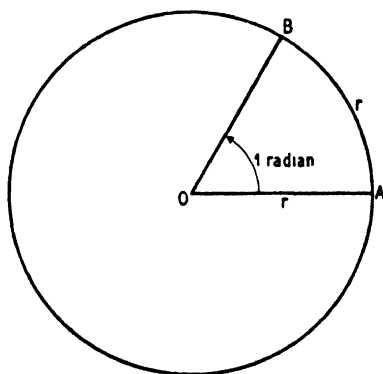


FIG. 4-3.

We are now faced with the problem of expressing the measure of any given angle, say θ , in radians. If we place θ with its vertex at the center O of the circle, and its initial side on the initial side OA of angle AOB

(Fig. 4-4), then angle θ will intercept on the circumference of the circle an arc whose length we shall call s . Then from the theorem in geometry

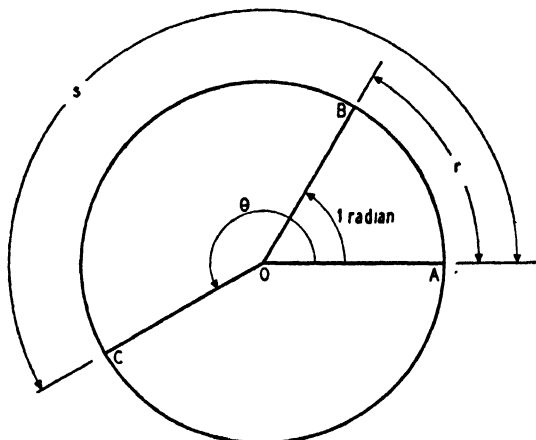


FIG. 4-4.

which states that *on the same circle, central angles are proportional to their intercepted arcs*, we may write

$$\frac{\text{Angle } \theta \text{ (measured in radians)}}{\text{Angle } AOB \text{ (measured in radians)}} = \frac{\text{Arc } AC}{\text{Arc } AB}.$$

However, since angle AOB equals 1 radian and since the arcs AC and AB are of lengths s and r respectively, we obtain

$$\frac{\theta \text{ (in radians)}}{1} = \frac{s}{r} \quad \text{or} \quad \theta = \frac{s}{r}.$$

Usually this is written

$$(1) \quad s = r\theta.$$

In words, formula (1) states that *the length of an arc of a circle is equal to the radius of the circle multiplied by the measure in radians of the angle subtended by the arc at the center of the circle*.

Example 1. In a circle having a radius of 5.0 ft., how long is the arc intercepted by a central angle of 1.5 radians?

Since the angle is given in radians, formula (1) may be applied directly. Since $r = 5.0$, $\theta = 1.5$,

$$s = r\theta = 5 \times 1.5 = 7.5 \text{ ft.}$$

Since r is measured in feet, s is measured in feet also.

Example 2. If a point on the circumference of a rotating wheel is moving at the rate of 400 ft. per sec. and the radius of the wheel is 2 ft., at what angular velocity is the wheel rotating?

By the angular velocity of the wheel is meant the rate, in this case measured in radians per second, at which a spoke of the wheel is rotating. Taking one second as a basis, $s = 400$, $r = 2$, and formula (1) gives $400 = \theta \cdot (2)$, or $\theta = \frac{400}{2} = 200$ radians/second.

It is often necessary to convert an angle from degrees to radians or from radians to degrees. Since the circumference of a circle is equal to $2\pi r$ (where $\pi = 3.1416$ approximately), one complete revolution considered as an angle measured in radians is found by formula (1) to be

$$\theta \text{ (1 revolution)} = \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

Similarly, one complete revolution as an angle measured in degrees is given by

$$\theta \text{ (1 revolution)} = 360^\circ.$$

Hence

$$2\pi \text{ radians} = 360^\circ.$$

From this equation

$$(2) \quad 1 \text{ radian} = \frac{180^\circ}{\pi} = 57.2958 \dots \text{ degrees.}$$

$$(3) \quad 1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174533 \dots \text{ radians.}$$

These expressions may be used for converting an angle from one system of measurement to the other, as stated in the following rules.

1. To convert degrees to radians, divide the number of degrees by $\frac{180}{\pi}$, or multiply by $\frac{\pi}{180}$.

2. To convert radians to degrees, multiply the number of radians by $\frac{180}{\pi}$.

Note that the approximations $\frac{180}{\pi} = 57.2958$ and $\frac{\pi}{180} = 0.0174533$ may be used, the desired accuracy determining the number of decimal places.

Example 3. Convert to radians: (a) 60° (b) 76.24° .

Using Rule 1 in both cases,

$$(a) \quad 60^\circ \div \frac{180^\circ}{\pi} = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

$$(b) \quad 76.24^\circ \div 57.30^\circ = 1.330 \text{ radians.}$$

Example 4. Convert to degrees: (a) $\frac{\pi}{2}$ radians (b) 5.716 radians.

Using Rule 2 in both cases,

$$(a) \frac{\pi}{2} \times \frac{180^\circ}{\pi} = 90^\circ.$$

$$(b) 5.716 \times 57.3^\circ = 327.5^\circ.$$

Occasionally it may be desirable to maintain a certain accuracy of measurement in converting from one system of angle measurements to the other. From equality (2) of this section, 0.01 radian is approximately 0.5° . Hence, as far as relative accuracy is concerned, it can be shown that:

If an angle is measured in de- grees to	the nearest degree	1 decimal	2 decimals	etc.
Then the radian measure of the angle is accurate to	2 decimals	3 decimals	4 decimals	etc.

From the discussions of Chapter 1, it follows that computations involving radians are accurate to no more than the number of significant digits used in the approximation $\frac{180}{\pi} = 57.2958\dots$.

Example 5. Convert 76.24° to radians as accurately as is justifiable.

Since the angle is given in degrees to two decimal places, the result may be given in radians accurate to four places. Hence using 57.2958, we obtain by Rule 1,

$$\frac{76.24}{57.2958} = 1.33064 = 1.3306 \text{ radians.}$$

Example 6. Convert 5.716 radians to degrees as accurately as the given data in radians permit.

Since the angle is given in radians to three decimal places, the transformation to degrees is accurate, according to the table, to one decimal place. Thus,

$$5.716 \times 57.2958^\circ = 327.5027928^\circ = 327.5^\circ.$$

Converting angles from degrees to radians or from radians to degrees is greatly simplified by the use of *conversion tables* which are printed in many logarithmic and trigonometric tables. Table 6 in the Appendix is such a table.

EXERCISES

Convert the following angles to radians, assuming the given numbers to be exact. Give the results in terms of π .

- | | | |
|-----------------------------|-----------------------------|---------------------------------------|
| 1. $30^\circ, 60^\circ$. | 2. $45^\circ, 135^\circ$. | 3. $90^\circ, 180^\circ, 270^\circ$. |
| 4. $360^\circ, 720^\circ$. | 5. $150^\circ, 330^\circ$. | 6. $300^\circ, 210^\circ$. |
| 7. $50^\circ, 140^\circ$. | 8. 450° . | 9. 480° . |
| 10. 1260° . | 11. -120° . | 12. -215° . |
| 13. -170° . | 14. $5^\circ, -5^\circ$. | 15. $99^\circ, -99^\circ$. |

Convert from radians to degrees; give answers to the nearest hundredth of a degree.

- | | | |
|--------------------------------------|--|--|
| 16. $\frac{\pi}{6}, \frac{\pi}{3}$. | 17. $\frac{\pi}{4}, \frac{3\pi}{4}$. | 18. $\frac{3\pi}{2}, \frac{\pi}{2}$. |
| 19. $6\pi, 3\pi$. | 20. $\frac{5\pi}{4}, \frac{7\pi}{4}$. | 21. $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. |
| 22. $\frac{25\pi}{3}$. | 23. $\frac{\pi}{18}$. | 24. $\frac{11\pi}{6}$. |
| 25. $\frac{231\pi}{72}$. | 26. $\frac{-\pi}{9}$. | 27. $\frac{-7\pi}{6}$. |
| 28. $\frac{2\pi}{360}$. | 29. $-5\pi, 5\pi$. | 30. $\frac{-18\pi}{40}, \frac{18\pi}{40}$. |

Express as radians, using $\frac{180}{\pi} = 57.3$; give results to two decimal places.

- | | | |
|---------------------|---------------------|----------------------|
| 31. 62° . | 32. 127° . | 33. 435° . |
| 34. -213° . | 35. 18° . | 36. 2222.2° . |
| 37. 32.8° . | 38. 67.9° . | 39. -3.7° . |
| 40. 3.12° . | 41. 84.70° . | 42. 167.8° . |
| 43. 28.65° . | 44. 0.67° . | 45. 0.78° . |

Change from radians to degrees, using $\frac{180}{\pi} = 57.3$; carry out the work to one decimal place.

- | | | |
|-------------|-------------|---------------|
| 46. 10.21. | 47. 6.33. | 48. 172.05. |
| 49. 5.1. | 50. 3. | 51. -0.01 . |
| 52. 0.20. | 53. 1.259. | 54. 0.07. |
| 55. 0.071. | 56. 1.0001. | 57. 0.009. |
| 58. 6.2832. | 59. 100. | 60. 1000.12. |

Change from degrees to radians, or vice versa; give answers to the number of decimal places permitted by the data. Choose the proper approximation of $\frac{180}{\pi}$ for the desired accuracy. Check your results by using Table 6 in the Appendix.

- | | | |
|----------------------|----------------------|-----------------------|
| 61. 88.7° . | 62. 88.70° . | 63. 88.700° . |
| 64. 47° . | 65. 47.01° . | 66. 0.40° . |
| 67. 72.5° . | 68. 1651.0° . | 69. 10.210 radians. |
| 70. 10.2100 radians. | 71. 102.10 radians. | 72. 6.23 radians. |
| 73. 41.4 radians. | 74. 57.2958 radians. | 75. 0.1619267 radian. |

Solve the following problems, using $\frac{180}{\pi} = 57.3$ or the conversion table; give answers to one decimal place.

76. In a circle having a radius of 6 in., what is the length of the arc intercepted by an angle of 7.26 radians?

77. If a central angle intercepts an arc of 14 in. on the circumference of a circle with a 5-in. diameter, what is the size of the angle in radians? in degrees?

78. A meter has a scale 7.8 in. long which forms an arc of a circle with a 3-in. radius. Through how many degrees must a needle pivoted at the center of the circle be free to move in order to cover all parts of the scale?

79. Ten amperes of current cause an ammeter needle to deflect 38° from the position it takes when no current flows. If the needle is 4.5 in. long and a circular scale is to be placed at the tip of the needle, how long must the scale be in order to read a maximum current of 10 amperes?

80. A truck trailer has front and rear wheels of 3 and 2 ft. in diameter respectively. In traveling 100 ft. how many complete revolutions does each wheel make? What angle in radians does a spoke of the front wheel generate in that distance? a spoke of the rear wheel? If the distance of 100 ft. is covered in 5 seconds, what is the angular velocity of each wheel?

81. Using the diameter of the earth as 7930.0 miles, find the distance from New York to the equator if the latitude of New York is 40.72° north.

82. Find the distance from Moscow, 55.75° north latitude, to the equator; to the north geographic pole.

83. Find the distance from Paris, 48.83° north latitude, to the south geographic pole.

84. How far apart are two points on the same meridian of longitude, one at 37.68° north latitude, the other at 62.71° south latitude?

85. Find the velocity in miles per hour and feet per second of a point on the equator due to the revolution of the earth about its axis (the earth revolves about its axis once in 24 hours). Find the angular velocity of a line from the center of the earth to the equator. (In this problem give an answer to two significant figures.)

86. The angular velocity of a revolving wheel is $\frac{7}{3}$ radian per second. Find the number of revolutions per minute. If the radius of the wheel is 18.7 in., what is the velocity in feet per minute of a point on the circumference?

87. If the maximum cutting speed of a lathe is to be 450 ft. per minute, find the maximum diameter of a cylindrical piece that may be cut at 200 revolutions per minute.

88. Two pulleys of diameters 8.2 and 10.4 in. are belted together. If the first pulley is driven with an angular velocity of 100 radians per second, find the linear velocity of the belt and the angular velocity of the second pulley.

89. If M is the speed of a locomotive in miles per hour and d is the diameter of the drive wheels in inches, find a formula for expressing R , the revolutions per second, in terms of M and d . Express A , the angular velocity in radians per second, in terms of M and d . Find R and A for a locomotive with drive wheels 70 in. in diameter, traveling at 70 m.p.h.

90. Prove that $\frac{1}{2}r^2\theta$ is the formula for the area of a sector of a circle of radius r if the sector is formed by a central angle θ measured in radians. (Use the theorem that the area of a sector is half the product of the length of its arc and the radius of the circle.)

Use the formula developed in Exercise 90 for Exercises 91–93.

91. In a circle of radius 5 in. find the area of a sector formed by an angle of $\frac{\pi}{2}$ radians. An angle of $\frac{3\pi}{7}$ radians. An angle of 162° . An angle of 43.7° .

92. A beam of light from a flashlight sweeps an angle of 18° . If the range of the light is 100 ft., how large an area is illuminated by the flashlight?

93. A sector whose central angle is 217° is cut from a circular piece of aluminum of diameter 10 in. The radii bounding the sector are then brought together. The sector then forms a cone. What is the radius of the base of this cone?

In the following problems, maintain as much accuracy as is possible according to the data, using the conversion table or the proper approximation of $\frac{180}{\pi}$ in each problem. When giving results, give also the number of significant figures or correct decimal places.

94. A generator rotor is found to make 62.74 revolutions per second. Find the angular velocity in radians per second. Through how many degrees does the rotor turn in 1 second? What is the period, that is, the number of seconds required for one revolution?

95. A certain type of camera shutter consists of a circular disc with a sector cut out, which revolves in front of or behind the lens and allows light to pass through only when the open part of the disc is passing by the lens. A manufacturer wishes to build a movie camera which will make sixteen exposures a second, each exposure lasting 0.03 second. What will be the size in radians of the angle of the sector cut from the disc? in degrees?

96. If the diameter of the earth is 7930 miles (measured to the nearest 10 miles), find the distance between two points on the equator whose longitude differs by 1.000° ; by 1° measured to the nearest second.

97. A flywheel 14.83 in. in diameter is revolving at an angular velocity of 626.30 radians per second. Find the velocity of a point on the circumference.

4-4. The Definitions of the Trigonometric Functions. The concepts of the Cartesian coordinate system, of the function, and of the angle have been discussed as separate topics earlier in this text. Combining these three ideas in a convenient manner, we now introduce the **trigonometric functions** upon which the whole subject of trigonometry is based.

Any angle may be superimposed on a rectangular coordinate system with its vertex at the origin and its initial side lying along the positive x -axis. An angle so located is said to be in **standard position**. Angles in standard position, with certain exceptions, maybe classified according to the quadrants in which their terminal sides lie. For example, a positive angle of 205° is said to lie in the third quadrant, for in standard position its terminal side lies in the third quadrant (Fig. 4-5). We cannot classify in this way angles such as 0° , 90° , 180° , whose terminal sides lie upon one of the coordinate axes. These angles are called **quadrantal angles**.

Now consider any angle θ . This angle, in standard position, might lie in any one of the four quadrants (Fig. 4-6). Regardless of the

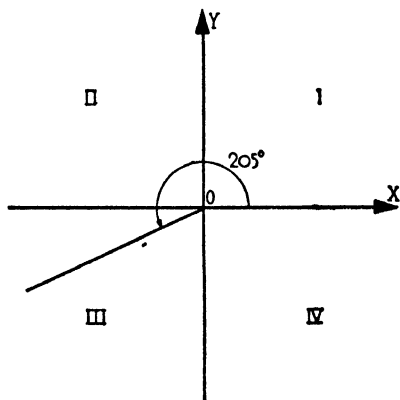


FIG. 4-5.

location of the angle, we select P , *any* point on the terminal side of the angle, and denote the coordinates of P by (x, y) . From P draw a line perpendicular to the x -axis. In each case this gives us what we call a **right triangle of reference** associated with θ . This triangle does not necessarily include the angle θ with which it is associated. The hypotenuse (denoted by r) of this right triangle of reference is the *distance* of P from

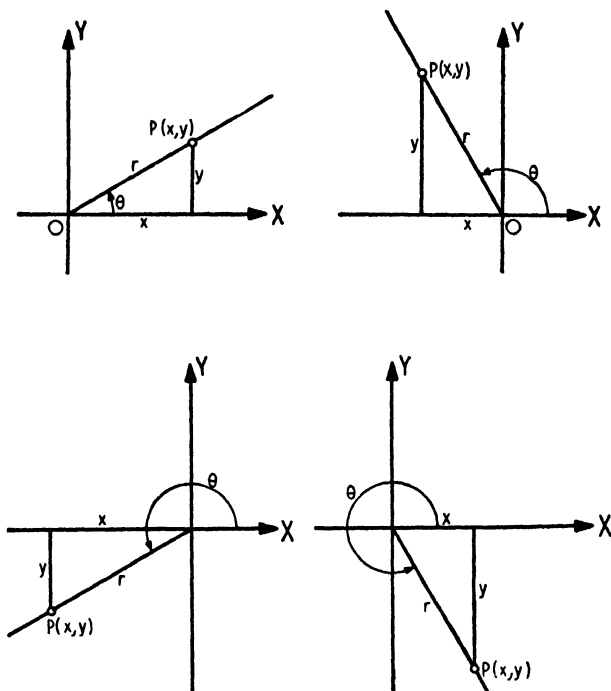


FIG. 4-6.

the origin, and this distance is always considered *positive*. The other two sides of the triangle are in every case the coordinates of the point P , and these sides are positive or negative according as x and y are positive or negative. It should be noted that for the quadrantal angles, a point P can still be chosen, but the right triangle of reference will degenerate into a triangle which has one side of length zero and the other two sides equal in length but possibly opposite in sign.

From the three quantities or lengths, x , y , and r which we have introduced, six different ratios may be defined. Regardless of the quadrant in which the angle lies, and even if it is a quadrantal angle, these definitions are the same.

Given any angle θ in standard position, we define:

$$\text{sine } \theta = \frac{y}{r},$$

$$\text{cosecant } \theta = \frac{r}{y},$$

$$\text{cosine } \theta = \frac{x}{r},$$

$$\text{secant } \theta = \frac{r}{x},$$

$$\text{tangent } \theta = \frac{y}{x},$$

$$\text{cotangent } \theta = \frac{x}{y},$$

where x , y , and r are the sides of any right triangle of reference associated with θ . These six names are usually abbreviated to $\sin \theta$, $\cos \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$. These are the trigonometric functions of θ .

For any given angle θ the values of the trigonometric functions are the same regardless of the position of P on the terminal side of the reference triangle, for different choices of P result in similar triangles of reference, and from plane geometry, the ratios of corresponding sides of similar triangles are equal.

For example, let θ be an angle in the second quadrant (Fig. 4-7). Choose any two points, P_1 and P_2 , on the terminal side of θ and form

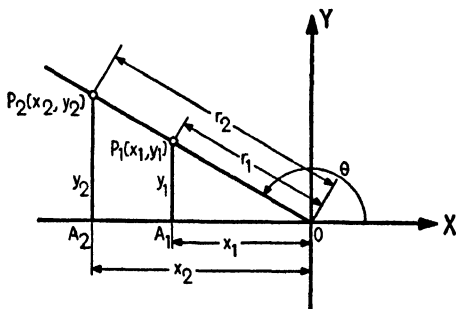


FIG. 4-7.

the right triangles of reference, OP_1A_1 and OP_2A_2 , associated with these points. Since these triangles are similar

$$(1) \quad \frac{y_1}{r_1} = \frac{y_2}{r_2}, \quad \frac{x_1}{r_1} = \frac{x_2}{r_2}, \quad \text{etc.}$$

Now,

$$\sin \theta \text{ (determined from } \triangle OP_1A_1) = \frac{y_1}{r_1},$$

and

$$\sin \theta \text{ (determined from } \triangle OP_2A_2) = \frac{y_2}{r_2}.$$

Therefore, substituting in relation (1),

$$\sin \theta \text{ (from } \triangle OP_1A_1) = \sin \theta \text{ (from } \triangle OP_2A_2).$$

In like manner for the other functions,

$$\cos \theta \text{ (from } \triangle OP_1A_1) = \frac{x_1}{r_1} = \frac{x_2}{r_2} = \cos \theta \text{ (from } \triangle OP_2A_2),$$

$$\tan \theta \text{ (from } \triangle OP_1A_1) = \frac{y_1}{x_1} = \frac{y_2}{x_2} = \tan \theta \text{ (from } \triangle OP_2A_2), \text{ etc.}$$

Thus, for a given angle θ , the value of the trigonometric function is independent of the triangle of reference which is used. However, for different angles, it is easy to see that, except for certain special cases which will be discussed later, the triangles of reference are not similar and hence the trigonometric ratios are not the same. Since for every value of the angle θ there corresponds one value for each of the trigonometric ratios, these ratios are called the **trigonometric functions of the angle**.

4-5. Signs of the Trigonometric Functions in the Various Quadrants.

On the basis of the definitions of the trigonometric functions it is easy to determine whether a given function of an angle is positive or negative.

For example, if θ is any angle in the second quadrant, $\cos \theta = \frac{x}{r}$ is a

negative quantity because r is always positive and x , in this case the abscissa of a point in the second quadrant, is negative. In like manner the reader may verify the information summarized in the following table.

	QUADRANT I	QUADRANT II	QUADRANT III	QUADRANT IV
$\sin \theta = \frac{y}{r}$	+	+	-	-
$\cos \theta = \frac{x}{r}$	+	-	-	+
$\tan \theta = \frac{y}{x}$	+	-	+	-
$\cot \theta = \frac{x}{y}$	+	-	+	-
$\sec \theta = \frac{r}{x}$	+	-	-	+
$\csc \theta = \frac{r}{y}$	+	+	-	-

4-6. Evaluation of the Trigonometric Functions When the Angle Is Known. Method 1. Construction. The six trigonometric functions of a given angle may be found approximately by the accurate construction

of a right triangle of reference. Suppose it is desired to find one of or all the functions of 72° . Constructing the angle by means of a protractor and dropping a perpendicular to the x -axis from any point on the terminal side, we have a right triangle of reference (Fig. 4-8). Then after measuring the sides of this triangle and obtaining $x = 1.25$ in., $y = 3.8$ in., $r = 4.0$ in., we can evaluate the ratios:

$$\sin 72^\circ = \frac{3.8 \text{ in.}}{4.0 \text{ in.}} = 0.95,$$

$$\cos 72^\circ = \frac{1.25 \text{ in.}}{4.0 \text{ in.}} = 0.31,$$

$$\tan 72^\circ = \frac{3.8 \text{ in.}}{1.25 \text{ in.}} = 3.0, \text{ etc.}$$

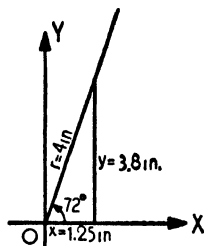


FIG. 4-8.

Since the lengths measured vary with the accuracy of construction and the type of scale used, the values found by this method are very rough approximations.

The above example furnishes an illustration of the fact that the trigonometric functions are pure numbers, that is, they are independent of the unit of measurement. No matter what unit of measurement is used the results are the same.

Method 2. Tables. By methods which are beyond the scope of this book, the trigonometric functions for any angle may be found to any desired degree of accuracy. These results are usually compiled in the form of tables. The use of these tables is discussed later in the chapter.

EXERCISES

By construction, find the six trigonometric functions of the following angles. Carry out all work to one decimal place.

- | | | |
|--------------------|------------------|------------------|
| 1. 30° . | 2. 45° . | 3. 65° . |
| 4. 123° . | 5. 217° . | 6. 345° . |
| 7. -15° . | 8. 402° . | 9. 42° . |
| 10. -210° . | | |

4-7. Evaluation of the Trigonometric Functions When One Function Is Known. This process can best be shown by the use of examples.

Example 1. Given $\sin \theta = \frac{3}{5}$, where θ is an angle in the first quadrant, find the other trigonometric functions of θ .

Since $\sin \theta = \frac{y}{r}$, then $\frac{y}{r} = \frac{3}{5}$ for the particular angle in which we are interested.

Then for some particular triangle of reference, the lengths of two sides will be $y = 3$,

$r = 5$ (Fig. 4-9). By the Pythagorean theorem, which states that the square of the hypotenuse of a right triangle equals the sum of the squares of the remaining two sides, we may find the third side x , for

$$3^2 + x^2 = 5^2,$$

$$x^2 = 25 - 9,$$

whence

$$x = \pm\sqrt{16},$$

or

$$x = \pm 4.$$

Since θ is in the first quadrant, we select the value $x = +4$. Now, knowing the three sides of a triangle of reference, we can find the other five trigonometric ratios: $\cos \theta = \frac{4}{5}$, $\tan \theta = \frac{3}{4}$, $\cot \theta = \frac{4}{3}$, $\sec \theta = \frac{5}{4}$, $\csc \theta = \frac{5}{3}$.

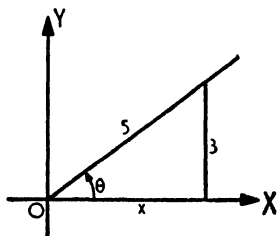


FIG. 4-9.

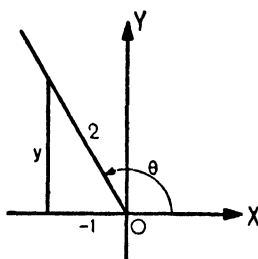


FIG. 4-10.

In this process it is not necessary to know the exact size of the angle θ , nor is it necessary that the right triangle of reference be very accurately drawn, for the length of the third side is found algebraically rather than geometrically.

Example 2. Given $\sec \theta = -2$, and $\sin \theta$ positive, find the other trigonometric functions of θ .

The angle θ must be in the second quadrant, for that is the only quadrant in which the secant is negative and the sine positive. Since $\sec \theta = \frac{r}{x}$, then $\frac{r}{x} = \frac{2}{-1}$, for r is always positive. Thus we take θ in the second quadrant, and for a triangle of reference we select one with $r = 2$ and $x = -1$ (Fig. 4-10). By the Pythagorean theorem:

$$2^2 = y^2 + (-1)^2,$$

or

$$y^2 = 4 - 1,$$

whence

$$y = \pm\sqrt{3}.$$

The positive root must be selected since y is positive in the second quadrant. Therefore the other five functions are: $\sin \theta = \frac{\sqrt{3}}{2}$, $\cos \theta = -\frac{1}{2}$, $\tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$, $\cot \theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$, $\csc \theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Example 3. Given $\cot \theta = -\frac{1}{3}$, where θ is between 0° and 360° , find the other trigonometric functions of θ .

Since the cotangent is negative in the second and fourth quadrants, there are two possible angles θ . Since $\cot \theta = \frac{x}{y}$, then in the second quadrant $\frac{x}{y} = \frac{-1}{3}$ and $x = -1$, $y = 3$, making $r = \sqrt{(3)^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$; whereas in the fourth quadrant $\frac{x}{y} = \frac{1}{-3}$ and therefore $x = 1$, $y = -3$, making $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$. Thus,

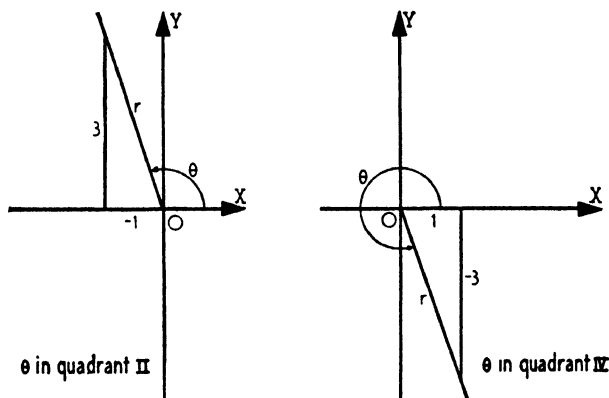


FIG. 4-11.

FOR θ IN THE SECOND QUADRANT

$$\sin \theta = \frac{3}{\sqrt{10}} \text{ or } \frac{3\sqrt{10}}{10}$$

$$\cos \theta = \frac{-1}{\sqrt{10}} \text{ or } -\frac{\sqrt{10}}{10}$$

$$\tan \theta = \frac{3}{-1} \text{ or } -3$$

$$\sec \theta = \frac{\sqrt{10}}{-1} \text{ or } -\sqrt{10}$$

$$\csc \theta = \frac{\sqrt{10}}{3}$$

FOR θ IN THE FOURTH QUADRANT

$$\sin \theta = \frac{-3}{\sqrt{10}} \text{ or } \frac{-3\sqrt{10}}{10}$$

$$\cos \theta = \frac{1}{\sqrt{10}} \text{ or } \frac{\sqrt{10}}{10}$$

$$\tan \theta = \frac{-3}{1} \text{ or } -3$$

$$\sec \theta = \frac{\sqrt{10}}{1} \text{ or } \sqrt{10}$$

$$\csc \theta = \frac{\sqrt{10}}{-3} \text{ or } -\frac{\sqrt{10}}{3}$$

The work of this section further illustrates that the trigonometric functions are numbers. Hence we may treat them as numbers. That is, the trigonometric functions may be added, subtracted, multiplied, divided, etc., as numbers. For example, $(\sin \theta) \cdot (\sin \theta) = (\sin \theta)^2$, which is usually written $\sin^2 \theta$. If $\sin \theta = \frac{3}{4}$, then $\sin^2 \theta = (\frac{3}{4})^2 = \frac{9}{16}$.

Example 4. Using the results of Example 2, evaluate $\frac{\sin \theta \cos \theta}{\tan^2 \theta} - 1$:

$$\begin{aligned}\frac{\sin \theta \cos \theta}{\tan^2 \theta} - 1 &= \frac{\frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right)}{(-\sqrt{3})^2} - 1 = \frac{-\frac{\sqrt{3}}{4}}{3} - 1 \\ &= -\frac{\sqrt{3}}{12} - 1 = -\left(\frac{\sqrt{3} + 12}{12}\right).\end{aligned}$$

EXERCISES

Find the six trigonometric functions of θ ; θ is in the standard position and its terminal side passes through the given point.

- | | | |
|--------------|--------------|----------------|
| 1. (3, 4). | 2. (6, 1). | 3. (-1, 2). |
| 4. (-2, -2). | 5. (17, 8). | 6. (1, -2). |
| 7. (3, 1). | 8. (-1, -3). | 9. (2, -2). |
| 10. (5, 2). | 11. (2, 0). | 12. (3, 0). |
| 13. (0, -6). | 14. (-2, 0). | 15. (-75, 40). |

List the six trigonometric functions of an angle θ if:

16. $\sin \theta = \frac{4}{5}$ and θ lies in the first quadrant.
17. $\cos \theta = -\frac{1}{2}$ and θ lies in the third quadrant.
18. $\sec \theta = -\frac{1}{\frac{3}{2}}$ and θ lies in the second quadrant.
19. $\tan \theta = \frac{5}{13}$ and θ lies in the first quadrant.
20. $\csc \theta = -\frac{1}{\frac{7}{8}}$ and θ lies in the fourth quadrant.
21. $\cot \theta = \frac{3}{4}$, $\sin \theta$ positive.
22. $\csc \theta = \frac{3}{2}$, $\cos \theta$ negative.
23. $\sin \theta = \frac{8}{15}$, $\cos \theta$ negative.
24. $\tan \theta = 2$, $\cos \theta$ negative.
25. $\csc \theta = \frac{1}{\frac{1}{3}}$, $\tan \theta$ negative.
26. $\sec \theta = -2$, $\tan \theta$ positive.
27. $\sin \theta = \frac{3}{4}$, $\sec \theta$ negative.
28. $\cos \theta = \frac{2}{3}$, $\sin \theta$ positive.

List the six trigonometric functions for each angle θ greater than or equal to 0° and less than 360° if:

- | | | |
|---|---|-------------------------------------|
| 29. $\csc \theta = \frac{1}{\frac{3}{5}}$. | 30. $\tan \theta = -\frac{4}{3}$. | 31. $\csc \theta = \pm 3$. |
| 32. $\sec \theta = \pm \frac{1}{\frac{7}{5}}$. | 33. $\cos \theta = \frac{5}{12}$. | 34. $\cos \theta = -\frac{5}{12}$. |
| 35. $\cot \theta = \pm \frac{1}{\frac{7}{9}}$. | 36. $\tan \theta = 0.3$. | 37. $\sin \theta = -0.5$. |
| 38. $\sec \theta = 4$. | 39. $\cot \theta = 5$. | 40. $\tan \theta = -1.2$. |
| 41. $\tan \theta = -1$. | 42. $\sec \theta = 0$. | 43. $\sin \theta = -1$. |
| 44. $\cos \theta = -\frac{\sqrt{3}}{2}$. | 45. $\csc \theta = \frac{5}{3}$. | 46. $\sin \theta = \frac{m}{n}$. |
| 47. $\cot \theta = m$. | 48. $\tan \theta = -\frac{\sqrt{1-a^2}}{a}$. | 49. $\cos \theta = -\frac{a}{2}$. |
| 50. $\sin \theta = 1.0$. | 51. $\sec \theta = -\frac{3}{2}$. | 52. $\tan \theta = 7500$. |
| 53. $\sin \theta = -0.2354$. | 54. $\cot \theta = 1.0009$. | 55. $\csc \theta = -5$. |
| 56. $\csc \theta = -1.124$. | | |

Compute the following expressions, assuming θ is an angle in the first or second quadrant.

57. $\frac{\sin \theta + \cos \theta}{1 + \cot \theta}$ if $\tan \theta = -\frac{4}{3}$.
58. $\frac{\sin^2 \theta + \cos^2 \theta}{1 - \tan^2 \theta}$ if $\sec \theta = -\frac{11}{4}$.
59. $\tan \theta - \sin \theta \sec \theta$ if $\sec \theta = -\frac{1}{5}$.
60. $\frac{(1 + \tan^2 \theta)(\cos^2 \theta)}{\tan^2 \theta(1 + \cot^2 \theta)}$ if $\cos \theta = \frac{3}{7}$.
61. $\frac{\sin \theta \cos \theta}{1 - \tan \theta} + 2 \cot \theta$ where $\sec \theta = \frac{18}{7}$.
62. $\frac{\sin \theta \sec \theta \cos \theta}{3 - \tan^2 \theta + \sec^2 \theta}$ where $\cot \theta = \frac{1}{4}$.
63. $\frac{\sin \theta \sec \theta}{\tan \theta \cot \theta} + \frac{\cos \theta}{1 - \sin^2 \theta}$ where $\sec \theta = -\frac{13}{9}$.
64. $\frac{1 + \sin \theta + \cos \theta}{1 + \sin \theta - \cos \theta} \cdot \sin \theta$ where $\cos \theta = \frac{\sqrt{3}}{3}$.

Use θ as an angle in the third or fourth quadrant.

65. $\frac{7}{8} \sin \theta - \cos^2 \theta$ where $\tan \theta = \frac{3}{4}$.
66. $\frac{I_1 \sin \theta + I_1 \cos \theta}{\frac{5}{13} \tan \theta}$ where $\cot \theta = \frac{5}{12}$.
67. $\frac{2R_1 \tan \theta \cos \theta}{9} + 4R_1 \csc \theta \cos \theta$ where $\sin \theta = -\frac{6}{7}$.
68. $\frac{4}{\sin^2 \theta} + \frac{9}{\cos^2 \theta} - \frac{3}{\tan^2 \theta}$ where $\cos \theta = \frac{5}{6}$.
69. $\frac{1 + \sin \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{1 + \sin \theta}$ where $\sec \theta = -5$.
70. $\frac{\tan \theta - \sin \theta}{\sin^3 \theta} - \frac{\sec \theta}{1 + \cos \theta}$ where $\cot \theta = -1.7$.
71. $\frac{3.716E_1 \cot^2 \theta}{(1 - \cos^2 \theta) \sin \theta} + E_1$ where $\cos \theta = 0$.
72. $\frac{E_1 + E_2 + 5 \sin \theta}{7 \cos^2 \theta}$ where $\cot \theta = 0$.

4-8. Determination of the Trigonometric Functions of Special Angles.

There are certain angles whose trigonometric functions can be found very easily with the aid of a few theorems from plane geometry.

The Functions of 30° . Locating an angle of 30° in standard position on the coordinate axes and forming a triangle of reference, we find that we have a 30° , 60° right triangle. In plane geometry, it is shown that a 30° , 60° right triangle has a hypotenuse which is twice as long as the

side opposite the 30° angle. Since the triangle of reference is used only in forming ratios, we may conveniently assume that $r = 2$, $y = 1$. Then,

$$x = \pm\sqrt{r^2 - y^2} = \pm\sqrt{4 - 1} = \pm\sqrt{3}.$$

Thus the three sides of the triangle of reference are $r = 2$, $y = 1$, $x = \sqrt{3}$, and the six trigonometric functions are:

$$\sin 30^\circ = \frac{1}{2},$$

$$\csc 30^\circ = \frac{2}{1} \text{ or } 2;$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2},$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3};$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}, \quad \cot 30^\circ = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}.$$

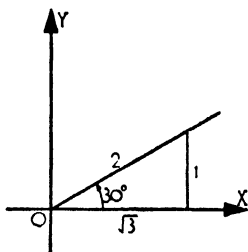


FIG. 4-12.

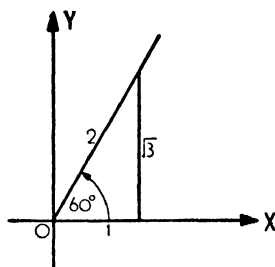


FIG. 4-13.

The Functions of 60° . Locating an angle of 60° in standard position on the coordinate axes and forming a triangle of reference, we have a 30° , 60° right triangle. However, now the side x is opposite the 30° angle, and hence it may be assumed that $r = 2$, $x = 1$. From the Pythagorean theorem we obtain $y = \sqrt{3}$. The six trigonometric functions are:

$$\sin 60^\circ = \frac{\sqrt{3}}{2},$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3};$$

$$\cos 60^\circ = \frac{1}{2},$$

$$\sec 60^\circ = \frac{2}{1} \text{ or } 2;$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}, \quad \cot 60^\circ = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

The Functions of 120° , etc. Any angle in standard position whose terminal side forms an angle of 30° or 60° with the x -axis may be treated in a similar fashion. For example, 120° , in standard position, forms an angle of 60° with the negative x -axis. Hence, the triangle of reference is a 30° , 60° right triangle. Since the side x in this triangle is opposite

the 30° angle, we may take $r = 2$, $x = -1$, x being negative because the angle lies in the second quadrant. From the Pythagorean theorem $y = \sqrt{3}$, making

$$\sin 120^\circ = \frac{\sqrt{3}}{2}, \quad \csc 120^\circ = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3};$$

$$\cos 120^\circ = \frac{-1}{2}, \text{ or } -\frac{1}{2} \quad \sec 120^\circ = \frac{2}{-1} \text{ or } -2;$$

$$\tan 120^\circ = \frac{\sqrt{3}}{-1} \text{ or } -\sqrt{3}, \quad \cot 120^\circ = \frac{-1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}.$$

It is easily seen that we can treat in this way any other angle which gives rise to a 30° , 60° right triangle of reference.

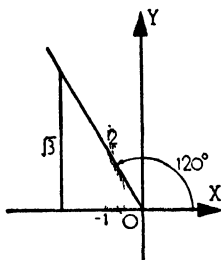


FIG. 4-14.

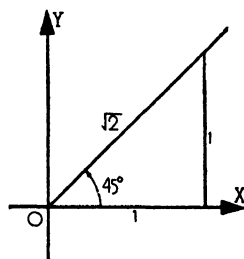


FIG. 4-15.

The Functions of 45° . Locate an angle of 45° in standard position and form a triangle of reference. The acute angles of this triangle of reference are each 45° in size, for the acute angles of right triangles are complementary. From plane geometry, then, the sides opposite these equal angles are of equal length. Thus for convenience, we may select $x = 1$, $y = 1$, $r = \sqrt{2}$. The functions are:

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}, \quad \csc 45^\circ = \frac{\sqrt{2}}{1} \text{ or } \sqrt{2};$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}, \quad \sec 45^\circ = \frac{\sqrt{2}}{1} \text{ or } \sqrt{2};$$

$$\tan 45^\circ = \frac{1}{1} \text{ or } 1, \quad \cot 45^\circ = \frac{1}{1} \text{ or } 1.$$

It is easily seen that the functions of any other angle whose terminal side forms an angle of 45° with the x -axis may be evaluated in this way.

The Functions of 0° . When an angle of 0° is placed in standard position, the terminal side of the angle coincides with the positive x -axis

(Fig. 4-16). Thus any triangle of reference chosen has its side $y = 0$, since the perpendicular distance from any point P on the terminal side to the x -axis is zero. Such a triangle is often called a **degenerate triangle**. The other two sides of this degenerate triangle are of equal length. For convenience we may select them as $r = 1$, $x = 1$. Thus

$$\sin 0^\circ = \frac{0}{1} \text{ or } 0, \quad \csc 0^\circ = \frac{1}{0} \text{ (which is undefined);}$$

$$\cos 0^\circ = \frac{1}{1} \text{ or } 1, \quad \sec 0^\circ = \frac{1}{1} \text{ or } 1;$$

$$\tan 0^\circ = \frac{0}{1} \text{ or } 0, \quad \cot 0^\circ = \frac{1}{0} \text{ (which is undefined).}$$

The definition of $\csc 0^\circ$ and $\cot 0^\circ$ will be discussed further in Chapter 5.

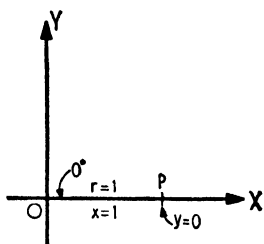


FIG. 4-16.

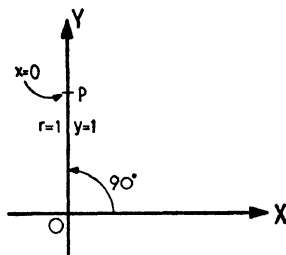


FIG. 4-17.

The Functions of 90° . When an angle of 90° is placed in standard position, the terminal side of the angle coincides with the positive y -axis. Any triangle of reference is degenerate, with the side $x = 0$. We may conveniently assume that the other two sides, being equal, are both of length 1, that is, $y = 1$, $r = 1$ (Fig. 4-17). Thus:

$$\sin 90^\circ = \frac{1}{1} \text{ or } 1,$$

$$\csc 90^\circ = \frac{1}{1} \text{ or } 1;$$

$$\cos 90^\circ = \frac{0}{1} \text{ or } 0,$$

$$\sec 90^\circ = \frac{1}{0} \text{ (which is undefined);}$$

$$\tan 90^\circ = \frac{1}{0} \text{ (which is undefined),} \quad \cot 90^\circ = \frac{0}{1} \text{ or } 0.$$

The definition of $\sec 90^\circ$ and $\tan 90^\circ$ will be discussed further in Chapter 5. It is evident that the trigonometric functions of any other quadrantal angle may be discussed and evaluated in the same manner.

EXERCISES

In the exercises of this section, assume 30° , 90° , etc., to mean exactly 30° , 90° , etc. Find the six trigonometric functions of each of the following angles.

1. 150° .

2. 300° .

3. 120° .

4. 240° .

5. 600° .

6. -210° .

7. -60° .

8. 135° .

9. 225° .

10. 405° .

11. -45° .

12. 765° .

- | | | | |
|--------------------------------|---------------------------------|-------------------------------|-------------------------------|
| 13. 180° . | 14. 270° . | 15. 360° . | 16. 1440° . |
| 17. -270° . | 18. 330° . | 19. -30° . | 20. $\frac{7\pi}{6}$ radians. |
| 21. $\frac{4\pi}{3}$ radians. | 22. $\frac{10\pi}{12}$ radians. | 23. $\frac{\pi}{4}$ radians. | 24. $\frac{5\pi}{4}$ radians. |
| 25. $-\frac{3\pi}{4}$ radians. | 26. $\frac{3\pi}{2}$ radians. | 27. 3π radians. | 28. π radians. |
| 29. $-\pi$ radians. | 30. $\frac{11\pi}{4}$ radians. | 31. $-\frac{\pi}{3}$ radians. | 32. -4π radians. |

Simplify the following trigonometric expressions as much as possible. For example,

$$33. \frac{Z \sin 30^\circ \cos 30^\circ}{1 - \sin 30^\circ} = \frac{Z \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{Z \cdot \frac{\sqrt{3}}{4}}{\frac{1}{2}} = Z \frac{\sqrt{3}}{2}.$$

$$34. \sin^2 45^\circ + \cos^2 45^\circ.$$

$$35. \sin^2 210^\circ + \cos^2 210^\circ.$$

$$36. \frac{\sec 300^\circ}{\tan 300^\circ} - \frac{2 \csc 45^\circ}{\sin 30^\circ \cos 360^\circ}.$$

$$37. \frac{\sin \frac{\pi}{2}}{2 - \cos^2 \pi}.$$

$$38. \frac{\cot \frac{3\pi}{4} - \cos 0^\circ}{\sec \pi} \cdot \frac{\sin \frac{7\pi}{6}}{\cos \frac{7\pi}{6}}.$$

$$39. 5 \sin \frac{\pi}{6} \left(\cos \frac{3\pi}{2} - 2 \tan \frac{11\pi}{4} \right).$$

$$40. \frac{\frac{1}{2} \csc \frac{7\pi}{4}}{\left(\sqrt{2} - 2 \cos \frac{7\pi}{4} \right)}.$$

$$41. \frac{5 \tan 45^\circ}{\cos 30^\circ \sin 150^\circ} - \frac{3}{2}.$$

$$42. I_1 \sin 270^\circ + I_2 \cos 180^\circ.$$

$$43. \frac{\cot 90^\circ \cot x}{1 + \tan 225^\circ} + 3 \tan 315^\circ.$$

$$44. \frac{\sin \theta \cos 0^\circ}{1 + \csc 90^\circ} + \frac{3 \sin \theta}{2}.$$

$$45. 2Z_1 \sin \frac{\pi}{4} + 3Z_2 \tan \frac{\pi}{6} + 2Z_2 \cos \frac{2\pi}{3}.$$

$$46. \left[\frac{\tan \frac{4\pi}{3} \sin wt}{2 \csc \frac{3\pi}{2}} \right] \div \left[\frac{\csc \frac{\pi}{2} \cos wt}{4} \right].$$

4-9. Functions of Complementary Angles. In trigonometry it is occasionally convenient to speak of the cosine, the cotangent, and the cosecant as the complementary functions or the **co-functions** of the sine, the tangent, and the secant respectively. Conversely, the sine, the tangent, and the secant are called the co-functions of the cosine, the cotangent, and the cosecant. Recalling that two angles are said to be

complementary if their sum is 90° , we shall now prove the theorem that *a trigonometric function of an angle is equal in value to the co-function of its complementary angle*.

We shall prove the theorem for complementary angles which are positive and less than 90° . The proof for other angles is very similar and need not be taken up here.

Let α_1 and α_2 be any two complementary angles which are positive and less than 90° . These angles may be placed in standard position, and equal distances r may be measured on the terminal side of each (Fig. 4-18). The two right triangles of reference formed by using r as

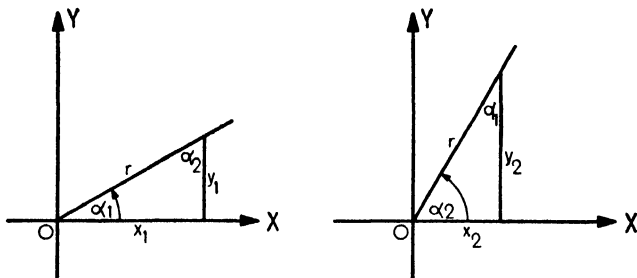


FIG. 4-18.

hypotenuse are congruent, for the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other. Since the corresponding sides of congruent triangles are equal, it follows that $r = r$, $x_1 = y_2$, and $y_1 = x_2$. Hence,

$$\sin \alpha_1 = \frac{y_1}{r} = \frac{x_2}{r} = \cos \alpha_2;$$

$$\cos \alpha_1 = \frac{x_1}{r} = \frac{y_2}{r} = \sin \alpha_2;$$

$$\tan \alpha_1 = \frac{y_1}{x_1} = \frac{x_2}{y_2} = \cot \alpha_2.$$

$$\cot \alpha_1 = \frac{x_1}{y_1} = \frac{y_2}{x_2} = \tan \alpha_2;$$

$$\sec \alpha_1 = \frac{r}{x_1} = \frac{r}{y_2} = \csc \alpha_2;$$

$$\csc \alpha_1 = \frac{r}{y_1} = \frac{r}{x_2} = \sec \alpha_2.$$

and the theorem is proved.

This theorem permits a considerable saving of space in compiling tables of trigonometric functions, for the functions of angles from 45° to 90° may be found by finding the functions of angles from 0° to 45° . For example,

$$\sin 72.3^\circ = \cos (90^\circ - 72.3^\circ) = \cos 17.7^\circ;$$

$$\cos 72.3^\circ = \sin (90^\circ - 72.3^\circ) = \sin 17.7^\circ;$$

$$\tan 72.3^\circ = \cot (90^\circ - 72.3^\circ) = \cot 17.7^\circ.$$

A converse of this theorem is also true in some cases. We shall state this converse theorem without proof.

If two positive, acute angles are known to have a trigonometric function of one angle equal to the trigonometric co-function of the other, then the angles are complementary.

This theorem is sometimes used in solving equations.

Example. Find an angle θ for which

$$\csc (100^\circ - 4\theta) = \sec (2\theta).$$

By the above theorem, if $100^\circ - 4\theta$ and 2θ are acute angles,

$$100^\circ - 4\theta + 2\theta = 90^\circ,$$

$$-2\theta = -10^\circ,$$

$$\theta = 5^\circ.$$

We must check to see if $100^\circ - 4\theta$ and 2θ are acute angles. Substituting

$$\theta = 5^\circ.$$

$$100^\circ - 4\theta = 80^\circ,$$

$$2\theta = 10^\circ,$$

which are both acute and positive. Therefore $\theta = 5^\circ$ is the desired angle.

EXERCISES

Find a function of another positive angle less than or equal to 90° which has the same value as the given function.

1. $\cos 65^\circ$.

2. $\sin 10^\circ$.

3. $\sec 84^\circ$.

4. $\cot 77^\circ$.

5. $\tan 2^\circ$.

6. $\tan 44.1^\circ$.

7. $\sec 72.35^\circ$.

8. $\cos 90^\circ$.

9. $\sin 14.43^\circ$.

10. $\csc 59.8^\circ$.

11. $\csc 0.013^\circ$.

12. $\cot 0^\circ$.

13. $\cos 178^\circ$.

14. $\sin 178^\circ$.

15. $\csc 121^\circ$.

Determine from each of the following equations, if possible, a value of θ which satisfies the equation. If θ makes the angles under consideration larger than 90° , verify the given equation by substitution.

16. $\sin \theta = \cos (\theta + 10^\circ)$.

17. $\sin (\theta + 10^\circ) = \cos \theta$.

18. $\sec (5\theta - 12^\circ) = \csc (2^\circ - 2\theta)$.

19. $\tan 3\theta = \cot 2\theta$.

20. $\tan 3\theta = \cot (-\frac{5}{2}\theta)$.

21. $\csc (5\theta - 5^\circ) = \sec (\theta - 55^\circ)$.

22. $\cos (\theta + 5^\circ) = \sin \theta$.

23. $\sec (3\theta + 18^\circ) = \csc 3\theta$.

24. $\cot 9\theta = \tan 0^\circ$.

25. $\sec \theta = \csc 90^\circ$.

26. $\sin \frac{1^\circ}{\theta} = \cos 89^\circ$.

27. $\cot \frac{40^\circ}{\theta} = \tan 82^\circ$.

28. $\sec \left(\frac{5^\circ}{\theta} + 30^\circ \right) = \csc 40^\circ$.

29. $\tan 89^\circ = \cot \frac{1}{\theta}$.

30. $\sin (\theta + 100^\circ) = \cos (-16\theta + 50^\circ)$.

31. $\sin (\theta + 45^\circ) = \cos (7\theta - 15^\circ)$.

32. $\tan 2\theta = \cot (6\theta - 18^\circ)$.

33. $\cos (2\theta + 31^\circ) = \sin (31^\circ + 2\theta)$.

34. $\cos (2\theta + 31^\circ) = \sin (31^\circ - 2\theta)$.

35. $\cos (\theta + 5^\circ) = \sin (15^\circ - \theta)$.

4-10. Tables of Values of the Trigonometric Functions. We have already discussed several methods for obtaining the trigonometric functions of a given angle. However, all the methods taken up were either inaccurate or restricted to special angles. Consequently, the usual practice is to compute the trigonometric functions for many angles between 0° and 90° by more accurate methods, which are beyond the scope of this book, and to compile these results in the form of a table to which the student may conveniently refer. Table 3 in the Appendix is of this type.

This table consists of angles from 0° to 45° listed by tenths of a degree in the left-hand column, reading downward, and of angles from 45° to 90° listed by tenths of a degree in the right-hand column, reading upward. The angles horizontally opposite each other are complementary. For example, the angle 3° on the left has opposite it the angle 87° . In the four columns between are the sines, the cosines, the tangents, and the cotangents of these two angles. The same numbers have been used for the functions of each of the two complementary angles, for as was shown in a previous section the functions of an angle are equal to the co-functions of its complement. For the angles on the left, the trigonometric headings at the top of the table indicate the column in which each function may be found; the labels at the bottom identify the functions for the angles at the right.

In general, the tables of trigonometric functions are used for one of two purposes:

1. To find the functional value when the angle is known.
2. To find the angle when the functional value is known.

We shall illustrate these procedures separately, by the use of examples.

1. *Finding the functional values of a given angle.*

Example 1. From the table, obtain $\cos 21.7^\circ$.

Locate 21.7° at the left on page 692, and note that the cosine is found in the fourth column. We obtain

$$\cos 21.7^\circ = 0.9291.$$

Example 2. Find $\tan 63.6^\circ$.

Since this angle is greater than 45° , it is found in the right-hand column on page 692, and its tangent, according to the labels at the foot of the page, will be found in the third column, giving

$$\tan 63.6^\circ = 2.014.$$

If the angle is given to the nearest hundredth of a degree, as, for example, in $\sin 20.27^\circ$, the nearest angle to it in the tables, 20.3° , may be used as an approximation. However, if a more accurate approximation is desired, we may use the process of **interpolation**.

Interpolation, in this case, is the process of finding a function of an acute angle not listed in the table. We assume that for small differences in angles, the difference in their trigonometric functions are proportional. This assumption is not absolutely correct, but the subsequent error is negligible except for angles very close to 0° and 90° .

Example 3. By interpolation, evaluate $\sin 41.74^\circ$.

From the table, the angles between which 41.74° lies have the following sines:

$$0.10^\circ \left[\begin{array}{l} 0.04^\circ \left[\begin{array}{l} \sin 41.70^\circ = 0.6652 \\ \sin 41.74^\circ = \\ \sin 41.80^\circ = 0.6665 \end{array} \right] d \end{array} \right] 0.0013$$

From the differences of the angles and the functions as indicated by the brackets on the left and right respectively, we note that as the angle increases 0.10° the sine changes by 0.0013. Therefore, as the angle increases 0.04° , the sine changes by

$$d = \frac{0.04}{0.10} (0.0013) = 0.00052 \text{ units.}$$

Interpolated values are not accurate to more decimal places than values given in the table. Therefore we shorten d to four places, that is, to $d = 0.0005$, and hence

$$\sin 41.74^\circ = 0.6652 + 0.0005 = 0.6657.$$

To save time, the computation of d can be made by omitting the decimal points, as

in $d = \frac{4}{10} \times 13 = 5.2$ or 5 units. Then the whole number 5 so obtained is added to the fourth decimal place of 0.6652.

Example 4. Evaluate $\cos 17.23^\circ$.

From the table

$$10 \left[\begin{array}{l} 3 \left[\begin{array}{l} \cos 17.20^\circ = 0.9553 \\ \cos 17.23^\circ = \\ \cos 17.30^\circ = 0.9548 \end{array} \right] d \end{array} \right] 5$$

Therefore $d = 0.3 \times 5 = 1.5$. The question here arises as to whether to make $d = 1$ or 2 . In all such cases, we shall adopt the convention of rounding off d to the nearest even number. In this case then, $d = 2$. Noting that the cosine decreases from 0.9553 to 0.9548 , we must in this case subtract $d = 2$ from 0.9553 , obtaining

$$\cos 17.23^\circ \approx 0.9553 - 0.0002 = 0.9551.$$

EXERCISES

From Table 3 in the Appendix find the values of the following functions; use interpolation when necessary.

- | | | |
|--------------------------|--------------------------|---------------------------|
| 1. $\sin 8.7^\circ$. | 2. $\cos 15.5^\circ$. | 3. $\tan 0.9^\circ$. |
| 4. $\cot 37.2^\circ$. | 5. $\sin 30.0^\circ$. | 6. $\tan 45.0^\circ$. |
| 7. $\sin 49.1^\circ$. | 8. $\cos 85.4^\circ$. | 9. $\cot 72.2^\circ$. |
| 10. $\tan 89.9^\circ$. | 11. $\cos 60.0^\circ$. | 12. $\cot 65.7^\circ$. |
| 13. $\sec 49.0^\circ$. | 14. $\csc 31.9^\circ$. | 15. $\sin 125.2^\circ$. |
| 16. $\sin 14.23^\circ$. | 17. $\sin 55.41^\circ$. | 18. $\tan 45.00^\circ$. |
| 19. $\cos 19.84^\circ$. | 20. $\cot 72.25^\circ$. | 21. $\tan 72.25^\circ$. |
| 22. $\sin 44.44^\circ$. | 23. $\cos 44.44^\circ$. | 24. $\sin 20.05^\circ$. |
| 25. $\sin 30.17^\circ$. | 26. $\cos 59.83^\circ$. | 27. $\cos 61.97^\circ$. |
| 28. $\sec 61.97^\circ$. | 29. $\csc 30.17^\circ$. | 30. $\sin 243.13^\circ$. |

Evaluate the following expressions; give all results to four decimal places.

- | | |
|---|---|
| 31. $\sin 14.2^\circ + \sin 63.71^\circ$. | 32. $2(7.16)(4.3)(\cos 21.26^\circ)$. |
| 33. $3.2 \sin 49.84^\circ$. | 34. $\frac{3.2 \sin 49.84^\circ}{\sin 37.72^\circ}$. |
| 35. $\frac{\tan 21.7^\circ}{\tan 43.1^\circ}$. | 36. $(\sin 32.71^\circ)(\cos 77.75^\circ) - 0.01276$. |
| 37. $(7.16)^2 + (4.3)^2 - 2(7.16)(4.3)(\cos 21.26^\circ)$. | |
| 38. $14.71 \tan 72.5^\circ$. | 39. $105.7 \cos 66.72^\circ + 105.7 \sin 23.28^\circ$. |
| 40. $\frac{(\cot 46.2^\circ)(\sin 71.74^\circ)(\tan 35.4^\circ)}{24.85(\tan 35.4^\circ)}$. | |

Evaluate, giving results to the proper number of significant figures.

- | | |
|---|---|
| 41. $2067.2 \sin 41.2^\circ$. | 42. $9.81 \tan 72^\circ - 5.65$. |
| 43. $\frac{\tan 43.1^\circ}{\tan 21.7^\circ}$. | 44. $(17.654 \cos 31.62^\circ)(\sin 17.77^\circ)$. |
| 45. $2.0000(7.16)(4.3)(\cos 21.26^\circ)$. | 46. $\frac{3.2 \sin 49.54^\circ}{\sin 37.72^\circ}$. |

2. Finding the angle having a given functional value.

Example 5. If $\tan \theta = 3.398$ and θ is in the first quadrant, find θ .

The tangents for angles between 0° and 90° are listed in the second and third columns of the table. Locating 3.398 in the third column on page 691, we know that for this column the label *tan* is at the bottom. Hence the angle is found in the right-hand column and is 73.6° .

If the angle cannot be read directly from the table, interpolation is frequently used. With four-place tables interpolation for more than hundredths of a degree is of no value.

Example 6. Find θ in the first quadrant if $\cos \theta = 0.4302$.

From the table, the cosines between which 0.4302 lies are

$$10 \left[x \begin{bmatrix} \cos 64.50^\circ = 0.4305 \\ \cos \theta = 0.4302 \\ \cos 64.60^\circ = 0.4289 \end{bmatrix} 3 \right] 16$$

Noting the differences denoted by the brackets, we find that as the cosine changes from 0.4305 to 0.4302, the change in the angle will be $x = (\frac{3}{18})(10) = 1.9$. Actually this change is 0.019° or to the nearest hundredth 0.02° . Thus

$$\theta = 64.5^\circ + 0.02^\circ = 64.52^\circ.$$

Where the proportional part x in a computation comes out to be half-way between two possible hundredths of a degree, the nearest even hundredth is used for x . To illustrate, suppose in a computation

$$x = \frac{6}{8} \cdot 10 = 7.5 \quad \text{or} \quad 0.075^\circ.$$

Then the nearest even hundredth, namely, $x = 8$ or 0.08° , is used.

EXERCISES

From Table 3 in the Appendix find the positive acute angle θ having the given functional value. Use interpolation when the given function is not found exactly in the table and give θ to the nearest hundredth of a degree.

- | | | |
|------------------------------|------------------------------|------------------------------|
| 1. $\sin \theta = 0.15471$. | 2. $\cos \theta = 0.9755$. | 3. $\tan \theta = 0.3779$. |
| 4. $\cot \theta = 1.1067$. | 5. $\cot \theta = 0.7898$. | 6. $\sin \theta = 0.5548$. |
| 7. $\tan \theta = 2.605$. | 8. $\cos \theta = 0.2957$. | 9. $\tan \theta = 6.940$. |
| 10. $\cot \theta = 1.0000$. | 11. $\sin \theta = 0.7157$. | 12. $\cos \theta = 0.5105$. |
| 13. $\cot \theta = 0.4899$. | 14. $\cos \theta = 0.8126$. | 15. $\cos \theta = 0.8128$. |
| 16. $\cos \theta = 0.8124$. | 17. $\sin \theta = 0.8124$. | 18. $\cot \theta = 2.209$. |
| 19. $\sin \theta = 0.4325$. | 20. $\tan \theta = 0.3909$. | 21. $\tan \theta = 0.3905$. |
| 22. $\tan \theta = 0.3906$. | 23. $\cot \theta = 0.4812$. | 24. $\cot \theta = 0.4957$. |
| 25. $\sin \theta = 0.9280$. | 26. $\cos \theta = 0.4430$. | 27. $\sin \theta = 0.9004$. |
| 28. $\sec \theta = 6.4152$. | 29. $\sec \theta = 2.165$. | 30. $\csc \theta = 2.165$. |
| 31. $\sin \theta = 2.165$. | 32. $\cos \theta = 0.0102$. | 33. $\cos \theta = 10.000$. |
| 34. $\tan \theta = 10.000$. | 35. $\sec \theta = 0.9432$. | 36. $\csc \theta = 1.0495$. |

4-11. The Functions of Angles not in the First Quadrant. As we have seen, tables of trigonometric functions are made up only for angles between 0° and 90° . The next problem which arises is the determination of the trigonometric ratios of angles which do not lie in the first quadrant. It is convenient to develop a method for obtaining the functions

of any angle in terms of the functions of a positive, acute angle which can be found in the table.

Consider any angle θ in standard position (Fig. 4-19). In the reference triangle ($\triangle OPA$) of θ , the acute angle α with vertex at the origin is called the acute angle associated with θ . This angle is always taken as positive.

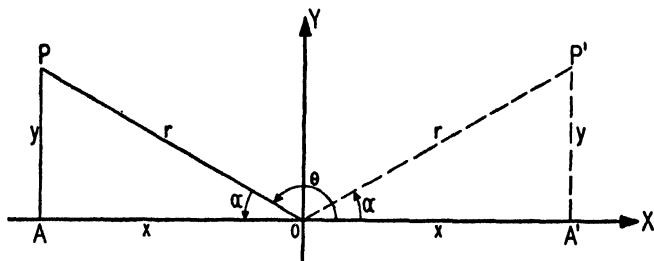


FIG. 4-19.

Thus, for example, the acute angles associated with 125° , 217° , 331° , and -47° are 55° , 37° , 29° , and 47° respectively. It is possible to demonstrate that the trigonometric functions of any angle θ may be found in terms of the functions of this positive, acute angle α .

Consider a second quadrant angle θ , with its reference triangle OPA , and the positive, acute angle α associated with θ (Fig. 4-19). The trigonometric functions of θ may be found by placing α in standard position

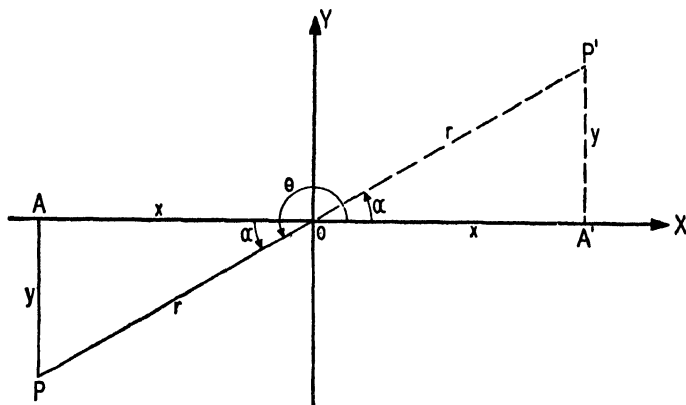


FIG. 4-20.

and constructing its triangle of reference $OP'A'$ with hypotenuse of length r . The two triangles of reference are therefore congruent, the hypotenuse and an acute angle of one being equal to the hypotenuse and an acute angle of the other. Hence the absolute value of each function of θ is equal to the corresponding function of α . For example,

since the $\sin \theta$ is positive and the $\sin \alpha$ is positive, $\sin \theta = \sin \alpha$. Again, since $\cos \theta$ is negative and $\cos \alpha$ is positive, $\cos \theta = -\cos \alpha$. In short, if θ is a second quadrant angle and α is the acute angle associated with θ , then:

θ in Quadrant II

$$\sin \theta = \sin \alpha$$

$$\csc \theta = \csc \alpha$$

$$\cos \theta = -\cos \alpha$$

$$\sec \theta = -\sec \alpha$$

$$\tan \theta = -\tan \alpha$$

$$\cot \theta = -\cot \alpha$$

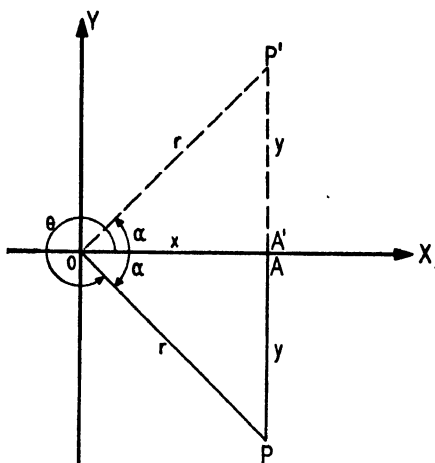


FIG. 4-21.

When θ is a third or fourth quadrant angle (Figs. 4-20, 4-21), the same reasoning is followed, resulting in the relationships:

θ in Quadrant III

θ in Quadrant IV

$$\sin \theta = -\sin \alpha$$

$$\sin \theta = -\sin \alpha$$

$$\cos \theta = -\cos \alpha$$

$$\cos \theta = \cos \alpha$$

$$\tan \theta = \tan \alpha$$

$$\tan \theta = -\tan \alpha$$

$$\cot \theta = \cot \alpha$$

$$\cot \theta = -\cot \alpha$$

$$\sec \theta = -\sec \alpha$$

$$\sec \theta = \sec \alpha$$

$$\csc \theta = -\csc \alpha$$

$$\csc \theta = -\csc \alpha$$

In every case the function of the given angle θ is equal to the same function of the acute angle α associated with θ , and the proper sign is prefixed.

Thus we may state the following rule.

To evaluate the trigonometric function of an angle θ which is not a positive acute angle:

1. Determine the acute angle α associated with θ , and in the table find the value of the same function of α .

2. From the function and quadrant of the given angle θ , determine the sign of the final result.

3. The desired trigonometric function of θ is then given by the number found in (1) prefixed by the sign obtained in (2).

Example 1. Evaluate $\sin 173.3^\circ$.

Since the sine is positive in the second quadrant, and since the acute angle associated with 173.3° is $\alpha = 180^\circ - 173.3^\circ = 6.7^\circ$, then by the rule,

$$\sin 173.3^\circ = +\sin 6.7^\circ.$$

From the tables, $\sin 6.7^\circ = 0.11667$. Therefore,

$$\sin 173.3^\circ = 0.11667.$$

Example 2. Evaluate $\cos 233.1^\circ$.

Since the cosine is negative in the third quadrant, and since $\alpha = 53.1^\circ$ is the acute angle associated with 233.1° , then by the rule,

$$\cos 233.1^\circ = -\cos 53.1^\circ.$$

Since $\cos 53.1^\circ = 0.6004$, $\cos 233.1^\circ = -0.6004$.

Example 3. Evaluate $\cot \frac{7\pi}{4}$.

Since $\frac{7\pi}{4}$ radians is a fourth quadrant angle, the cotangent is negative. Also,

$\alpha = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$ is the acute angle associated with $\frac{7\pi}{4}$. Therefore, from the rule,

$$\cot \frac{7\pi}{4} = -\cot \frac{\pi}{4} = -1.$$

Example 4. Evaluate $\sin 590^\circ$.

Since 590° is in the third quadrant, and since $\alpha = 50^\circ$,

$$\sin 590^\circ = -\sin 50^\circ = -0.7660.$$

Example 5. Evaluate $\tan (-38.2^\circ)$.

The angle $\theta = (-38.2)^\circ$ is in the fourth quadrant, and its associated angle $\alpha = 38.2^\circ$. Hence, $\tan (-38.2)^\circ = -\tan 38.2^\circ = -0.7869$.

EXERCISES

Using Table 3 in the Appendix find the value of the following functions, interpolating when necessary.

1. $\sin 178^\circ$.

2. $\cos 132^\circ$.

3. $\tan 147^\circ$.

4. $\sec 166^\circ$.

5. $\cos 144.7^\circ$.

6. $\tan 211^\circ$.

7. $\cot 256.2^\circ$.

8. $\sin 197.4^\circ$.

9. $\cos 234.4^\circ$.

- | | | |
|------------------------------|-----------------------------|------------------------------|
| 10. $\csc 200^\circ$. | 11. $\cot 311.1^\circ$. | 12. $\tan 311.1^\circ$. |
| 13. $\sin 286.9^\circ$. | 14. $\cos 359.1^\circ$. | 15. $\sin 340^\circ$. |
| 16. $\sin 400.1^\circ$. | 17. $\cos 511.6^\circ$. | 18. $\cos 43.2^\circ$. |
| 19. $\cos (-43.2^\circ)$. | 20. $\sin (-127.2^\circ)$. | 21. $\tan 4691.7^\circ$. |
| 22. $\cot 711.2^\circ$. | 23. $\sin 131.27^\circ$. | 24. $\cos 188.88^\circ$. |
| 25. $\tan 188.88^\circ$. | 26. $\cot 299.17^\circ$. | 27. $\sin (-211.26^\circ)$. |
| 28. $\cos (-416.12^\circ)$. | 29. $\cos 416.12^\circ$. | 30. $\csc 370.15^\circ$. |

Evaluate the following trigonometric expressions to as many significant figures as is permissible.

31. $\cos 147.21^\circ \cdot \cos 214.2^\circ - \sin 147.21^\circ \cdot \sin 214.2^\circ$.
32. $2.00 \cdot \sin 311.71^\circ \cdot \cos 311.71^\circ$. 33. $2.00 \cdot \sin (-75^\circ) \cdot \cos (-75^\circ)$.
34. $\frac{10.7 \sin 131.29^\circ}{\sin 11.14^\circ}$. 35. $13.2 \cdot \tan 411.72^\circ$.
36. $E_1 \cos 142.7^\circ - E_2 \sin (-142.7^\circ)$.
37. $\cos x \sin 344.42^\circ + \sin x \cos 344.42^\circ$.
38. $1765 \csc 247.2^\circ$. 39. $714.3 \cos \frac{31}{18}$.
40. $\frac{\sin \frac{17}{42} - 3.76 \cos \frac{17}{42}}{\cot \frac{117}{42}}$.

4-12. Angles for a Given Trigonometric Function. In the preceding section it was found that the trigonometric functions of angles not in the first quadrant have the same absolute value as the same functions of their associated acute angles. We now investigate the inverse procedure, that is, the determination of all angles, particularly those between 0° and 360° , having a given trigonometric function.

If α is a given positive acute angle, then in each quadrant there is one angle for which α is the associated acute angle. For example, $\alpha = 35^\circ$ is the acute angle associated with 35° (itself), 145° , 215° , and 325° (Fig. 4-22). Since by the previous section the trigonometric functions of these angles are equal in absolute value to the same functions of α , it is obvious that if α is a positive acute angle, then in each quadrant there is one and only one angle between 0° and 360° whose trigonometric functions are equal in absolute value to the corresponding functions of α .

Hence, to find all positive angles between 0° and 360° having a given, signed value for a particular trigonometric function:

1. Find the first quadrant angle α having the given functional value without regard to sign.

2. From the sign of the given value and the given function determine the quadrants in which the angles may lie.

3. In each possible quadrant, find the angle for which α is the associated acute angle.

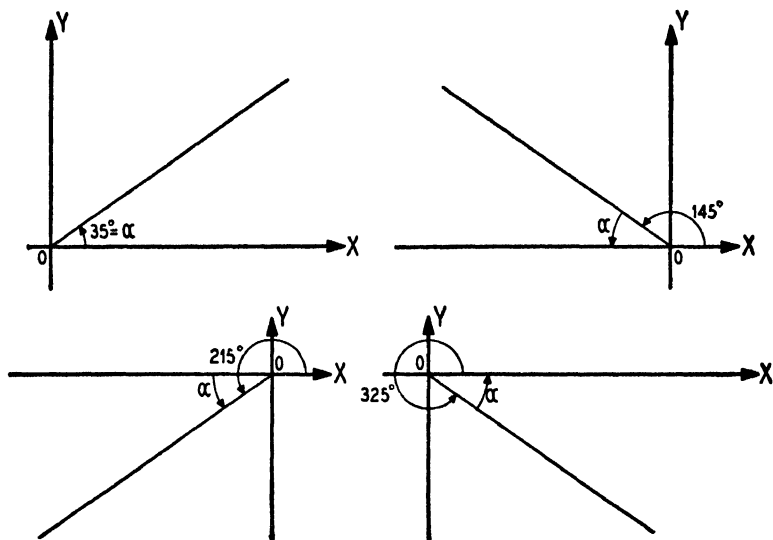


FIG. 4-22.

Example 1. Find all angles θ between 0° and 360° such that $\cos \theta = -0.4540$.

The angle in the first quadrant having 0.4540 as its cosine value is 63° . Since the cosine is negative only in the second and third quadrants, we must find the one angle in each of these quadrants having 63° as the associated acute angle. The desired angles are

$$180^\circ - 63^\circ = 117^\circ$$

and

$$180^\circ + 63^\circ = 243^\circ.$$

If it is desired to find other angles, negative or greater than 360° , having cosines of -0.4540 , it is merely necessary to find other angles having the same terminal sides as 117° and 243° .

Example 2. Find four other angles satisfying the condition that $\cos \theta = -0.4540$ of Example 1.

Four other angles having the same terminal sides as 117° and 243° are:

- (a) $117^\circ + 360^\circ = 477^\circ$.
- (b) $243^\circ + 360^\circ = 603^\circ$.
- (c) $117^\circ - 360^\circ = -243^\circ$.
- (d) $243^\circ - 360^\circ = -117^\circ$.

As an abbreviation for the expression " θ is an angle between 0° and 360° " we often write $0^\circ < \theta < 360^\circ$. If θ can be equal to 0° as well, we write $0^\circ \leq \theta < 360^\circ$. The student may infer the similar meanings of the expressions $0^\circ < \theta \leq 360^\circ$ and $0^\circ \leq \theta \leq 360^\circ$.

Example 3. Find $0^\circ \leq \theta < 360^\circ$ such that $\sin \theta = 0.4325$.

The first quadrant angle having a sine of 0.4325 is 25.62° . Since the sine is positive in the first and second quadrants, the values of θ are

$$\theta = 25.62, \quad \theta = 180 - 25.62^\circ = 154.38^\circ.$$

EXERCISES

In each case find all angles θ , where $0^\circ \leq \theta < 360^\circ$, having the given functional value. Interpolate where necessary, giving angles to the nearest hundredth of a degree.

- | | | |
|----------------------------------|--|---|
| 1. $\sin \theta = 0.4179$. | 2. $\cos \theta = 0.7242$. | 3. $\tan \theta = 1.3916$. |
| 4. $\sin \theta = -0.4179$. | 5. $\cos \theta = -0.7242$. | 6. $\tan \theta = -1.3916$. |
| 7. $\cot \theta = -0.9896$. | 8. $\sin \theta = -\frac{\sqrt{3}}{2}$. | 9. $\cos \theta = \frac{\sqrt{2}}{2}$. |
| 10. $\tan \theta = 0$. | 11. $\cos \theta = \pm 1$. | 12. $\sin \theta = \pm 0.4179$. |
| 13. $\tan \theta = \pm 1.0686$. | 14. $\sin \theta = 0.8970$. | 15. $\cos \theta = 0.8970$. |
| 16. $\sin \theta = -0.9327$. | 17. $\tan \theta = -3.436$. | 18. $\cos \theta = \pm 0.9080$. |
| 19. $\sin \theta = \pm 0.4315$. | 20. $\csc \theta = -1.7963$. | 21. $\csc \theta = \pm 0.7142$. |
| 22. $\cos \theta = -0.146$. | 23. $\sin \theta = 114.59$. | 24. $\tan \theta = 114.59$. |

In each case find six angles having the given functional value.

- | | | |
|-------------------------------|-------------------------------|-------------------------------|
| 25. $\sin \theta = 0.4909$. | 26. $\cos \theta = -0.8171$. | 27. $\tan \theta = 0.5961$. |
| 28. $\sin \theta = -0.8114$. | 29. $\cot \theta = -0.6811$. | 30. $\cos \theta = -0.7270$. |

Find all angles θ , $0^\circ \leq \theta < 360^\circ$, which are possible, given that,

$$31. \sin \theta = \frac{4.62 \sin 48.2^\circ}{10.72}.$$

Since

$$\sin 48.2^\circ = 0.7455,$$

$$\sin \theta = \frac{4.62 \times 0.7455}{10.72} = 0.321.$$

Thus

$$\theta = 18.7^\circ \quad \text{or} \quad 161.3^\circ.$$

- | | |
|--|---|
| 32. $\sin \theta = \frac{7.15 \sin 81.21^\circ}{34.02}$. | 33. $\tan \theta = \frac{1.347}{9.269} \tan 60.84^\circ$. |
| 34. $\cos \theta = \frac{(10.2)^2 + (7.15)^2 - (3.76)^2}{(2.000)(10.2)(7.15)}$. | 35. $\sin \theta = \frac{7.15 \sin 81.21^\circ}{1.312}$. |
| 36. $\cot \theta = \frac{1}{\tan 46.25^\circ}$. | 37. $\cot \theta = \frac{10.72}{71.62 \tan 17.85^\circ}$. |
| 38. $\tan \theta = \frac{71.62}{10.72} \tan 17.85^\circ$. | 39. $17.61 \cos \theta = \frac{7.12}{11.7} \sin 44.1^\circ$. |
| 40. $\frac{\tan \theta}{4.84} = 6.80 \frac{\sin 35.4^\circ}{\cos 84.77^\circ}$. | |

4-13. Functions of Negative Angles. In Sec. 4-11 we discussed a general method for evaluating the trigonometric functions of all angles, including those which are negative. In this section we shall develop a set of formulas which permit a slightly different approach to the same problem when the angles involved are negative.

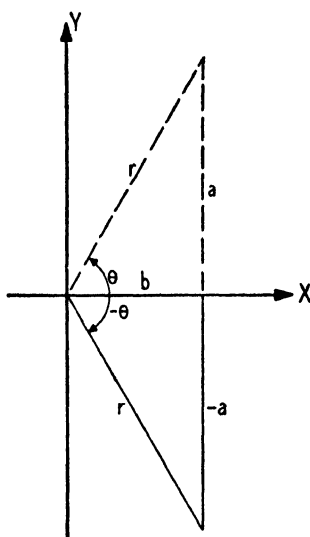


FIG. 4-23.

Let $(-\theta)$ be a given negative angle whose terminal side lies in the fourth quadrant. Drawing a triangle of reference (Fig. 4-23) with sides, $r, -a, b$, where r, a, b are positive, we find

$$\begin{aligned} \sin(-\theta) &= -\frac{a}{r}, & \csc(-\theta) &= -\frac{r}{a}; \\ (1) \quad \cos(-\theta) &= \frac{b}{r}, & \sec(-\theta) &= \frac{r}{b}; \\ \tan(-\theta) &= -\frac{a}{b}, & \cot(-\theta) &= -\frac{b}{a}. \end{aligned}$$

Now, since $(-\theta)$ is negative, θ is positive. If θ is placed in standard position and a length r is measured on its terminal side, then the triangle of reference for θ is congruent to the triangle for $(-\theta)$; and thus the sides of the new triangle have the lengths given in the figure. Hence

$$\sin \theta = \frac{a}{r}, \quad \csc \theta = \frac{r}{a};$$

$$(2) \quad \cos \theta = \frac{b}{r}, \quad \sec \theta = \frac{r}{b};$$

$$\tan \theta = \frac{a}{b}, \quad \cot \theta = \frac{b}{a}.$$

Equating the results from (1) and (2):

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \csc(-\theta) &= -\csc \theta \\ (3) \quad \cos(-\theta) &= \cos \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \end{aligned}$$

The equations (3), which we have proved to hold when $(-\theta)$ is a fourth quadrant angle, can be shown in the same manner to hold for $(-\theta)$ in any other quadrant.

Example 1. Evaluate the six trigonometric functions of -38.2° .

Using the proper formulas from (3):

$$\sin(-38.2)^\circ = -\sin 38.2^\circ = -0.6184.$$

$$\cos(-38.2)^\circ = \cos 38.2^\circ = 0.7859.$$

$$\tan(-38.2)^\circ = -\tan 38.2^\circ = -0.7869.$$

$$\cot(-38.2)^\circ = -\cot 38.2^\circ = -1.2708.$$

$$\sec(-38.2)^\circ = \sec 38.2^\circ = \frac{1}{\cos 38.2^\circ} = \frac{1}{0.7859} = 1.2725.$$

$$\csc(-38.2)^\circ = -\csc 38.2^\circ = -\frac{1}{\sin 38.2^\circ} = -\frac{1}{0.6184} = -1.6171.$$

Example 2. Evaluate $\cos(-163.7)^\circ$.

From the proper formula in (3), $\cos(-163.7)^\circ = \cos 163.7^\circ$, and by Sec. 4-11, since the acute angle associated with 163.7° is 16.3° , $\cos 163.7^\circ = -\cos 16.3^\circ = -0.9598$. Therefore, $\cos(-163.7)^\circ = -0.9598$.

EXERCISES

Evaluate the following functions by the methods of this section.

- | | | |
|---------------------------------|-----------------------------------|---|
| 1. $\sin(-37.25^\circ)$. | 2. $\cos(-37.25^\circ)$. | 3. $\tan(-37.25^\circ)$. |
| 4. $\cot(-142.7^\circ)$. | 5. $\sec(-10.1^\circ)$. | 6. $\csc(-90.0^\circ)$. |
| 7. $\sin(-546.2^\circ)$. | 8. $\cos(-330.7^\circ)$. | 9. $\tan(-135^\circ)$. |
| 10. $\cot(-75.75^\circ)$. | 11. $\cos(-2\theta + 10^\circ)$. | 12. $\sin(-2\theta + 10^\circ)$. |
| 13. $\sin(-2\phi + 10^\circ)$. | 14. $\tan(10^\circ - 2\phi)$. | 15. $\cot\left(-\frac{\pi}{2}\right)$. |
| 16. $\sec(-\pi)$. | | |

Prove the following:

$$17. I_1 \sin \theta + I_1 \sin (-\theta) + \frac{3}{2}I_1 = \frac{3}{2}I_1.$$

For example, since $\sin (-\theta) = -\sin \theta$, the above becomes

$$I_1 \sin \theta - I_1 \sin \theta + \frac{3}{2}I_1 = \frac{3}{2}I_1.$$

$$18. 10[\cos (-\theta) + j \sin (-\theta)] = 10(\cos \theta - j \sin \theta).$$

$$19. 10[\cos \theta - j \sin \theta] = 10[\cos \theta + j \sin (-\theta)].$$

$$20. \frac{1}{\sqrt{10}} \left[\cos \left(-\frac{\theta}{2} \right) + j \sin \left(-\frac{\theta}{2} \right) \right] = \frac{\sqrt{10}}{10} \left(\cos \frac{\theta}{2} - j \sin \frac{\theta}{2} \right).$$

$$21. 5 \cos^2 \phi = 5 \cos (-\phi) \cos \phi.$$

$$22. \text{ If } \cos (-\phi) = 0.7181, \text{ then } \phi = 44.1^\circ \text{ or } 315.9^\circ.$$

$$23. \text{ If } \sin (-\phi) = 0.7181, \text{ then } \phi = 225.9^\circ \text{ or } 314.1^\circ.$$

$$24. \text{ If } \tan (-\phi) = -0.8332, \text{ then } \phi = 39.8^\circ \text{ or } 219.8^\circ.$$

$$25. \text{ If } \tan (-\phi) = 0.8332, \text{ then } \phi = 140.2^\circ \text{ or } 320.2^\circ.$$

4-14. The Trigonometric Functions for an Acute Angle of a Right Triangle. In discussing the trigonometric functions of one of the acute angles of a right triangle, it is often advantageous to use a modification of the original definitions.

If α is an acute angle of a right triangle, then

$$\sin \alpha = \frac{\text{Side opposite } \alpha}{\text{Hypotenuse}}, \quad \csc \alpha = \frac{\text{Hypotenuse}}{\text{Side opposite } \alpha};$$

$$(1) \quad \cos \alpha = \frac{\text{Side adjacent } \alpha}{\text{Hypotenuse}}, \quad \sec \alpha = \frac{\text{Hypotenuse}}{\text{Side adjacent } \alpha};$$

$$\tan \alpha = \frac{\text{Side opposite } \alpha}{\text{Side adjacent } \alpha}, \quad \cot \alpha = \frac{\text{Side adjacent } \alpha}{\text{Side opposite } \alpha}.$$

These statements are immediately obvious if the right triangle is placed on the coordinate axes with α in standard position, applying the original definitions (Fig. 4-24).

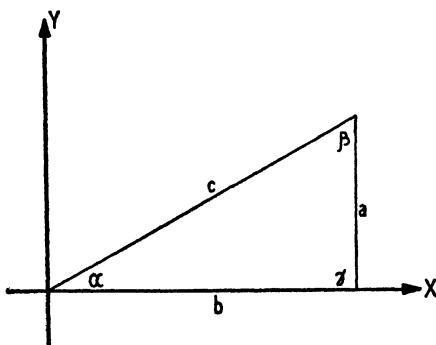


FIG. 4-24.

In the following discussions, angles of a right triangle are denoted, as a rule, by α , β , γ , and the sides opposite these angles by a , b , c , respectively. γ is always the right angle.

EXERCISES

Give the six trigonometric functions for each of the acute angles, α and β , of a right triangle having sides:

- | | |
|------------------------------|-------------------------------------|
| 1. $a = 3, b = 4, c = 5$. | 2. $a = 7, b = 4, c = \sqrt{65}$. |
| 3. $a = 6, b = 8, c = 10$. | 4. $a = 8, b = 15, c = 17$. |
| 5. $a = 40, b = 9, c = 41$. | 6. $a = 6, b = 6$. |
| 7. $a = 6, b = 5$. | 8. $a = 10, b = 11$. |
| 9. $a = 6, c = 13$. | 10. $b = 7, c = 10$. |
| 11. $a = 2, c = 2\sqrt{2}$. | 12. $b = \sqrt{7}, c = \sqrt{30}$. |
| 13. $b = 5, c = 5$. | 14. $a = 10, c = 9$. |

Obtain α and β correct to the nearest hundredth of a degree for the right triangle in:

- | | | |
|-----------------|------------------|------------------|
| 15. Exercise 1. | 16. Exercise 5. | 17. Exercise 6. |
| 18. Exercise 8. | 19. Exercise 10. | 20. Exercise 12. |

Obtain α and β correct to the nearest hundredth of a degree for the right triangle having sides:

- | | | |
|---|------------------------|------------------------|
| 21. $a = 8, b = 7$. | 22. $a = 9, c = 13$. | 23. $b = 1, c = 1.6$. |
| 24. $a = 2, b = 3$. | 25. $b = 30, c = 50$. | |
| 26. $a = 6, b = 5$ (without finding c). | | |
| 27. $b = 10, c = 14$ (without finding a). | | |
| 28. $a = 16, b = 23$ (without finding c). | | |
| 29. $a = 7.6, c = 69$ (without finding b). | | |
| 30. $b = 12, c = 16$ (without finding a). | | |

4-15. The Solution of Right Triangles.

In plane geometry it is shown that if two sides, or one side and one acute angle, of a right triangle are given, the triangle can be constructed, and the unknown sides and angles found by measurement. The same results can be obtained much more accurately by means of the modified definitions of the trigonometric functions given in (1) of Sec. 4-14. Each of these expressions involves three parts of the triangle. By selecting an expression involving the two known parts and an unknown part which is to be found, an equation is obtained which can be solved for the unknown part.

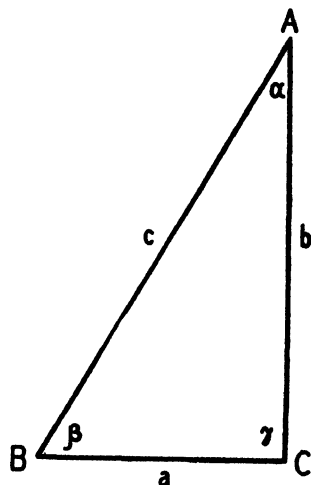


FIG. 4-25.

Briefly, the following rules can be used in solving for an unknown part of a right triangle.

1. To find the acute angle α , knowing the acute angle β , use the formula

$$\alpha = 90^\circ - \beta.$$

2. To find an unknown acute angle, knowing two sides but not the other acute angle, select from (1) of Sec. 4-14 the proper relation involving the unknown angle and the two known sides.

3. To find an unknown side, knowing one side and one acute angle, from the relations below select the one most easily used.

$$\begin{aligned}\text{Unknown side} &= \text{Hypotenuse} \cdot \text{Sine of angle opposite unknown side,} \\ &= \text{Hypotenuse} \cdot \text{Cosine of angle adjacent to unknown side;}\end{aligned}$$

$$\begin{aligned}\text{Unknown side} &= \text{Known side} \cdot \text{Tangent of angle opposite unknown side,} \\ &= \text{Known side} \cdot \text{Cotangent of angle adjacent to unknown side;}\end{aligned}$$

$$\begin{aligned}\text{Hypotenuse} &= \text{Known side} \div \text{Sine of angle opposite known side,} \\ &= \text{Known side} \div \text{Cosine of angle adjacent to known side.}\end{aligned}$$

These relations are merely convenient restatements of those in (1) of Sec. 4-14.

Example 1. Given a right triangle ABC in which $a = 10$, $\alpha = 30^\circ$, find c .

Since the side a and α , the angle opposite, are known, the fifth relation in (3) is used, giving

$$c = \frac{a}{\sin \alpha}$$

or

$$c = \frac{10}{\frac{1}{2}} = 20.$$

Obviously, in solving a right triangle problem it is advisable to make a sketch and carry out the computations in an orderly fashion. Occasionally, it may be necessary to solve a right triangle completely. In such work, after all the parts have been found, the Pythagorean theorem may be used as a check. The form below is a convenient one to use in such problems.

Example 2. In the right triangle ABC , $a = 7.32$, $c = 10.67$. Find the other parts of the triangle.

(1) *Given*

$a = 7.32$

$c = 10.67$

To find

$b =$

$\alpha =$

$\beta =$

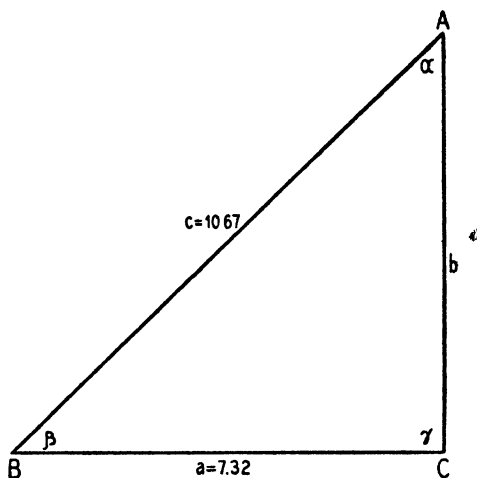
Sketch

FIG. 4-26.

(2) *Formulas:*

(a) To find α , $\sin \alpha = \frac{a}{c}$.

(b) To find β , $\beta = 90 - \alpha$.

(c) To find b , $b = c \cos \alpha$.

(3) *Substitution and Computations:*

(a) $\sin \alpha = \frac{7.32}{10.67} = 0.6860$, $\alpha = 43.32^\circ$.

(b) $\beta = 90 - 43.32^\circ = 46.68^\circ$, $\beta = 46.68^\circ$.

(c) $b = 10.67 \times \cos 43.32^\circ$, $b = 10.67 \times 0.7276 = 7.763$.

(4) *Check:* Using the Pythagorean theorem and the rules for computation introduced in Chapter 1,

$$a^2 + b^2 = c^2.$$

$$(7.32)^2 + (7.763)^2 = (10.67)^2,$$

$$53.6 + 60.26 = 113.9,$$

$$113.9 = 113.9.$$

The check shows the results to be reasonably correct.

The solution of right triangles has many practical applications, particularly in obtaining the lengths of inaccessible distances by means of angles and other lengths which are more easily measured. In preparation for these applications, it will be valuable to discuss some of the terminology which is frequently used.

The acute angle between the line of sight from the observer O to the object P and the horizontal line which is in the same vertical plane as the line of sight is called the angle of **elevation** or the angle of **depression**

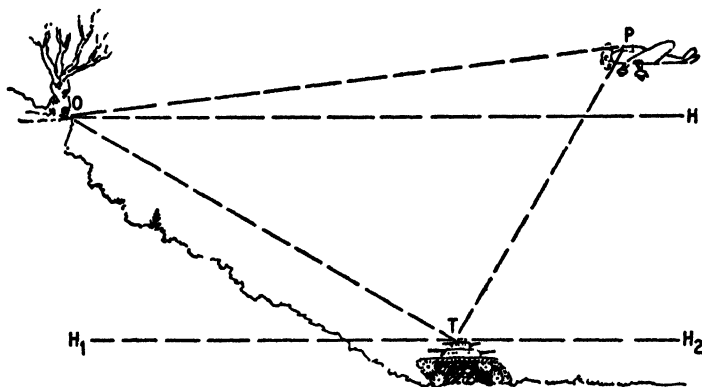


FIG. 4-27.

of P from O , according as P is at a higher or lower elevation (height above sea level) than O . For example, if a man O at the top of a hill (Fig. 4-27) observes an enemy tank T in the valley below an enemy plane P in the sky above: $\angle HOT$ is the angle of depression of T from O ; $\angle HOP$ is the angle of elevation of P from O ; $\angle H_1TO$ is the angle of elevation of O from T ; $\angle H_2TP$ is the angle of elevation of P from T , etc.

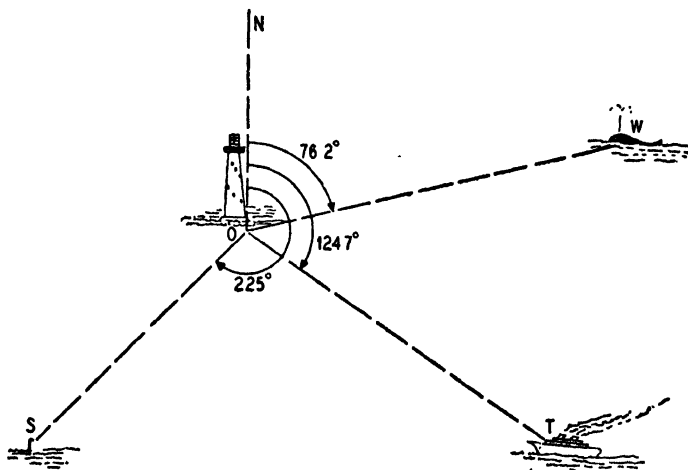


FIG. 4-28.

In navigation, if an observer at O sights an object W , and ON is the line from O directly north, then angle NOW , always measured in a clockwise direction, is called the **bearing** of W from (at) O . Thus (Fig. 4-28),

if from a lighthouse O the keeper spots a submarine S directly south-west, then at O the bearing of S is 225° . Also, at O , the bearings of a whale W and a transport T as shown would be 76.2° and 124.7° respectively.

EXERCISES

In the following exercises, find the desired parts of the right triangle, giving sides and angles to the nearest hundredth.

1. Given $a = 15.76$, $c = 31.2$, find α .
2. Given $b = 11.05$, $\beta = 42.3^\circ$, find a .
3. Given $a = 762.14$, $b = 466.66$, find α and β .
4. Given $a = 46.49$, $\alpha = 9.60^\circ$, find β and b .
5. Given $b = 0.265$, $\alpha = 83.31^\circ$, find c and a .
6. Given $a = 1.23$, $b = 0.076$, find α .
7. Given $a = 76.214$, $b = 466.66$, find α and β .
8. Given $a = 145.6$, $\beta = 74^\circ$, find b and c .
9. Given $a = 362.2$, $c = 271.7$, find b and β .
10. Given $\alpha = 36.27^\circ$, $\beta = 47.8^\circ$, find a and b .

Solve each of the following right triangles for the three unknown parts, giving results to two decimal places. It is suggested that the student follow an orderly procedure such as the one suggested in this section.

- | | |
|---|--|
| 11. $a = 1.72$, $b = 3.65$. | 12. $a = 31.34$, $c = 44.723$. |
| 13. $b = 6146$, $c = 10,112$. | 14. $a = 10.021$, $\alpha = 67.92^\circ$. |
| 15. $b = 21.13$, $\beta = 11.26^\circ$. | 16. $a = 110.2$, $\beta = 17.34^\circ$. |
| 17. $b = 0.9156$, $\alpha = 44.78^\circ$. | 18. $c = 7.426$, $\alpha = 59.3^\circ$. |
| 19. $c = 7.426$, $\beta = 30.7^\circ$. | 20. $\alpha = 59.3^\circ$, $\beta = 30.7^\circ$. |

By the use of right triangles, find the desired quantities in the following exercises to one decimal place.

21. A 50-ft. ladder resting against a wall makes an angle of 62.7° with the ground. How far up the wall does the top of the ladder touch?

22. A man wishes to measure the height of a tree. To do so, he first obtains the angle of elevation of the sun to be 42.7° and then measures the shadow of the tree as 126.7 ft. Find the height of the tree.

23. An insurance company figures that the maximum safe angle which a ladder can make with the ground is 65° . What is the shortest ladder a painter should use to paint at a height of 50 ft.?

24. According to Exercise 23, will an eloping couple have trouble in using a 30-ft. ladder to climb down from a window which is 27 ft. above the ground?

25. Two buildings are 20 ft. apart. A ladder which is placed 5 ft. from the base of one building reaches a height of 42 ft. on that building. How far will the ladder reach up the other building if its foot is held in the same position?

26. An observer in a balloon 900 ft. above a battery X observes that the angle of depression of an enemy battery is 21.34° . What is the distance from battery X to the enemy battery?

27. A captive balloon is floating directly above a point A . At point B , which is 625 ft. from A , the angle of elevation of the balloon is 37.3° . How high is the balloon above A ?

28. A captive balloon floats 1500 ft. above a point A . At a point B which is 972 ft. from A , what will be the angle of elevation of the balloon? What will be the angle of depression of B from the balloon?

29. An observer A is 13,620 ft. due east of an observer B . Simultaneously, they observe an enemy bomber directly west. At A and B the angles of elevation of the bomber are 20.6° and 33.4° , respectively. What is the altitude of the bomber, and how far is it from each observation post?

30. An engineer wishes to measure the width of a river without getting his feet wet. From a point A on the north bank, he sights a point B directly across the river, and then measures off on the north bank a distance AC of 500 ft., perpendicular to the line AB . He finds the angle ABC to be 67.32° . How wide is the river?

31. From a 200-ft. tower on one side of a lake, the angle of depression of a boat-house on the opposite side of the lake is 12.7° . How far is the boathouse from the tower?

32. A vertical antenna 320 ft. high is to be held in position on a level plain by guy wires extending from the top of the antenna to the ground. If proper precautions against wind resistance require that the angle formed by the guy wire and the axis of the antenna be at least 42.5° , find the minimum length possible for each guy wire. Also, how far from the base of the antenna must these wires be secured?

33. According to the minimum angle of safety given in Exercise 32 as 42.5° , determine whether or not guy wires which are 212 ft. long will safely hold a 150-ft. antenna tower in place.

34. What angle does a rafter 25 ft. long make with the horizontal if it attains a height of 7.2 ft. at its maximum point?

35. A railroad track is built on a slope where the vertical rise is 2.4 ft. for a horizontal distance of 100 ft. At what angle with the horizontal is the track built?

36. A boat A is 12 miles directly north of a lighthouse B and directly west of town C . If the lighthouse is 17.2 miles from the town, find the bearings of A and C from B . Of A and B from C ; of C and B from A .

37. If a boat travels west from a town for a distance of 7.2 miles and then turns and goes directly north for 1.16 miles, what is the bearing of the town from the boat?

38. A boat at A finds that the bearing of two other boats B and C are 212.2° and 252.4° . If the bearing of B at C is 162.4° , find the bearings of A and C from B . If $BC = 8.6$ miles, find the distance of A from B and C .

39. Two P.T. boats set out from a dock at the same time. The first boat bears 22.7° , traveling at 60 m.p.h. The second bears 112.7° at a rate of 40 m.p.h. After 16 minutes, how far apart will the boats be? What will be the bearing of the first boat from the second?

40. A boat is traveling due southwest at 20 m.p.h. At 9 A.M. the bearing of a lighthouse A is 168.8° . At 10:15 A.M. the bearing of the lighthouse is 78.8° . How far was the boat from the lighthouse at 9 A.M.?

41. A Curtiss P-40, traveling on a straight line, climbs 1750 ft. while making an airline distance of 4 miles. Find the angle of climb.

42. A flying fortress is approaching its home base at an altitude of 10,000 ft. When 12 miles away the plane begins to descend. If the plane flies in a straight line, what angle must its line of flight make with the horizontal in order to be at an altitude of 100 ft. when it reaches the field?

43. A fighter plane can climb at an angle of 27° while making 210 m.p.h. How long will it take for the plane to reach an altitude of 15,000 ft.?

44. How much fence will a farmer need for a field which is in the shape of a right triangle if the shortest side is 300 yd. long and makes an angle of 52° with the longest side?

45. Two men wish to climb to a hilltop which is 2050 ft. above the point from which they start. If one man takes a road which slopes upward at 21° and the other takes a road which has a slope of 34° , give the distance each will travel in reaching the top.

46. An observation plane flying at 2000 ft. notes an enemy battery due south with an angle of depression of 42° and a friendly battery due north with an angle of depression of 22° . Assuming that the batteries are on the same horizontal plane, what is the distance between them?

47. An observer is located at the top of a cliff 315 ft. above his battery on the beach below. He sights a group of enemy landing barges with an angle of depression of 11° . At what range should his battery start firing immediately? If 2 minutes later the angle of depression is 23° , how fast are the barges approaching the battery? How soon will they reach the shore?

48. An observer sights a squadron of enemy planes coming directly toward him. He finds that the angle of elevation is 5.2° and that they are flying at 10,000 ft. Exactly 5 minutes later, their angle of elevation is 11.5° and altitude still 10,000 ft. How far have the planes come between observations? What is their speed in m.p.h. and in how many minutes will they be directly overhead?

49. A radio tower stands on top of a building. From a point 2400 ft. from the base of the building, the angles of elevation of the top of the building and the top of the tower are 24.7° and 31.2° respectively. How high is the building? the tower?

50. A P.T. boat sets out from a dock A, bearing 276.7° at 50 m.p.h. Ten minutes later a second P.T. boat sets out from dock A, bearing 310° at 45 m.p.h. What will be the distance between the boats half an hour after the second boat leaves the dock?

51. An engineer wishes to build a bridge across a river. He has a choice of two bridge heads, A and B, on the south bank but only one point C on the north bank. To find the distances AC and BC, he measures the distance AB = 655 ft., and the angles $ABC = 90.0^\circ$, $BAC = 32.7^\circ$. What will be the lengths of AC and BC?

52. A navigator in a plane observes two cities A and B both directly south. The angles of depression of A and B are 32.7° and 45.3° . If A and B are known to be 11.6 miles apart, how high is the plane and how far is the plane from each of the cities?

53. An aircraft carrier sights a group of planes, bearing 172.3° , angle of elevation 6.3° . Thirty seconds later the planes, still at the same altitude, are observed to have a bearing of 172.3° and angle of elevation of 26.2° . What is the apparent objective of the planes and how soon will they reach it?

54. What will be the width, measured from one face to the opposite face of an octagonal machine nut, if each face is 0.425 in. long? What will be the width measured from one corner to the opposite corner?

PROGRESS REPORT

In this chapter the trigonometric functions of an angle were introduced as certain ratios which were defined after placing the angle in standard position on a rectangular coordinate system. There are only

a few angles for which the trigonometric ratios can be found by geometrical considerations, and therefore tables must be used from which the values of the trigonometric functions can be taken. These tables give, as a rule, the functions for angles between 0° and 45° , but a few theorems about the trigonometric functions of complementary and related angles permit us to find from these tables the functions of any angle.

The last few sections of this chapter discussed the solution of right triangles, one of the most important applications of the trigonometric functions.

Two other topics discussed in this chapter are particularly worth mentioning. The first is the possibility of measuring an angle in different units and the examination of the two most frequently used units: the degree and the radian. The second topic is the interpolation between two values given in a table, which is not only used for trigonometric tables but in all cases where numerical tables are used and values have to be found which are between two values in the table.

CHAPTER 5

THE GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

In Chapter 3, the graphs of various types of functions were studied. It was shown there how useful the graphs are in revealing properties of functions. In the present chapter, we shall see that the trigonometric ratios are functions and we shall plot their graphs and study their properties. We shall also study the graphs of various combinations of trigonometric functions.

A knowledge of the trigonometric functions and their graphs is a very important basis and tool for much scientific and engineering work. Almost all electric current is generated by rotating machines which produce alternating voltages. These voltages alternate like the sine function and are called **sine wave** voltages or **sinusoidal** alternating voltages. The pendulum of a clock swings back and forth sinusoidally. Sound waves can be analyzed as a combination of superimposed sine waves. Vibrating strings, reeds, and diaphragms execute motions which can be resolved into sine waves or combinations of sine waves superimposed on each other. The sound track recorded at the edge of a motion picture film is a representation of a group of such superimposed sine waves. Radio stations send out electric waves composed of varying combinations of sine waves. Receiving sets pick up and change these waves into the sound we hear from the loudspeaker.

5-1. Introduction. The six trigonometric ratios were defined in the preceding chapter for every angle. Thus the sine ratio defines a correspondence between angles and numbers as follows. To every angle there corresponds a certain number, termed the sine of the angle. A similar correspondence between angles and numbers is defined by each of the other five trigonometric ratios.

In order to measure an angle, a unit of measurement must be chosen. In Sec. 4-3, two different units were discussed, the degree and the radian, and the relation between them was given by the formula

$$(1) \qquad 1 \text{ degree} = \frac{\pi}{180} \text{ radians.}$$

Thus if θ is any number,

$$(2) \qquad \theta \text{ degrees} = \frac{\pi}{180} \theta \text{ radians.}$$

When it is important that a given expression indicate the units of angular measure, we make use of the following agreement, which is adopted for the sake of convenience.

Consider an angle which contains x units.

(a) *If the unit is the degree, the measure of the angle is written with the degree sign ($^{\circ}$), as x° .*

(b) *If the unit is the radian, the measure of the angle is written without any sign, as x .*

Thus relation (1) can be written

$$(3) \quad 1^{\circ} = \frac{\pi}{180}$$

and relation (2) as

$$(4) \quad \theta^{\circ} = \frac{\pi}{180} \theta.$$

From (4), for example, 90° can be given in radians by $90^{\circ} = \frac{\pi}{180} \cdot 90$
 $= \frac{\pi}{2} = 1.5708.$

In trigonometric relations in which the unit of angular measurement is of no immediate interest, as in Chapter 10, the convention given above is ignored.

5-2. The Graph of $y = \sin x$. To each angle, and therefore to the number giving its measure, there corresponds a number which is the sine of the angle. Therefore, the functional notation and the graphical representation given in Chapter 3 can be used. Thus, if x° denotes a variable angle measured in degrees, $y = \sin x^{\circ}$ is a function of the angle x° , and we may write

$$y = f(x^{\circ}) = \sin x^{\circ}.$$

The graph of the function $y = \sin x^{\circ}$ can be drawn (see Fig. 5-1), using the values given in the following table.

x°	-180°	-150°	-120°	-90°	-60°	-30°	0°	30°	60°	90°	120°	150°	180°
$\sin x^{\circ}$	0	-0.50	-0.87	-1.00	-0.87	-0.50	0	0.50	0.87	1.00	0.87	0.50	0

In finding the above table of values for $\sin x^{\circ}$, the angle x° was measured in degrees. In this chapter, however, we shall use almost exclusively the radian measure of an angle, since this is more convenient for theoretical demonstrations and very useful in practical work.

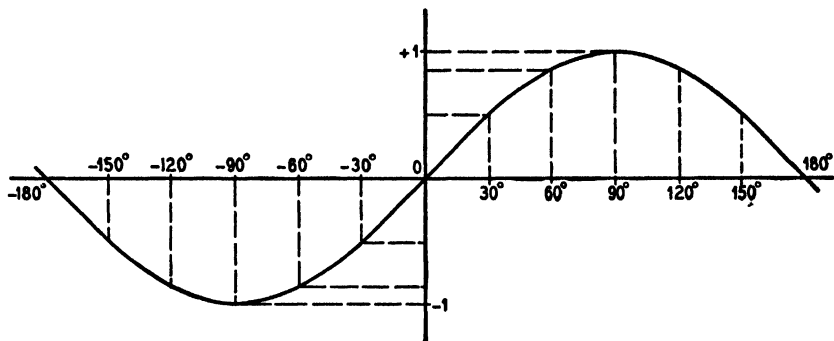


FIG. 5-1.

If x is an angle measured in radians, $y = \sin x$ is a function of the angle x and we have, as above,

$$y = f(x) = \sin x.$$

The graph of the function $y = \sin x$ will now be plotted from a table of values in which the angle x is measured in radians. In order to construct such a table of pairs of numbers from which to plot the graph of $y = \sin x$, it is convenient to pick the values of x corresponding to the principal angles. Without using the tables in the appendix, the values of y can be found by the method explained in Sec. 4-8. The table of values from which the curve of Fig. 5-2 was plotted is given below.

x	y	x	y
0	0 = 0.000	$\frac{7\pi}{6}$	$-\frac{1}{2} = -0.500$
$\frac{\pi}{6}$	$\frac{1}{2} = 0.500$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} = -0.707$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} = 0.707$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} = -0.866$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{3\pi}{2}$	$-1 = -1.000$
$\frac{\pi}{2}$	1 = 1.000	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} = -0.866$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} = -0.707$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} = 0.707$	$\frac{11\pi}{6}$	$-\frac{1}{2} = -0.500$
$\frac{5\pi}{6}$	$\frac{1}{2} = 0.500$	2π	0 = 0.000
π	0 = 0.000		

In Fig. 5-2 the unit on the x -axis is the same as the unit on the y -axis, whereas in Fig. 5-1 the unit on the y -axis is much larger than the unit on the x -axis.

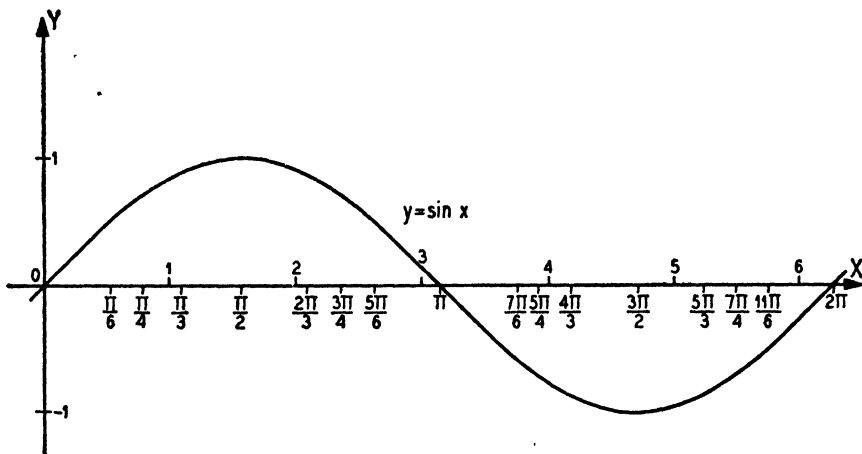


FIG. 5-2.

For values of x from 2π to 4π the function repeats the cycle of values given in the table, and hence the curve repeats between 2π and 4π the pattern shown in Fig. 5-2 between 0 and 2π . The reader should verify this statement by extending the table of values given above. The graph also repeats this pattern between -2π and 0, and continues to repeat

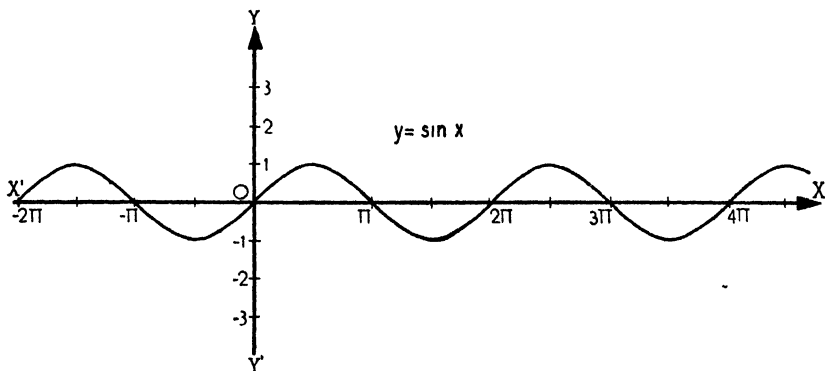


FIG. 5-3.

this pattern in both directions from the origin as far as we choose to plot the graph. A more extended graph is shown in Fig. 5-3. From these graphs it is seen that the maximum value of the function is $+1$, and the minimum is -1 .

5-3. The Graph of $y = \cos x$. The curve is shown in Fig. 5-4. The reader should construct a table of values and verify, by plotting the curve himself, that the curve shown is the correct one. That the pattern

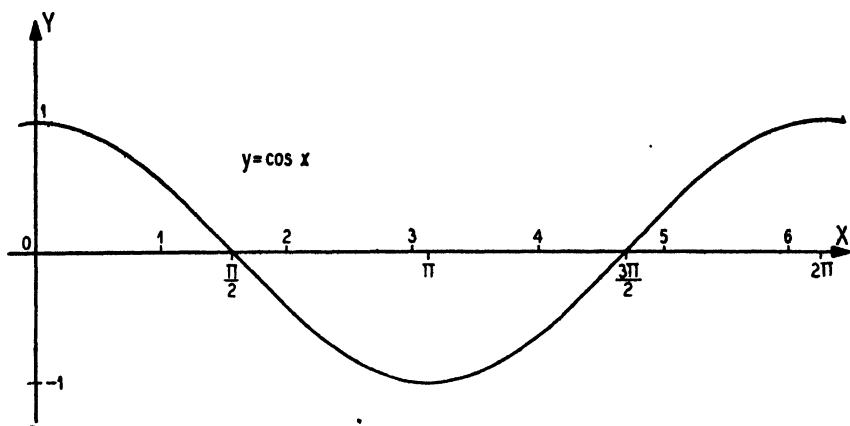


FIG. 5-4.

of the curve repeats is shown in Fig. 5-5. From these graphs it is seen that the maximum value of the function is $+1$ and the minimum value is -1 .

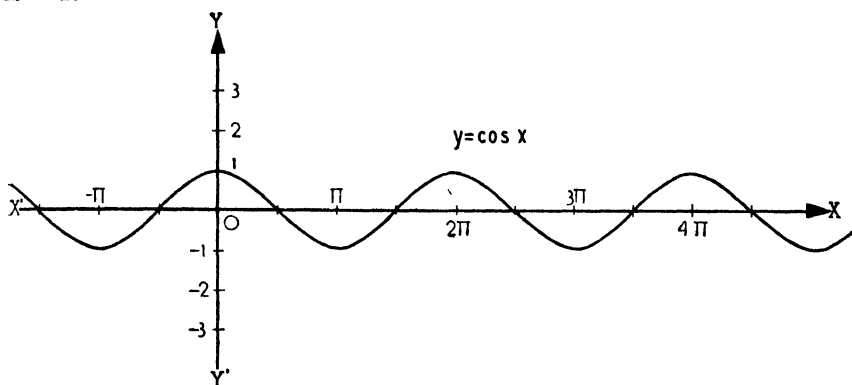


FIG. 5-5.

5-4. The Graph of $y = \tan x$. A table of values of y for values of x corresponding to the principal angles is given below. The reader should verify the values given in this table by the method of Sec. 4-8.

An examination of this table reveals that if the graph is to be accurately drawn for values of x between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, the behavior of the

TABLE 1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
y	Not defined	-1.732	-1.000	-0.577	0.000	0.577	1.000

x	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
y	1.732	Not defined	-1.732	-1.000	-0.577	0.000	0.577

x	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
y	1.000	1.732	Not defined	-1.732	-1.000	-0.577	0.000

function in this interval must be studied in greater detail. In order to investigate the behavior of this function here, Table 2 is constructed

TABLE 2

CORRESPONDING
ANGLE IN
DEGREES

x	DEGREES	y
1.047	60°	1.732
1.222	70°	2.747
1.309	75°	3.732
1.396	80°	5.671
1.449	83°	8.144
1.484	85°	11.43
1.518	87°	19.08
1.536	88°	28.64
1.553	89°	57.29
1.562	89.5°	114.6
1.567	89.8°	286.5
1.5706	89.99°	5730.

TABLE 3

CORRESPONDING
ANGLE IN
DEGREES

x	DEGREES	y
2.094	120°	-1.732
1.920	110°	-2.747
1.833	105°	-3.732
1.745	100°	-5.671
1.693	97°	-8.144
1.658	95°	-11.43
1.623	93°	-19.08
1.606	92°	-28.64
1.588	91°	-57.29
1.580	90.5°	-114.6
1.574	90.2°	-286.5
1.5710	90.01°	-5730.

from tables of trigonometric functions. By the method of Sec. 4-11, it is possible to obtain Table 3 from Table 2.

From the above Table 2, as x approaches $\frac{\pi}{2} = 1.570796$ from the left, through values less than $\frac{\pi}{2}$, $\tan x$ increases very rapidly, getting extremely large when x is close to $\frac{\pi}{2}$. Similarly, from Table 3, as x approaches $\frac{\pi}{2}$ from the right, through values greater than $\frac{\pi}{2}$, $\tan x$ decreases very rapidly, assuming extremely large negative values when x

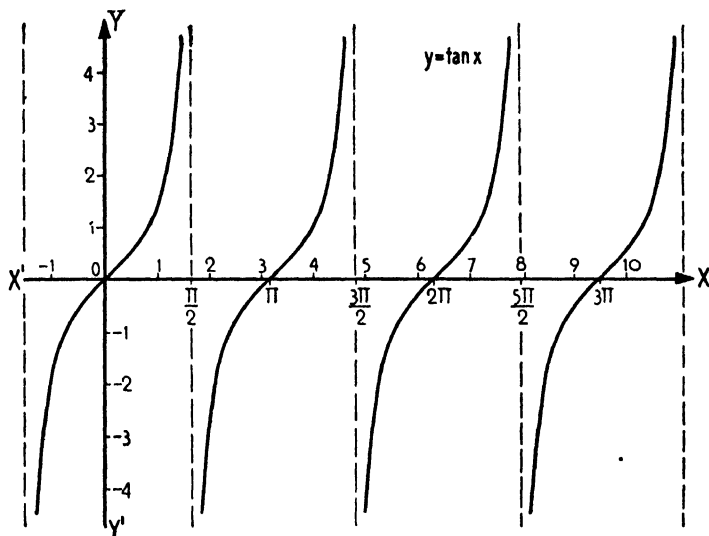


FIG. 5-6.

is close to $\frac{\pi}{2}$. Since $\tan \frac{\pi}{2}$ is not defined, we conclude that the curve is not connected over this point. Discussions paralleling this one can be given for the behavior of $\tan x$ near $x = -\frac{\pi}{2}$, and $x = \frac{3\pi}{2}$. The curve is plotted in Fig. 5-6. A dotted line is drawn perpendicular to the x -axis at each point where $\tan x$ is not defined.

To describe the behavior of the function $y = \tan x$ for values of x close to $\frac{\pi}{2}$ we say that *the function $\tan x$ becomes positively infinite as x approaches $\frac{\pi}{2}$ from the left, and that it becomes negatively infinite as x approaches $\frac{\pi}{2}$ from the right.* It is also sometimes stated by saying that

$\tan x$ approaches positive infinity as x approaches $\frac{\pi}{2}$ from the left. The phrase **positive infinity** can be replaced by the symbol $+\infty$, and then the second statement given above, while read as before, can be written as $\tan x$ approaches $+\infty$ as x approaches $\frac{\pi}{2}$ from the left. Similarly we can write that $\tan x$ approaches $-\infty$ as x approaches $\frac{\pi}{2}$ from the right.

In general a function $f(x)$ may show, in the neighborhood of a point x_0 , a behavior similar to that of $\tan x$ at $x = \frac{\pi}{2}$, in four ways.

(a) $f(x)$ may become positively infinite as x approaches x_0 from both the right and the left (Fig. 5-7a).

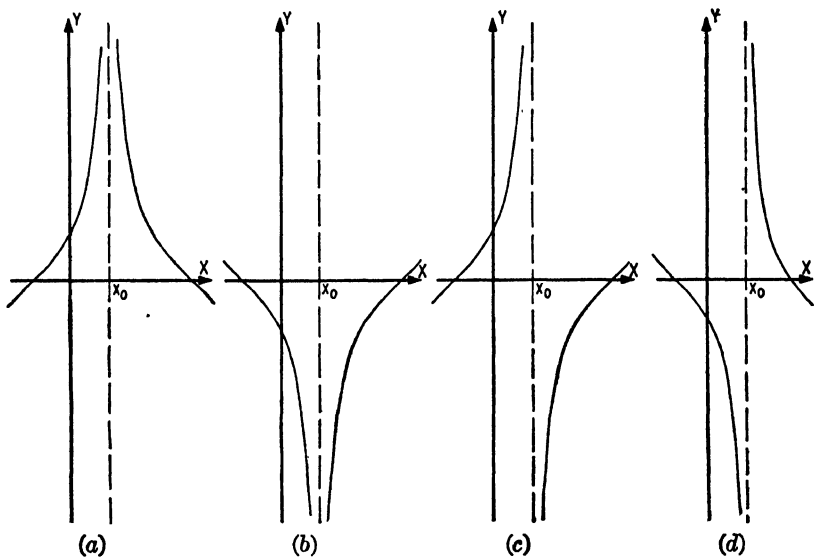


FIG. 5-7.

(b) $f(x)$ may become negatively infinite as x approaches x_0 from both the right and the left (Fig. 5-7b).

(c) $f(x)$ may become positively infinite as x approaches x_0 from the left and negatively infinite as x approaches x_0 from the right (Fig. 5-7c).

(d) $f(x)$ may become positively infinite as x approaches x_0 from the right and negatively infinite as x approaches x_0 from the left (Fig. 5-7d).

In all these cases the absolute value of $f(x)$ becomes positively infinite as x approaches x_0 from both right and left. In all these cases we say that $f(x)$ becomes infinite or approaches infinity at the point x_0 . [We

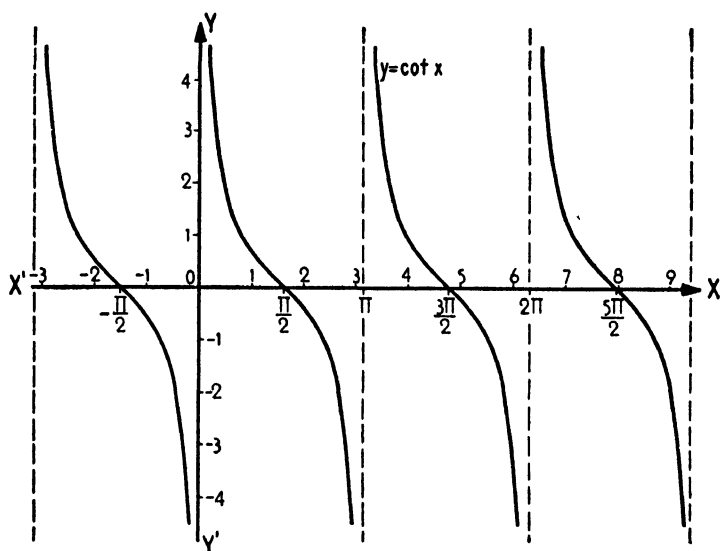


FIG. 5-8.

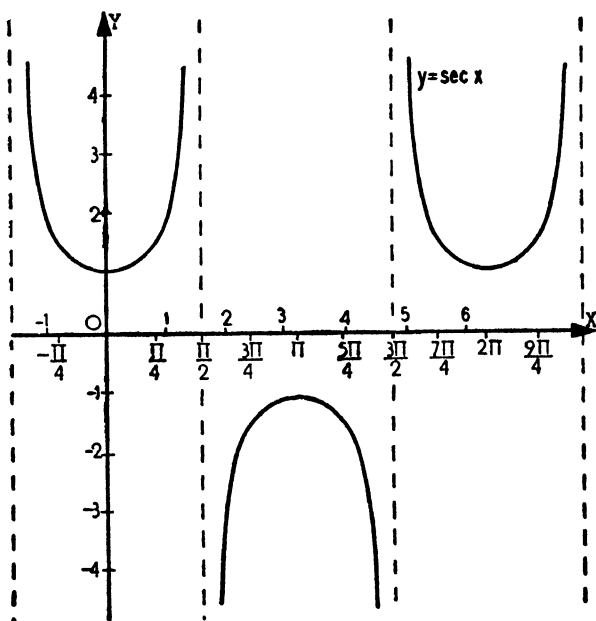


FIG. 5-9.

abbreviate this statement by the notation $f(x_0) = \infty$.] Thus we may write

$\tan \frac{\pi}{2} = \infty$ and the graphs in the next section will show that $\cot 0 = \infty$,

$\sec \frac{\pi}{2} = \infty$, and $\csc 0 = \infty$.

The symbol ∞ is not a number. It furnishes simply a convenient abbreviation which can be used in describing the behavior of a function.

5-5. The Graphs of the Cotangent, Secant, and Cosecant Curves. The graph of $y = \cot x$ is shown in Fig. 5-8. A discussion similar to

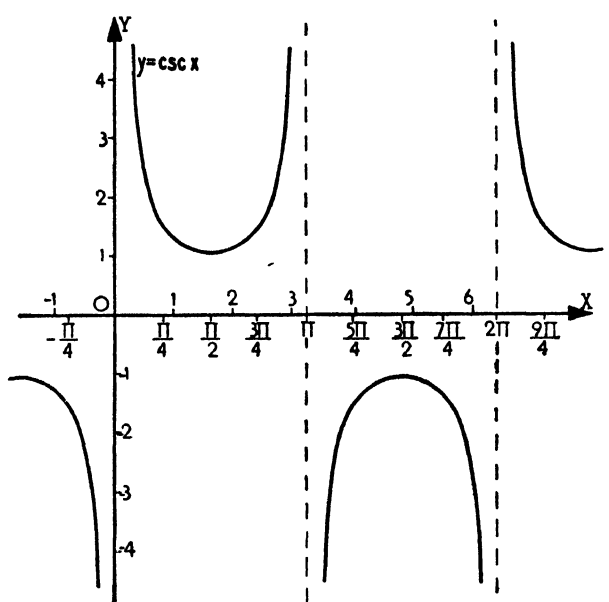


FIG. 5-10.

that for the tangent curve can be made to establish that the function has the graph shown. Similar discussions apply for the functions $y = \sec x$ and $y = \csc x$, whose graphs are shown in Fig. 5-9 and Fig. 5-10 respectively.

EXERCISES

1. Verify that Fig. 5-3 gives the correct graph of $y = \sin x$ by extending the table of values in Sec. 5-2 to include values of x between -2π and 0, and between 2π and 6π . Use the values of x corresponding to the principal angles in setting up the table. Explain why the pattern of the curve given for x between 0 and 2π is repeated to the right and left.

2. Construct a table of values for $y = \cos x$ for the same values of x given in the table in Sec. 5-2. Measure off these values on Fig. 5-4 to verify that this curve is correctly plotted.

3. Extend the table of values for $y = \cos x$ obtained in Exercise 2 to include values of x between -2π and 0, and between 2π and 6π . Use the values of x corresponding to the principal angles in setting up the table. Use this table to verify that Fig. 5-5 is correctly drawn. Explain why the pattern of the curve given for x between 0 and 2π is repeated to the right and left.

4. Verify Table 1 of Sec. 5-4 by constructing the principal angles corresponding to the given values of x and by finding from these figures the values of $\tan x$.

5. Study the behavior of $\tan x$ between $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ by constructing tables like Table 2 and Table 3 of Sec. 5-4. What conclusions do you draw from this data? Verify that Fig. 5-6 agrees with your conclusions.

6. Prepare a table of values for $y = \cot x$ like Table 1 of Sec. 5-4. Measure off the values you obtain on Fig. 5-8 to verify that this curve is correctly plotted.

7. Study the behavior of $\cot x$ between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$ by constructing tables like Table 2 and Table 3 of Sec. 5-4. Does Fig. 5-8 bear out your conclusions from these tables?

8. Prepare a table of values for $y = \sec x$ like Table 1 of Sec. 5-4. Measure off the values you obtain on Fig. 5-9 to verify that this curve is correctly plotted.

9. Study the behavior of $\sec x$ between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ by constructing tables like Table 2 and Table 3 of Sec. 5-4. Does Fig. 5-9 bear out your conclusions from these tables?

10. Prepare a table of values for $y = \csc x$ like Table 1 of Sec. 5-4. Measure off the values you obtain on Fig. 5-10 to verify that this curve is correctly plotted.

11. Study the behavior of $\csc x$ between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$ by constructing tables like Table 2 and Table 3 of Sec. 5-4. Does Fig. 5-10 bear out your conclusions from these tables?

Plot the curve of each of the following functions on a full page, using exactly the same units on both axes. To plot the curves, make a table using the values of x corresponding to the principal angles between $-\frac{\pi}{2}$ and $\frac{5\pi}{2}$.

12. $y = \sin x$.

13. $y = \cos x$.

14. $y = \tan x$.

15. $y = \cot x$.

16. $y = \sec x$.

17. $y = \csc x$.

5-6. The Trigonometric Functions as Coordinates of Points. Consider the angle θ in standard position shown in Fig. 5-11, θ being measured in radians. The terminal side of θ which falls in the first quadrant intersects the circle with unit radius and center at the origin at $P(x, y)$. This circle is called the **unit circle**. By dropping the perpendicular PL from P to the x -axis we obtain a triangle of reference OPL for the angle θ .

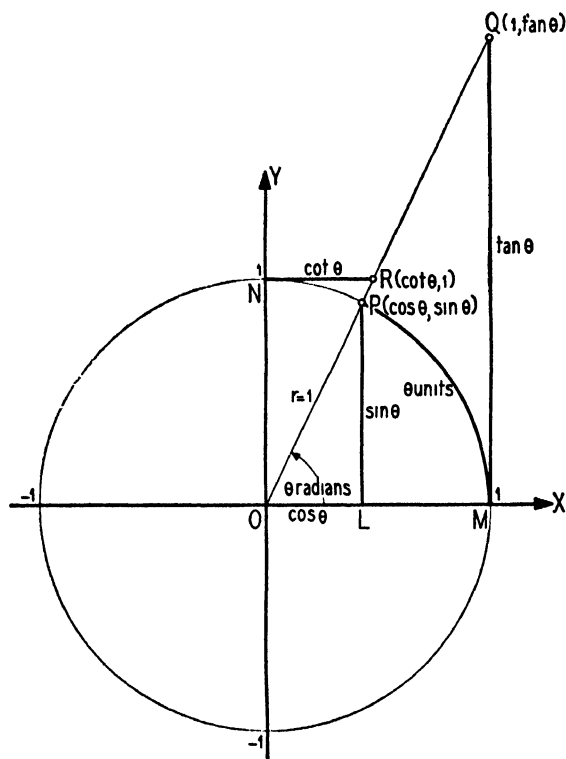


FIG. 5-11.

Now, since the circle has unit radius, $r = \sqrt{x^2 + y^2} = 1$, and the arc PM has length θ , then by definition

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y,$$

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x.$$

Thus the ordinate of P is $\sin \theta$, the abscissa of P is $\cos \theta$, and we can write $P(\cos \theta, \sin \theta)$. The argument and its results are precisely the same regardless of the quadrant in which the terminal side of θ falls. The reader may verify this by repeating it exactly as given above in connection with the triangles OPL in Figs. 5-12, 5-13, and 5-14.

At M , the intersection of the unit circle and the positive x -axis, erect a perpendicular. The intersection of this perpendicular and the terminal side of θ is $Q(x_1, y_1)$. By a theorem of plane geometry QM is then tangent to the circle. We see also that $x_1 = 1$. These statements apply

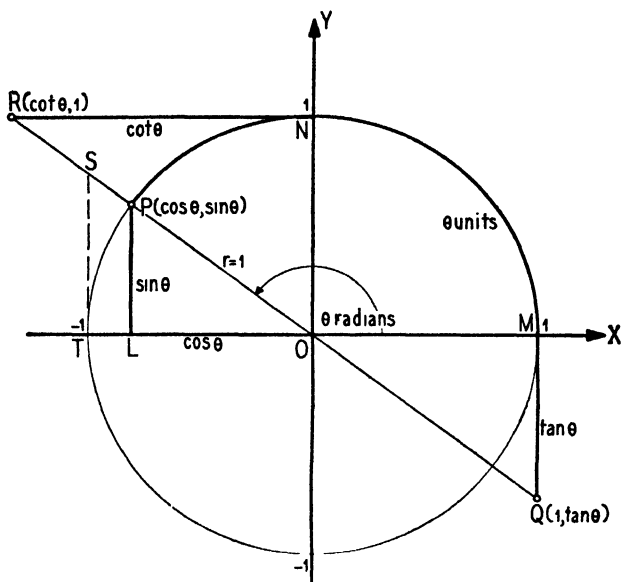


FIG. 5-12.

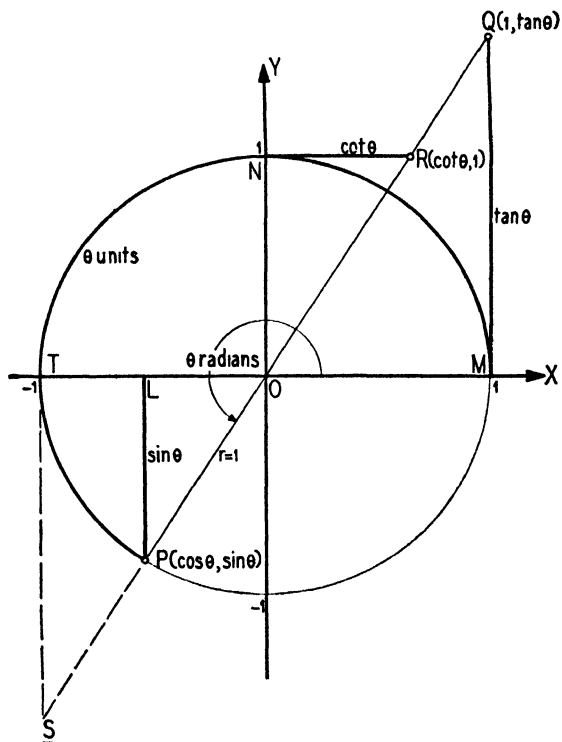


FIG. 5-13.

to Figs. 5-11, 5-12, 5-13, and 5-14. Now, in Fig. 5-11 and Fig. 5-14, the triangle OQM is a triangle of reference for θ . Then

$$\tan \theta = \frac{y_1}{x_1} = \frac{y_1}{1} = y_1.$$

In Figs. 5-12 and 5-13, erect a perpendicular at T , the intersection of the unit circle and the negative x -axis. Let the intersection of the

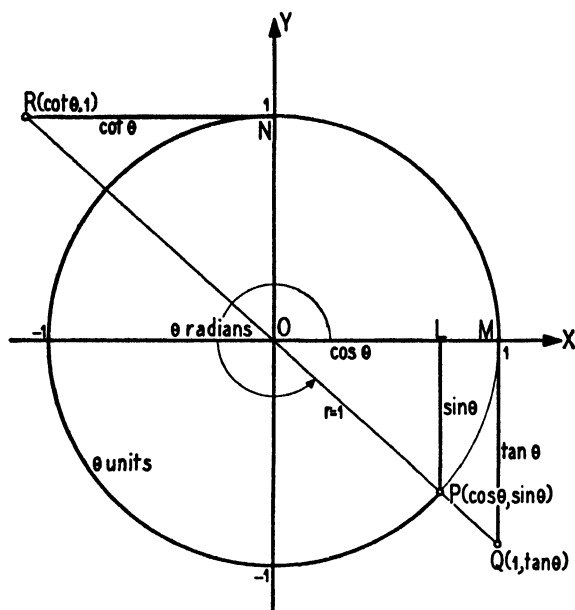


FIG. 5-14.

terminal side of θ and this perpendicular be $S(x_2, y_2)$. We see that $x_2 = -1$, and that the triangle OST forms a triangle of reference for θ . Then

$$\tan \theta = \frac{y_2}{x_2} = \frac{y_2}{-1} = -y_2.$$

Obviously triangle OST is congruent to triangle OQM , consequently the side ST has the same length as the side MQ . Then the ordinate of Q and the ordinate of S have the same absolute value but differ in sign, or $y_1 = -y_2$. Since this is so,

$$\tan \theta = -y_2 = y_1.$$

Hence in all four cases,

$$\tan \theta = y_1,$$

or the ordinate of Q is $\tan \theta$, and we may write $Q(1, \tan \theta)$.

At N , the intersection of the positive y -axis and the unit circle, erect a perpendicular. The intersection of this perpendicular and the terminal side of θ is $R(x_3, y_3)$. We see that RN is tangent to the circle, and that for any θ , $y_3 = 1$. It can be shown easily that triangle OPL is similar to triangle ORN . Using triangle OPL as a reference triangle for θ ,

$$|\cot \theta| = \left| \frac{x}{y} \right| = \left| \frac{NR}{ON} \right| = \left| \frac{x_3}{1} \right|,$$

or

$$|\cot \theta| = |x_3|.$$

In the first and third quadrants $\cot \theta$ is positive, and we see that x_3 is positive from Fig. 5-11 and Fig. 5-13; in the second and fourth quadrants $\cot \theta$ is negative, and we see that x_3 is negative from Fig. 5-12 and Fig. 5-14. Since $|\cot \theta| = |x_3|$, and x_3 and $\cot \theta$ have always the same sign, we may write

$$\cot \theta = x_3.$$

Thus the abscissa of R is $\cot \theta$, and we can write $R(\cot \theta, 1)$.

The reader can show easily that $\sec \theta$ equals the length of OQ , if we measure OQ as positive when Q and P are on the same side of the origin and as negative when the origin lies between P and Q . It can also be shown that $\csc \theta$ equals the length OR , if we measure OR as positive when R and P are on the same side of the origin and as negative when the origin lies between R and P .

In summary, we have:

$\sin \theta$ equals the ordinate of the point where the terminal side of θ intersects the unit circle.

$\cos \theta$ equals the abscissa of the point where the terminal side of θ intersects the unit circle.

$\tan \theta$ equals the ordinate of the point where the terminal side of θ intersects the line which is tangent to the unit circle at the point where this circle intersects the positive x -axis.

$\cot \theta$ equals the abscissa of the point where the terminal side of θ intersects the line which is tangent to the unit circle at the point where this circle intersects the positive y -axis.

$\sec \theta$ equals the length of the segment from the origin to the point where the terminal side of θ intersects the line which is tangent to the unit circle at the point where this circle intersects the positive x -axis, this length being positive if θ terminates on this segment, negative otherwise.

$\csc \theta$ equals the length of the segment from the origin to the point where the terminal side of θ intersects the line which is tangent to the unit circle

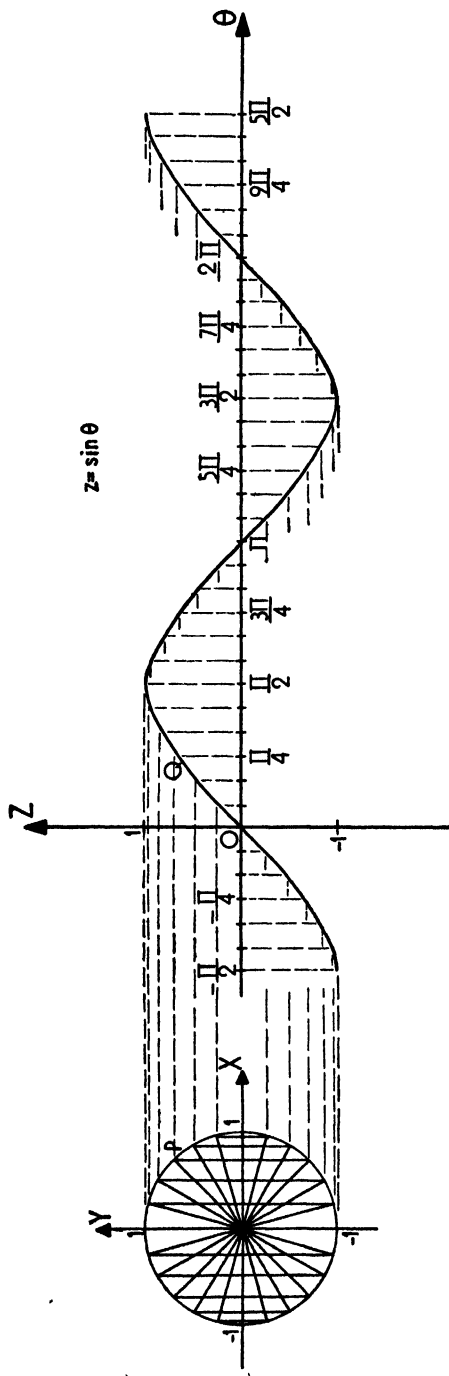


Fig. 5-15.

at the point where this circle intersects the positive y -axis, this length being positive if θ terminates on this segment, negative otherwise.

In this way all the trigonometric functions can be interpreted as line lengths, indicated by the dark lines on the figures. This interpretation is called **the line representation of the trigonometric functions**.

Historically, definitions of the trigonometric functions were given much as we have established them in this section long before the ratio definitions were established. The origin of the names tangent and secant becomes apparent from this interpretation of the functions. It also indicates why the trigonometric functions are called the **circular functions**. This graphical representation of the functions will be of much use in studying their properties.

5-7. Geometrical Method of Plotting the Graphs of the Trigonometric Functions. The information developed in the preceding section can be used very effectively in constructing the graphs of the trigonometric functions. To illustrate, we shall plot the graph of $z = \sin \theta$. Place the x - y coordinate system and the z - θ coordinate system

(as shown in Fig. 5-15) so that the x -axis and the θ -axis are on the same line. Then use the same unit on the axes of both coordinate systems. Draw a unit circle with center at the origin of the x - y system. Now if an angle θ is drawn in standard position on the x - y system, $\sin \theta$ is the ordinate of the point where the terminal side of θ intersects the unit circle. Thus this ordinate value is the distance above the θ -axis at which the point of the graph of $z = \sin \theta$ should be found which corresponds to the number θ on the θ -axis. The graph is constructed as shown in Fig. 5-15.

Consider, for example, $\theta = \frac{\pi}{4}$. The terminal side of this angle intersects the unit circle at P . Draw a line through P parallel to the horizontal axes. Draw another line perpendicular to the θ -axis at $\theta = \frac{\pi}{4}$.

The intersection Q of these two lines is a point on the graph of $z = \sin \theta$.

This method, with suitable modifications, may be used to plot each of the trigonometric functions.

EXERCISES

Use the method described in this section to plot the graphs given in the exercises below. Plot each graph on a full page, using ruler and compasses.

1. Plot the graph of $z = \sin \theta$, as shown in Fig. 5-15.
2. Plot the graph of $z = \cos \theta$. In order to use the method described, it will be convenient to rotate the x - and y -axes of Fig. 5-15 through $+90^\circ$.
3. Plot the graph of $z = \tan \theta$.
4. Plot the graph of $z = \cot \theta$. It will be convenient to use the same arrangement of axes as in Exercise 2.
5. Plot the graph of $z = \sec \theta$. In this case a direct construction cannot be made as in the previous exercises. However, a compass or divider can be used to transfer the distances from diagram to graph.
6. Plot the graph of $z = \csc \theta$. The remarks in Exercise 5 apply here.

5-8. Variation of the Trigonometric Functions. From the graph of the sine function in Fig. 5-2, it is apparent that as θ increases from 0 to $\frac{\pi}{2}$, $\sin \theta$ increases from 0 to 1. As θ increases from $\frac{\pi}{2}$ to π , $\sin \theta$ decreases

from 1 to 0. As θ increases from π to $\frac{3\pi}{2}$, $\sin \theta$ decreases from 0 to -1 .

As θ increases from $\frac{3\pi}{2}$ to 2π , $\sin \theta$ increases from -1 to 0. These facts should be carefully verified also from figures like those of Figs. 5-11, 5-12, 5-13, and 5-14. Figure 5-15 can also be used in this connection. Similar studies will verify the information in the following table.

	FROM	TO	FROM	TO	FROM	TO	FROM	TO
θ	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	π	π	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	1	1	0	0	-1	-1	0
$\cos \theta$	1	0	0	-1	-1	0	0	1
$\tan \theta$	0	$+\infty$	$-\infty$	0	0	$+\infty$	$-\infty$	0
$\cot \theta$	$+\infty$	0	0	$-\infty$	$+\infty$	0	0	$-\infty$
$\sec \theta$	1	$+\infty$	$-\infty$	-1	-1	$-\infty$	$+\infty$	1
$\csc \theta$	$+\infty$	1	1	$+\infty$	$-\infty$	-1	-1	$-\infty$

EXERCISES

1. Make a table of the variations of the trigonometric functions like the table of Sec. 5-8 as θ ranges from 2π to 4π . Compare this table with the table in Sec. 5-8. Explain the similarities you notice.

2. Make a table as directed in Exercise 1 as θ ranges from -2π to 0, and discuss as above. Explain the similarity of the tables arrived at in Exercises 1 and 2.

5-9. Amplitude of the Sine and Cosine. This chapter has indicated that the largest value assumed by $\sin \theta$ and $\cos \theta$ is $+1$, and that the smallest value assumed by these functions is -1 . Thus the maximum absolute value of these functions is $+1$.

Consider the function

$$(1) \quad z = A \sin \theta$$

where A is a constant. By this relation the values of $\sin \theta$ are multiplied by A , and hence the graph of (1) is obtained by plotting the usual sine curve with the ordinates multiplied by A . Since the maximum absolute value of $\sin \theta$ is 1, the maximum absolute value of $A \sin \theta$, called its **amplitude** or **peak value**, is $|A|$. In like manner the amplitude of $A \cos \theta$ is also $|A|$.

The maximum absolute value of $z = A \sin \theta$ or $z = A \cos \theta$, called the *amplitude*, is $|A|$.

Since the other trigonometric functions become both positively and negatively infinite, it serves no useful purpose to define an *amplitude* for these functions.

The graph of $z = A \sin \theta$ or $z = A \cos \theta$ can be drawn by simply sketching the usual sine or cosine curve with the ordinates multiplied by A . Since the general shapes of these curves are well known, the sketch can be made very easily without plotting any points. Care should be taken, however, that the curves cross the horizontal axis in the proper places. Similar remarks apply to the other functions.

Example 1. Sketch the graph of $z = 3 \sin \theta$.

The amplitude of this function is 3, and the ordinates of its graph are 3 times those of the usual sine curve. Hence we can sketch the curve by drawing a sine curve, making it 3 times as tall as usual. The graph is shown in Fig. 5-16.

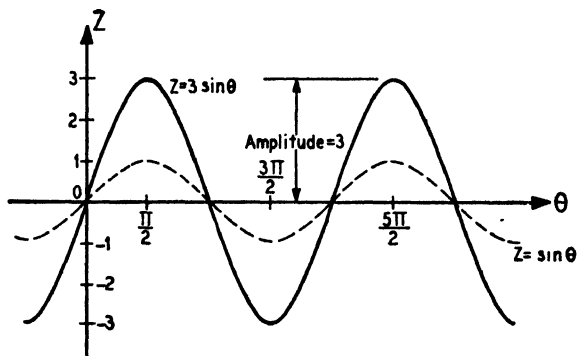


FIG. 5-16.

Example 2. Sketch the graph of $z = -\frac{1}{2} \cos \theta$.

The amplitude of this function is $\frac{1}{2}$. Since the minus sign is present, the cosine curve is turned upside down, with its ordinates halved. The graph is shown in Fig. 5-17.

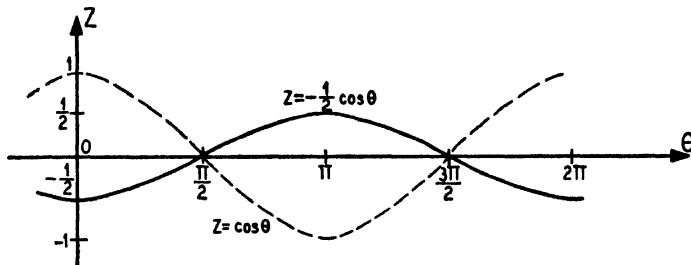


FIG. 5-17.

EXERCISES

State the amplitude of each function, if it has one. Sketch the graph of each function, labeling the axes carefully, especially those points at which the curve crosses either axis.

- | | | |
|-------------------------------------|----------------------------|-------------------------------------|
| 1. $z = 2 \sin \theta$. | 2. $z = -\sin \theta$. | 3. $z = 3 \cos \theta$. |
| 4. $z = -2 \cos \theta$. | 5. $z = 5 \sin \theta$. | 6. $z = \frac{1}{2} \cos \theta$. |
| 7. $z = -\frac{1}{2} \sin \theta$. | 8. $z = 10 \cos \theta$. | 9. $z = -6 \cos \theta$. |
| 10. $z = -4 \sin \theta$. | 11. $z = 3 \sin \theta$. | 12. $z = 10 \sin \theta$. |
| 13. $z = 2 \tan \theta$. | 14. $z = -3 \tan \theta$. | 15. $z = \frac{1}{2} \sec \theta$. |
| 16. $z = 4 \csc \theta$. | 17. $z = 2 \cot \theta$. | 18. $z = 5 \cot \theta$. |
| 19. $z = -3 \sec \theta$. | 20. $z = 4 \sec \theta$. | 21. $z = -2 \csc \theta$. |
| 22. $z = -\cot \theta$. | 23. $z = -3 \cot \theta$. | 24. $z = -10 \sec \theta$. |

5-10. The Periods of the Trigonometric Functions. We have already noticed that the pattern of certain parts of the graphs of the trigonometric functions is repeated in other sections of the graph. This paragraph will study this phenomenon more closely. The discussion will be based upon the following definition.

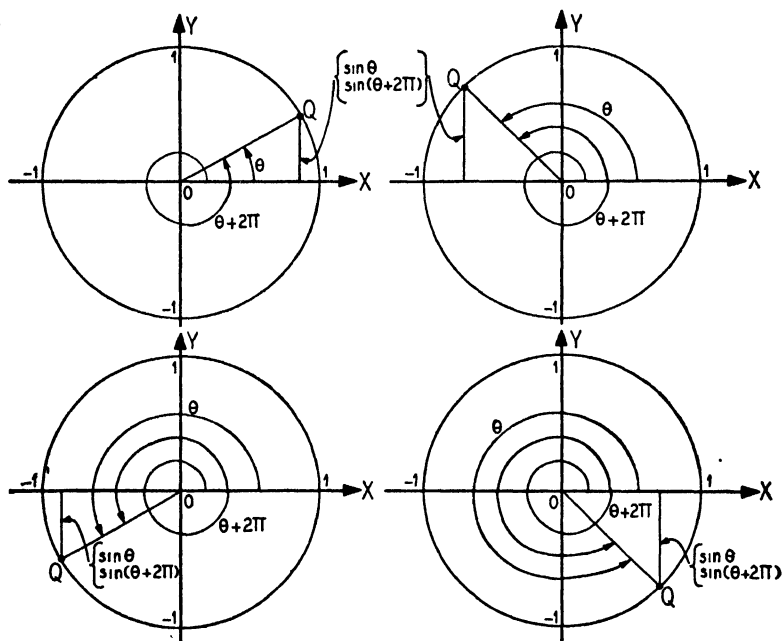


FIG. 5-18.

If $f(x)$ is such that $f(x + P) = f(x)$ for all values of x , then $f(x)$ is said to be periodic and have a period P . The smallest positive period is called the **fundamental period**.

Figure 5-18 shows that θ and $\theta + 2\pi$ have the same terminal side. This terminal side cuts the unit circle at Q , and the ordinate of Q is then $\sin \theta$ and also $\sin(\theta + 2\pi)$. Hence $\sin \theta = \sin(\theta + 2\pi)$, and since Fig. 5-18 gives us representatives of all possible positions of the terminal side, this equality is true for all values of θ . Hence $\sin \theta$ is periodic with period 2π .

It can also be shown that $\sin \theta$ has period -2π , 4π , 6π , and in general any integral multiple of 2π . However 2π is the smallest positive number which is a period of $\sin \theta$, and hence 2π is the **fundamental period** of $\sin \theta$.

Similar arguments will establish the information given in the table below.

Function	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
Fundamental period	2π	2π	π	π	2π	2π

Since we are most often concerned with the fundamental period of a trigonometric function, the adjective *fundamental* is often omitted, the context indicating the proper meaning to be attached to the term "period."

Consider the curve $z = \sin \theta$ in Fig. 5-19. Since the equation $\sin(\theta + 2\pi) = \sin \theta$ holds for all values of θ , the ordinate at θ on the graph is the same as the ordinate at $(\theta + 2\pi)$. In this way we see that the curve between any angle α and $\alpha + 2\pi$ is repeated between $(\alpha + 2\pi)$ and $(\alpha + 4\pi)$. Thus any section of horizontal length 2π can be used

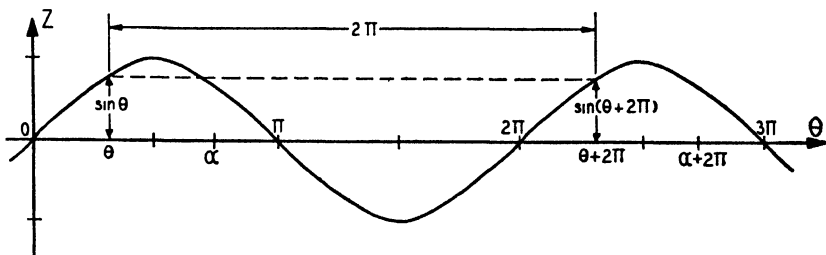


FIG. 5-19.

as a pattern for the rest of the curve. The period of the other trigonometric functions may also be interpreted in this way. The reader should construct figures for the other functions like Fig. 5-19, using the fundamental periods.

EXERCISES

By using the line representation of the trigonometric functions as in Fig. 5-18, show that the following functions have the periods given.

- | | | |
|----------------------------|----------------------------|----------------------------|
| 1. $\sin \theta, 2\pi$. | 2. $\cos \theta, 2\pi$. | 3. $\tan \theta, \pi$. |
| 4. $\cot \theta, \pi$. | 5. $\sec \theta, 2\pi$. | 6. $\csc \theta, 2\pi$. |
| 7. $\sin \theta, 4\pi$. | 8. $\cos \theta, 4\pi$. | 9. $\tan \theta, 2\pi$. |
| 10. $\cot \theta, 2\pi$. | 11. $\sec \theta, 4\pi$. | 12. $\csc \theta, 4\pi$. |
| 13. $\sin \theta, 6\pi$. | 14. $\cos \theta, -2\pi$. | 15. $\sin \theta, -2\pi$. |
| 16. $\tan \theta, -\pi$. | 17. $\cot \theta, -\pi$. | 18. $\sec \theta, -2\pi$. |
| 19. $\csc \theta, -2\pi$. | 20. $\sin \theta, 10\pi$. | |

In the following exercises, n is any integer.

- | | | |
|----------------------------|----------------------------|----------------------------|
| 21. $\sin \theta, 2n\pi$. | 22. $\cos \theta, 2n\pi$. | 23. $\tan \theta, n\pi$. |
| 24. $\cot \theta, n\pi$. | 25. $\sec \theta, 2n\pi$. | 26. $\csc \theta, 2n\pi$. |

5-11. The Period of $\sin \omega x$. Consider the function $\sin \omega x$ where ω is a constant greater than zero. Since the sine function has period 2π ,

$$(1) \quad \sin \omega x = \sin (\omega x + 2\pi),$$

or

$$(2) \quad \sin \omega x = \sin \omega \left(x + \frac{2\pi}{\omega} \right).$$

Equation (2) may be interpreted, if we let $f(x) = \sin \omega x$, as meaning that

$$f(x) = f\left(x + \frac{2\pi}{\omega}\right).$$

Thus $\sin \omega x$ has a period of $\frac{2\pi}{\omega}$. Furthermore, it can be shown that this is the fundamental period of the function.

In like manner the information in the following table can be established:

Function	$\sin \omega x$	$\cos \omega x$	$\tan \omega x$	$\cot \omega x$	$\sec \omega x$	$\csc \omega x$
Fundamental period	$\frac{2\pi}{\omega}$	$\frac{2\pi}{\omega}$	$\frac{\pi}{\omega}$	$\frac{\pi}{\omega}$	$\frac{2\pi}{\omega}$	$\frac{2\pi}{\omega}$

Now $\sin x$ and $\sin \omega x$ run through the same cycle of values, but $\sin x$ completes a cycle in a horizontal distance of 2π , and $\sin \omega x$ in a horizontal distance of $\frac{2\pi}{\omega}$. Since both cross the y -axis at the origin, the graph of the second function may be sketched very easily from what we already know about the first. The following examples will illustrate.

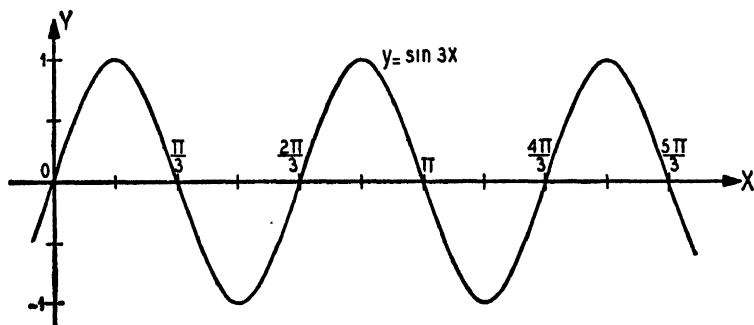


FIG. 5-20.

Example 1. Sketch the curve $y = \sin 3x$.

Here $\omega = 3$ and the function has period $\frac{2\pi}{3}$. This means that one cycle of the sine curve appears in a horizontal length of $\frac{2\pi}{3}$. The y -axis is crossed at the origin.

Thus to plot the graph we can lay off on the x -axis multiples of the length $\frac{2\pi}{3}$. Each of these segments can be divided in four equal parts by points corresponding to where the curve crosses the axis or reaches a maximum or minimum. The curve then can be sketched easily, as shown in Fig. 5-20.

Example 2. Sketch the curve $y = 2 \cos \frac{1}{2}x$.

Since $\omega = \frac{1}{2}$, the function has period $\frac{2\pi}{\frac{1}{2}} = 4\pi$. Its amplitude is 2. The curve crosses the y -axis at point $y = 2$. The graph is plotted in Fig. 5-21.

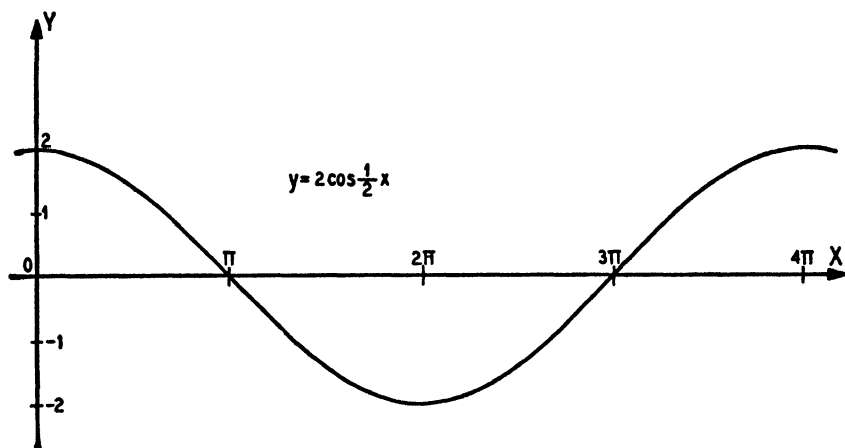


FIG. 5-21.

EXERCISES

State the fundamental period of each function and its amplitude, if it has one. Sketch the graph of each function, labeling the axes carefully, especially those points at which the curve crosses either axis.

- | | | |
|---------------------------------|--------------------------------------|--------------------------------|
| 1. $y = \sin 2x$. | 2. $y = \cos 2x$. | 3. $y = \tan 2x$. |
| 4. $y = \cot 2x$ | 5. $y = 2 \sin 3x$. | 6. $y = \frac{1}{2} \cos 3x$. |
| 7. $y = -2 \sin 3x$. | 8. $y = \cos 4x$. | 9. $y = 3 \sin 4x$. |
| 10. $y = 5 \sin \frac{1}{2}x$. | 11. $y = -3 \cos \frac{1}{2}x$. | 12. $y = -2 \cos 2x$. |
| 13. $z = 3 \tan 2\theta$. | 14. $z = 2 \sin \frac{3}{2}\theta$. | 15. $z = \sin \pi\theta$. |
| 16. $z = \cos \pi\theta$. | 17. $z = \tan \pi\theta$. | 18. $z = \cot \pi\theta$. |
| 19. $z = \sec \pi\theta$. | 20. $z = \csc \pi\theta$. | 21. $y = 3 \sin 2\pi x$. |
| 22. $y = -2 \cos 2\pi x$. | 23. $y = 10 \sin \frac{2}{3}x$. | 24. $y = -\pi \sin \pi x$. |
| 25. $y = 5 \cos 6x$. | 26. $z = \frac{1}{2} \tan 3x$. | 27. $z = 3 \cot 2x$. |
| 28. $z = 5 \sec 2x$. | 29. $z = 6 \csc 5x$. | 30. $z = 8 \sec 3x$. |

5-12. Phase and Displacement. Consider the function

$$(1) \quad y = \sin(\omega x + \alpha)$$

where ω is a positive constant and α is a constant which may be either positive or negative.

The expression $\omega x + \alpha$ is called the **phase** of the function. The quantity α , the value of the phase for $x = 0$, is called the **initial phase** of the function.

Now

$$\sin (\omega x + \alpha) = \sin \omega \left(x + \frac{\alpha}{\omega} \right),$$

which can be used to advantage in setting up the table below. As x

x	$x + \frac{\alpha}{\omega}$	$\omega \left(x + \frac{\alpha}{\omega} \right)$	$\sin \omega \left(x + \frac{\alpha}{\omega} \right)$
$-\frac{\alpha}{\omega}$	0	0	0
$\frac{\pi}{2\omega} - \frac{\alpha}{\omega}$	$\frac{\pi}{2\omega}$	$\frac{\pi}{2}$	1
$\frac{\pi}{\omega} - \frac{\alpha}{\omega}$	$\frac{\pi}{\omega}$	π	0
$\frac{3\pi}{2\omega} - \frac{\alpha}{\omega}$	$\frac{3\pi}{2\omega}$	$\frac{3\pi}{2}$	-1
$\frac{2\pi}{\omega} - \frac{\alpha}{\omega}$	$\frac{2\pi}{\omega}$	2π	0

increases from $-\frac{\alpha}{\omega}$ to $\frac{\pi}{2\omega} - \frac{\alpha}{\omega}$, $\sin (\omega x + \alpha)$ increases from 0 to 1.

Examining the rest of the table in this way, we conclude that as x increases from $-\frac{\alpha}{\omega}$ to $\frac{2\pi}{\omega} - \frac{\alpha}{\omega}$, the function $\sin (\omega x + \alpha)$ completes one

cycle of values. Hence it has a period $\frac{2\pi}{\omega}$, as we already know, and

intersects the x -axis during this cycle at $-\frac{\alpha}{\omega}$, $\frac{1}{\omega}(\pi - \alpha)$, and $\frac{1}{\omega}(2\pi - \alpha)$.

Hence the graph of (1) is the graph of $y = \sin \omega x$ displaced along the x -axis a distance $\frac{\alpha}{\omega}$. The displacement is to the left if $\alpha > 0$, to the right if $\alpha < 0$. The quantity $\frac{\alpha}{\omega}$ is called the **displacement**

Similar arguments can be made for the other functions with the phase $\omega x + \alpha$ to show that the displacement in each case is $\frac{\alpha}{\omega}$, to the left if $\alpha > 0$, to the right if $\alpha < 0$.

5-13. Sketching the Graphs of Trigonometric Functions. The information we have gained about the graphs of the trigonometric functions is summarized in the following table.

FUNCTION	AMPLITUDE	PERIOD	DISPLACEMENT
$A \sin(\omega x + \alpha)$ $A \cos(\omega x + \alpha)$ $A \sec(\omega x + \alpha)$ $A \csc(\omega x + \alpha)$ $A \tan(\omega x + \alpha)$ $A \cot(\omega x + \alpha)$	$ A $	$\frac{2\pi}{\omega}$ $\frac{\pi}{\omega}$	$\frac{\alpha}{\omega}$ To the left if $\alpha > 0$. To the right if $\alpha < 0$.
$A \neq 0, \omega > 0, \alpha \text{ any value}$			

The information for the sine and cosine is also summarized in Fig. 5-22.

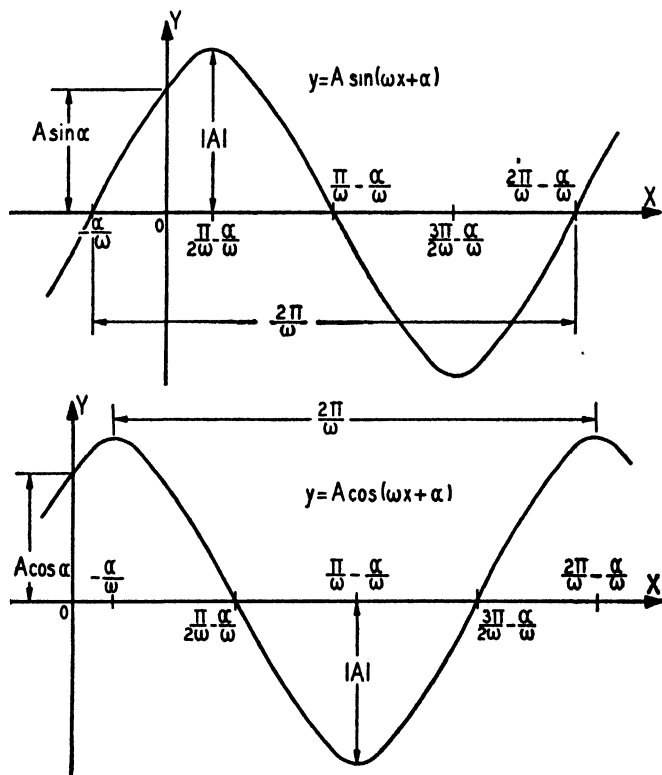


FIG. 5-22.

Using this information, it is possible to sketch easily and quickly the graphs of trigonometric functions. The procedure given below is suggested for the curves for $A \sin(\omega x + \alpha)$ and $A \cos(\omega x + \alpha)$. The reader

will infer immediately the slight modifications necessary in sketching the other functions.

To plot $y = A \sin (\omega x + \alpha)$ or $y = A \cos (\omega x + \alpha)$:

1. List the following data:

(a) Amplitude, $|A|$.

(b) Period, $\frac{2\pi}{\omega}$.

(c) Initial phase, α .

(d) Displacement, $\frac{\alpha}{\omega}$.

2. Locate $-\frac{\alpha}{\omega}$ on the x -axis and the point one period to its right, and then divide this segment into four equal parts. In this way we locate the five points with coordinates $-\frac{\alpha}{\omega}$, $\frac{1}{\omega}\left(\frac{\pi}{2} - \alpha\right)$, $\frac{1}{\omega}(\pi - \alpha)$, $\frac{1}{\omega}\left(\frac{3\pi}{2} - \alpha\right)$, $\frac{1}{\omega}(2\pi - \alpha)$. Between the end-points of this segment, the function completes the cycle that $A \sin \omega x$ (or $A \cos \omega x$) completes between 0 and $\frac{2\pi}{\omega}$.

3. Locate $\pm A$ on the y -axis.

4. Locate the maximum and minimum points on the curve, using the ordinates obtained in Step 3 and the appropriate abscissas obtained in Step 2. The abscissas obtained in Step 2 and not used here are the points at which the curve crosses the x -axis.

5. Sketch the graph of the function by drawing it freehand through the maximum and minimum points and the x -intercepts. By drawing the curve in lightly at first with short strokes of a pencil and then darkening as the shape becomes satisfactory, very good results can be achieved.

6. If more than one period of the function is desired, repeat the pattern obtained between $-\frac{\alpha}{\omega}$ and $\frac{1}{\omega}(2\pi - \alpha)$ to the right and left.

Example 1. Sketch the curve of $y = 2 \sin \left(3x + \frac{\pi}{2}\right)$.

For this example

Amplitude = 2.

Period = $\frac{2\pi}{3}$.

Initial phase = $\frac{\pi}{2}$.

Displacement = $\frac{\pi}{6}$.

First we locate $-\frac{\pi}{6}$ on the x -axis, and then $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$. This segment is divided into four parts by the points $0, \frac{\pi}{6}, \frac{\pi}{3}$. The maximum point is $(0, 2)$, and the minimum point is $(\frac{\pi}{3}, -2)$, and the curve crosses the x -axis at $-\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{\pi}{2}$. The

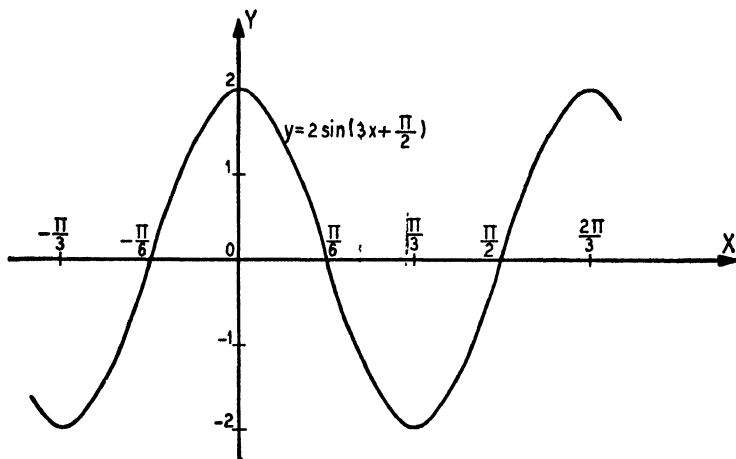


FIG. 5-23.

graph is shown in Fig. 5-23, with the unit on the x -axis twice that on the y -axis.

For $x = 0$, $y = 2 \sin \frac{\pi}{2} = 2$, which is verified by the graph.

Example 2. Sketch the curve of $y = \frac{3}{2} \cos \left(2\pi x - \frac{\pi}{3} \right)$.

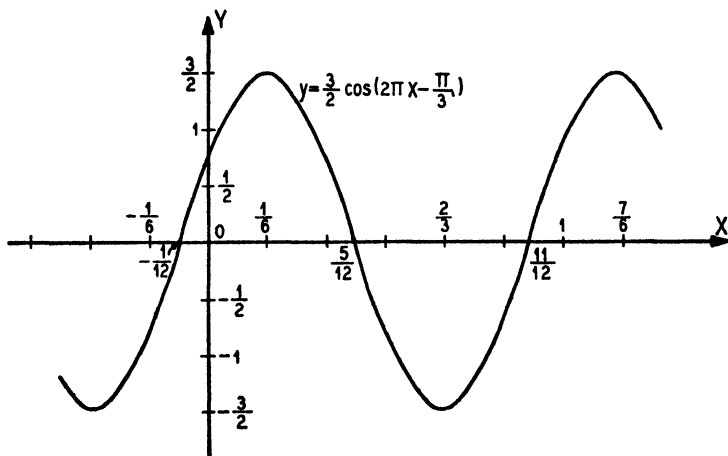


FIG. 5-24.

For this example

$$\text{Amplitude} = \frac{3}{2}.$$

$$\text{Period} = 1.$$

$$\text{Initial phase} = -\frac{\pi}{3}.$$

$$\text{Displacement} = -\frac{1}{8}.$$

First we locate $+\frac{1}{8}$ on the x -axis, and then $\frac{7}{8}$. This segment is divided into four parts by the points $\frac{5}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$. The maximum points are $(\frac{1}{8}, \frac{3}{2})$ and $(\frac{7}{8}, \frac{3}{2})$, and the minimum point is $(\frac{5}{8}, -\frac{3}{2})$. The curve crosses the x -axis at $\frac{3}{8}$ and $\frac{1}{8}$. The graph is shown in Fig. 5-24 with the unit on the x -axis three times that on the y -axis. For $x = 0$, $y = \frac{3}{2} \cos\left(-\frac{\pi}{3}\right) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$, which is verified by the graph.

EXERCISES

For each function given below list (a) the amplitude (if it has an amplitude), (b) the period, (c) the initial phase, and (d) the displacement. Sketch the curve according to the directions given above, labeling both axes carefully.

1. $y = 2 \sin(2x + \pi).$

2. $y = 3 \cos(3x + \pi).$

3. $y = \sin(4x + \pi).$

4. $y = 2 \cos\left(2x + \frac{\pi}{2}\right).$

5. $y = 2 \sin(2x - \pi).$

6. $y = 2 \cos\left(3x - \frac{\pi}{2}\right).$

7. $y = 3 \sin\left(4x - \frac{\pi}{2}\right).$

8. $y = -2 \sin(3x + \pi).$

9. $y = -\cos\left(2x - \frac{\pi}{2}\right).$

10. $y = \tan(2x + \pi).$

11. $y = \cot\left(x + \frac{\pi}{3}\right).$

12. $y = \tan(3x + \pi).$

13. $y = \cot\left(2x - \frac{\pi}{2}\right).$

14. $y = 2 \tan\left(x - \frac{\pi}{2}\right).$

15. $y = 3 \cot\left(4x - \frac{\pi}{2}\right).$

16. $z = \sec\left(x - \frac{\pi}{6}\right).$

17. $z = \csc\left(x + \frac{\pi}{4}\right).$

18. $z = 2 \sec(2\theta - \pi).$

19. $z = 2 \csc\left(2\theta - \frac{\pi}{2}\right).$

20. $z = 3 \sec\left(2\theta + \frac{\pi}{3}\right).$

21. $z = 3 \csc\left(3\theta - \frac{\pi}{2}\right).$

22. $y = \sin\left(\frac{3}{2}x - \frac{3}{8}\pi\right).$

23. $y = 3 \cos \left(\frac{3}{4}x + \frac{3\pi}{16} \right).$

24. $y = \frac{5}{2} \sin \left(\frac{1}{2}x - \frac{\pi}{8} \right).$

25. $y = 10 \sin \left(\pi x + \frac{\pi}{4} \right).$

26. $y = 4 \cos (4\pi x + \pi).$

27. $y = 2 \sin \left(\pi x - \frac{\pi}{3} \right).$

28. $z = 3 \cos \left(3\pi x + \frac{\pi}{2} \right).$

29. $z = 2 \sin (10\pi x + 2\pi).$

30. $z = \tan \left(\pi x + \frac{\pi}{2} \right).$

31. $z = \cot \left(5\pi x + \frac{5\pi}{2} \right).$

32. $z = 3 \sec \left(\pi \theta + \frac{\pi}{2} \right).$

33. $z = 4 \csc \left(2\pi \theta - \frac{\pi}{4} \right).$

34. $z = -2 \sin \left(\pi \theta - \frac{\pi}{4} \right).$

35. $z = -3 \cos \left(\pi \theta - \frac{\pi}{5} \right).$

36. $z = -4 \sin (4\theta - \pi).$

5-14. Plotting Graphs by Composition of Ordinates. When a function is given which is the sum or difference of two or more trigonometric functions, its graph may be plotted easily by using the information we have obtained so far. The graph of each trigonometric function in the sum or difference is plotted, and then the corresponding ordinates are added (or subtracted) graphically by a compass or dividers to obtain the ordinates of the desired graph. This process is called the **composition of ordinates**. The following examples will illustrate the method.

Example 1. Plot the graph of $y = 2 \sin x + \cos 2x$.

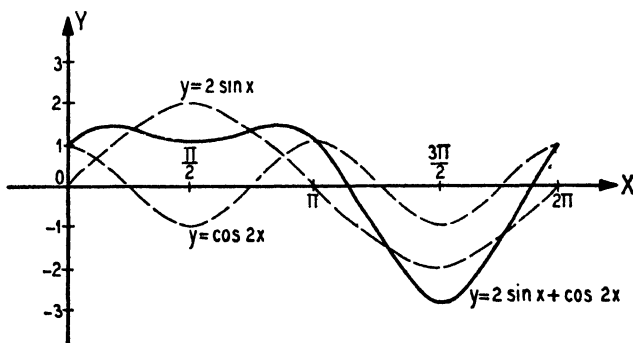


FIG. 5-25.

In order to plot the given function we shall plot the graph of $2 \sin x$ and the graph of $\cos 2x$ separately. They are shown in Fig. 5-25. Then using a compass or dividers the ordinates of $2 \sin x + \cos 2x$ are found by adding and subtracting the ordinates of the graphs already plotted.

Example 2. Plot the graph of $y = x + \sin x$.

We plot $y = x$ and $y = \sin x$ and then add the ordinates to obtain the graph of the given function, shown in Fig. 5-26.

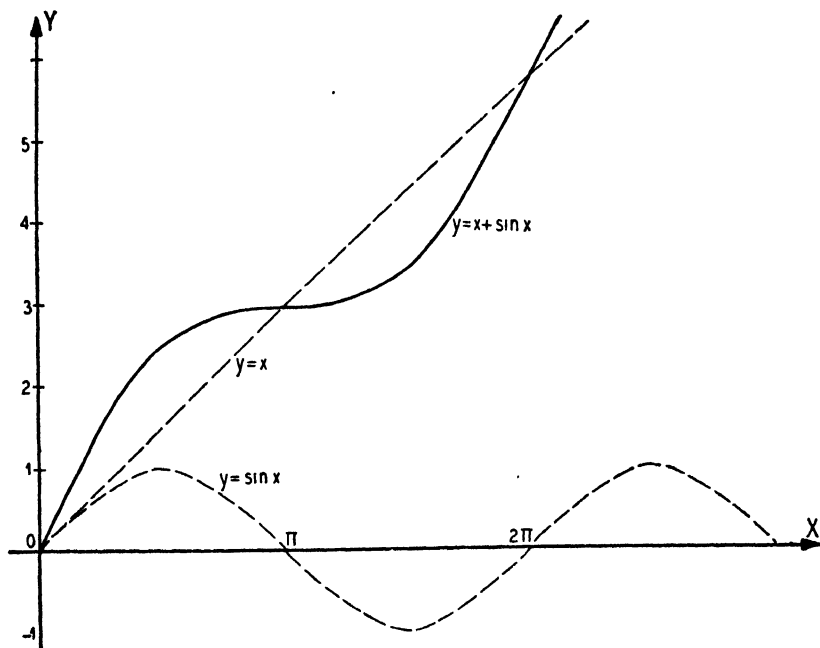


FIG. 5-26.

EXERCISES

Plot the following graphs by composition of ordinates.

1. $y = \sin x + \cos x$.
2. $y = 2 \sin x + 3 \cos x$.
3. $y = \cos x - \sin x$.
4. $y = \sin 2x + 2 \cos x$.
5. $y = \frac{1}{2} \sin 2x + 3 \cos 2x$.
6. $y = x - \sin x$.
7. $y = x + \cos x$.
8. $y = x - \cos x$.
9. $y = \sin x - x$.
10. $y = \cos x - x$.
11. $y = 2 \cos 3x + \sin \frac{3}{2}x$.
12. $y = 2 \sin 2\pi x + 3 \cos 2\pi x$.
13. $y = 3 \sin 2\pi x + \sin \frac{3}{2}x$.
14. $y = 4 \sin 4\pi x - 2 \cos 2\pi x$.
15. $y = \frac{1}{3}x^2 + \cos 4\pi x$.
16. $y = \frac{1}{3}x^2 - \sin 4\pi x$.
17. $y = 2 \sin 2x - 3 \cos 4x$.
18. $y = \sin \left(x + \frac{\pi}{4} \right) + \cos x$.
19. $y = \sin \left(x + \frac{\pi}{4} \right) + \cos \left(x - \frac{\pi}{4} \right)$.
20. $y = \sin \left(x + \frac{\pi}{4} \right) - \cos \left(x - \frac{\pi}{4} \right)$.
21. $y = 2 \sin \left(2x + \frac{2\pi}{3} \right) + \sin x$.
22. $y = 2 \cos 2x + 3 \sin 2 \left(x + \frac{\pi}{4} \right)$.
23. $y = 3 \cos 2\pi x - 2 \sin 2\pi \left(x + \frac{1}{4} \right)$.
24. $y = \sin (4\pi x - \pi) + \cos \left(\frac{\pi}{3}x \right)$.

5-15. Applications of Graphs of Trigonometric Functions: The Simple Pendulum. One of the applications of the trigonometric functions in engineering is to the study of periodic motions. Periodic motions, as the name suggests, are motions which are repeated after regular intervals of time. Some examples of such motion are the motion of the pendulum of a clock, the vibration of a taut string, the vibration of a reed, and the motion of a planet around the sun.

Each of the above motions can be expressed mathematically as some periodic function of time, but only a few of them can be expressed as simple trigonometric functions of the form $y = A \sin \omega t$. Others can be expressed as simple trigonometric functions only if simplifying assumptions are made, whereas still others may be very complicated periodic functions of time.

Consider the simple pendulum of length L feet shown in Fig. 5-27, which is suspended from the point P . The line PQ is the position of rest of the pendulum. The pendulum is started swinging by drawing it to the right an angle of θ_0 radians and then releasing it. Then, (a) if θ_0 is reasonably small, say less than 20° , (b) if we neglect air resistance and frictional effects, and (c) if we agree to measure the variable position angle θ of the pendulum as positive to the right of PQ and negative to the left, the position of the pendulum t seconds after it has been released can be shown to be given by

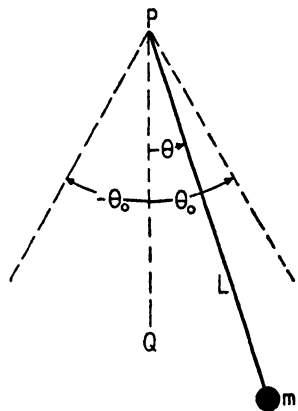


FIG. 5-27.

$$(1) \quad \theta = \theta_0 \cos \sqrt{\frac{g}{L}} t$$

where $g = 32.2$ is the acceleration of gravity in feet per sec. per sec. If L is measured in centimeters, then g is measured in centimeters per sec. per sec., and has the value 980.2.

The period T , the amount of time necessary to complete one swing from θ_0 to the left and back again, is the period of the function (1) which is

$$(2) \quad T = 2\pi \sqrt{\frac{L}{g}}.$$

We see that the motion of the pendulum given by the function (1) is independent of the mass m at the end of the pendulum, and that the period given in (2) depends only on the length of the pendulum.

By attaching a marking device to the lower end of the pendulum, a graph very similar to a sine curve will be drawn on a strip of paper which moves at a constant velocity perpendicular to the plane in which the pendulum is swinging (Fig. 5-28).

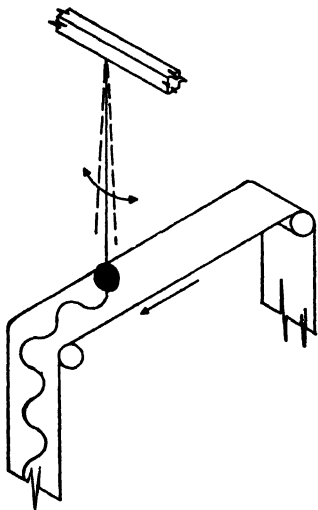


FIG. 5-28.

A complete swing of the pendulum from the position $\theta = \theta_0$ to the opposite position $\theta = -\theta_0$ and again to the original position $\theta = \theta_0$ is called a **cycle**, and the number of cycles in a unit of time is called **frequency**. These terms are used not only for motion of the pendulum but also for every periodic motion. A cycle is the smallest part of a periodic motion by whose repetition the motion is produced. If T is the period, the time needed for one cycle, and f the frequency, the number of cycles per unit of time, then we have the following fundamental relations:

$$fT = 1, \quad f = \frac{1}{T}, \quad T = \frac{1}{f}.$$

The period in seconds of a pendulum is given in (2). The frequency therefore is

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \text{ cycles per second.}$$

5-16. Projection of a Rotating Spoke. Consider the wheel of radius 10 ft. shown in Fig. 5-29, whose axle is taken as the center of a Cartesian coordinate system. The projection of the spoke OS on the line LL' which is perpendicular to the x -axis is QR .

Suppose the spoke starts from a position OT making an angle $\frac{\pi}{6}$ with the positive x -axis and rotates in a counterclockwise direction at a uniform speed of 3 revolutions per second (Fig. 5-29). Since in each revolution it rotates through 2π radians, its angular velocity is 6π radians per second, and in t seconds after starting the spoke will have rotated through an angle of $6\pi t$ radians. Hence, the angle in standard position, of which OS is the terminal side, is $\left(6\pi t + \frac{\pi}{6}\right)$. Thus the ordinate of S is $10 \sin \left(6\pi t + \frac{\pi}{6}\right)$.

If Q is the projection of O on LL' , and R is the projection of S , then QR is the projection of OS , and, further, has a length $\left| 10 \sin \left(6\pi t + \frac{\pi}{6} \right) \right|$. If we agree to measure distances up from Q as positive and those down from Q as negative, then the length p of the projection of OS on $L'L$ is given by

$$(1) \quad p = 10 \sin \left(6\pi t + \frac{\pi}{6} \right).$$

This function has amplitude 10, period $\frac{1}{3}$ second, initial phase $\frac{\pi}{6}$, and displacement $\frac{1}{3}\pi$. Hence by plotting the function (1), we have a curve showing the behavior of the projection as time goes on.

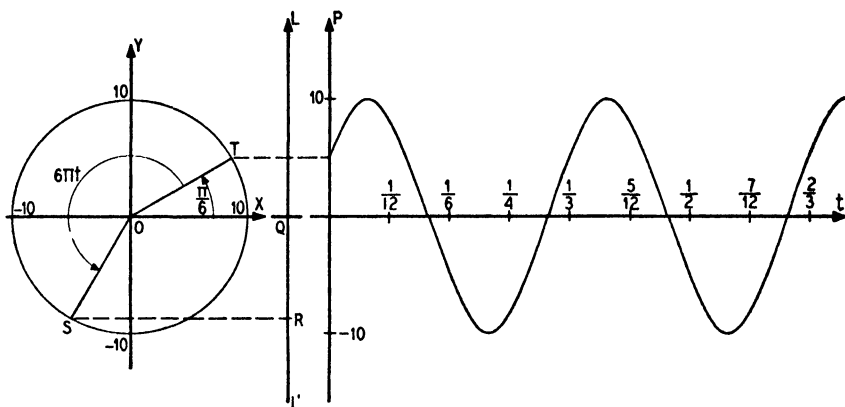


FIG. 5-29.

In order to derive a general formula, suppose that the angular velocity is ω radians per second, the amplitude A units of length and that the initial phase is α . Then,

$$p = A \sin (\omega t + \alpha).$$

The period $T = \frac{2\pi}{\omega}$ seconds; the frequency $f = \frac{\omega}{2\pi}$ cycles per second.

The formula for the frequency f is often written $\omega = 2\pi f$.

5-17. Alternating Current. The results of Sec. 5-16 can now be applied in order to examine the voltage produced by a rotating alternating-current generator. This voltage varies from instant to instant in the same way as the projection of a spoke of the rotating part on a straight line, which was discussed in the previous section. If e is the value of the voltage, measured in volts, at a certain moment (the so-called in-

stantaneous voltage), E_m the **maximum** or **peak voltage**, ω the angular velocity in radians of the conductor in which the voltage is induced, and t is the time, measured in seconds, then we have the formula

$$(1) \quad e = E_m \sin (\omega t + \alpha).$$

The initial phase α is often called **phase angle**. The curve corresponding to (1) is called a **sine wave**. Such a wave is plotted in Fig. 5-30

for the case $\alpha = 0$. The frequency is, according to Sec. 5-16, $f = \frac{\omega}{2\pi}$.

In reality the voltage produced by an alternating-current generator does not change exactly like a sine curve, but the curves obtained by

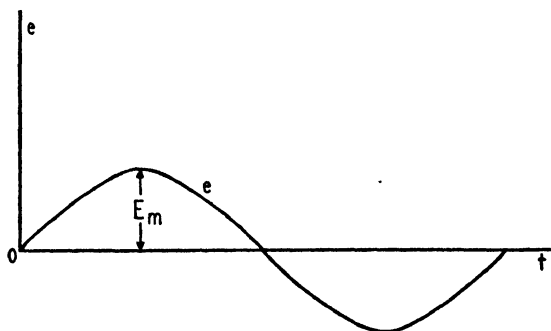


FIG. 5-30.

plotting the voltage against the time are often very similar to sine waves, and are then called **sinusoidal**. The wave form can be observed on an oscilloscope, an extremely sensitive electronic device which projects the wave, using an electron stream, on to a fluorescent screen.

An alternating voltage $e = E_m \sin \omega t$ produces in a circuit an electric current whose instantaneous value is

$$(2) \quad i = I_m \sin (\omega t + \alpha).$$

This current is represented by a sine wave of amplitude I_m , the same frequency $\frac{\omega}{2\pi}$ as the voltage, and a **phase difference** α compared with the voltage e . The current is said to **lead** the voltage if α is positive and to **lag** behind the voltage if α is negative.

If two or more voltages e_1, e_2, \dots are produced in the same circuit, the **resultant voltage** is the sum of them, and the resultant wave can be

found by plotting the waves corresponding to e_1 , e_2 , \dots , and adding their ordinates according to Sec. 5-14.

Example. Two voltages present in a circuit are given by

$$e_1 = E_{m1} \sin \omega t \quad \text{and} \quad e_2 = E_{m2} \sin 3\omega t.$$

The resultant wave is the sum

$$e_t = e_1 + e_2 \quad \text{or} \quad e_t = E_{m1} \sin \omega t + E_{m2} \sin 3\omega t.$$

This wave is plotted in Fig. 5-31 by first plotting the two components and then adding their ordinates. In the laboratory the resultant wave can be shown on an oscilloscope.

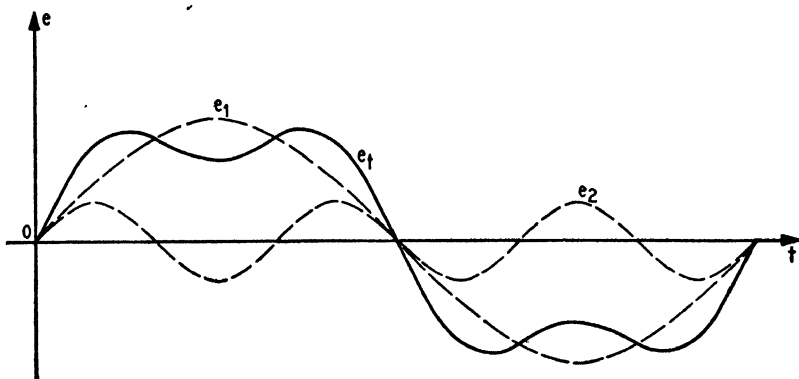


FIG. 5-31.

EXERCISES

1. A pendulum weighing 25 grams is suspended by a thin rod 50 cm. long. The pendulum is pulled aside 5° from the vertical and allowed to swing. If the acceleration of gravity is 980 cm. per sec. per sec., find the period of the pendulum, the frequency, and the graph of the angle θ which the rod makes with a vertical line against time. If the pendulum is carried to the top of a mountain, the period of the pendulum is found to be 1.7 seconds. What is the acceleration of gravity on the mountain top?

2. The length of a pendulum is 4 ft. Its oscillations are started by pulling it aside 10° from the vertical. If the acceleration of gravity at the point of the experiment is 32 ft. per sec. per sec., find the period of the pendulum and plot the graph of the angle θ , which the pendulum makes with a vertical line, against the time.

3. What is the length of a pendulum whose frequency is 60 cycles per minute at a place where the acceleration of gravity is 980 cm. per sec. per sec. Plot, against the time, the graph of the angle between the pendulum and a vertical line.

4. The driving wheel of the valve action mechanism of Fig. 5-32 makes 90 r.p.m. Describe the motion of the valve actuating the arm by expressing the distance y

between the crank pin and the horizontal axis as a function of the time. (Use the discussion of Sec. 5-16.) The distance between the centers of the wheel and the crank pin is 2 in.

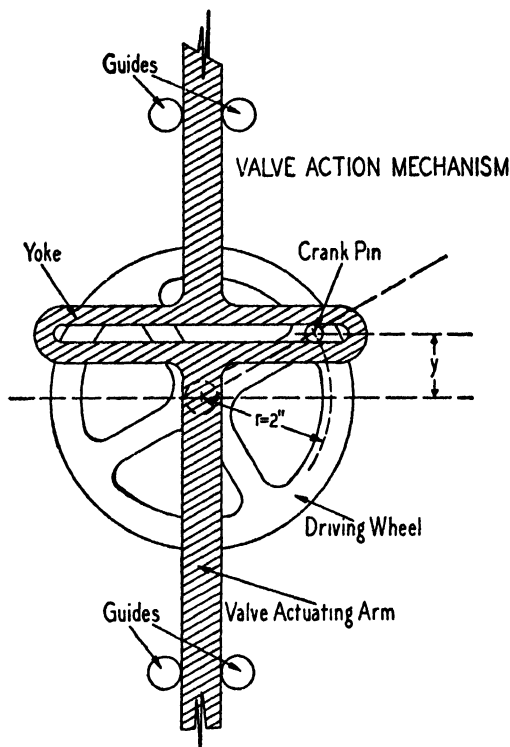


FIG. 5-32.

What is the angular velocity ω if the frequency f is:

5. 60 cycles per second.
6. 500 cycles per second.
7. 300 kilocycles per second (1 kilocycle = 1000 cycles).
8. 4×10^7 cycles per second.

What is the frequency f if the angular velocity ω is:

9. 377 radians per second.
10. 8×10^5 radians per second.
11. 5×10^8 radians per second.

Compute the period T of an oscillatory motion of frequency f and corresponding angular velocity ω if:

12. $f = 60$ cycles per second.
13. $f = 200$ cycles per second.
14. $f = 3000$ cycles per second.
15. $f = 20$ cycles per minute.

16. $f = 3$ cycles per minute.
17. $f = 10,000$ cycles per minute.
18. $\omega = 377$ radians per second.
19. $\omega = 0.5$ radian per second.
20. $\omega = 8 \times 10^3$ radians per second.

Plot the following waves. Note that the angle ωt is in radians and that phase angles are in degrees. Thus, for example, $\omega t + 60^\circ$ must be changed to either $\omega t + \frac{\pi}{3}$ radians or $\frac{180}{\pi} \omega t + 60$ degrees for consistent units.

21. $e = E_m \sin \omega t$, where $E_m = 114$ volts and $\omega = 377$ radians per second.
22. $i = I_m \sin \omega t$, where $I_m = 10$ amperes and $\omega = 377$ radians per second.
23. $e = E_m \sin (\omega t + 30^\circ)$, where $E_m = 228$ volts and $\omega = 188.5$ radians per second.
24. $i = I_m \sin (\omega t - 60^\circ)$, where $I_m = 5$ amperes and $\omega = 1000$ radians per second.
25. Plot the wave of a current which lags 60° behind the voltage of Exercise 21 if the maximum current I_m is 2 amperes.
26. Plot the wave of a current which leads the voltage of Exercise 21 by 45° if the maximum current is 10 amperes.
27. Plot the wave of a voltage which leads the current of Exercise 22 by 60° if the peak voltage is 80 volts.

Plot the following waves:

28. $e_t = e_1 + e_2 = E_{m1} \sin \omega t + E_{m2} \sin (\omega t + 90^\circ)$, where $E_{m1} = E_{m2} = 144$ volts and $\omega = 30$ radians per second.
29. $i_t = i_1 + i_2 + i_3 = I_{m1} \sin (\omega t - 30^\circ) + I_{m2} \sin (\omega t + 30^\circ) + I_{m3} \sin (\omega t + 90^\circ)$, where $I_{m1} = I_{m2} = I_{m3} = 4$ amperes and $\omega = 377$ radians per second.
30. $i_t = i_1 + i_2 = I_{m1} \sin \omega t + I_{m2} \sin 2\omega t$, where $I_{m1} = 5$ amperes, $I_{m2} = 10$ amperes, and $\omega = 377$ radians per second.
31. $e_t = e_1 + e_2 + e_3 = E_{m1} \sin t + E_{m2} \sin 2\omega t + E_{m3} \sin 4\omega t$, where $E_{m1} = E_{m2} = E_{m3} = 50$ volts and $\omega = 120$ radians per second.

5-18. Plotting Sine Functions Along the Vertical Axis and Along the Horizontal Axis Simultaneously (Lissajous Figures). An interesting application of two sinusoidal wave forms is obtained when two sine waves at right angles to each other are plotted. As shown in Fig. 5-33, a sine function is plotted along a vertical axis while another sine function of the same amplitude and the same frequency is plotted along the horizontal axis.

Both curves are plotted against the time t . For a particular value of t , two points, one on each of the curves, correspond to one another. Through the point on the vertical sine curve a vertical line is drawn, and through the corresponding point of the horizontal sine curve a horizontal line is drawn. These two projecting lines have a point of intersection. The locus of these points of intersection will be either a

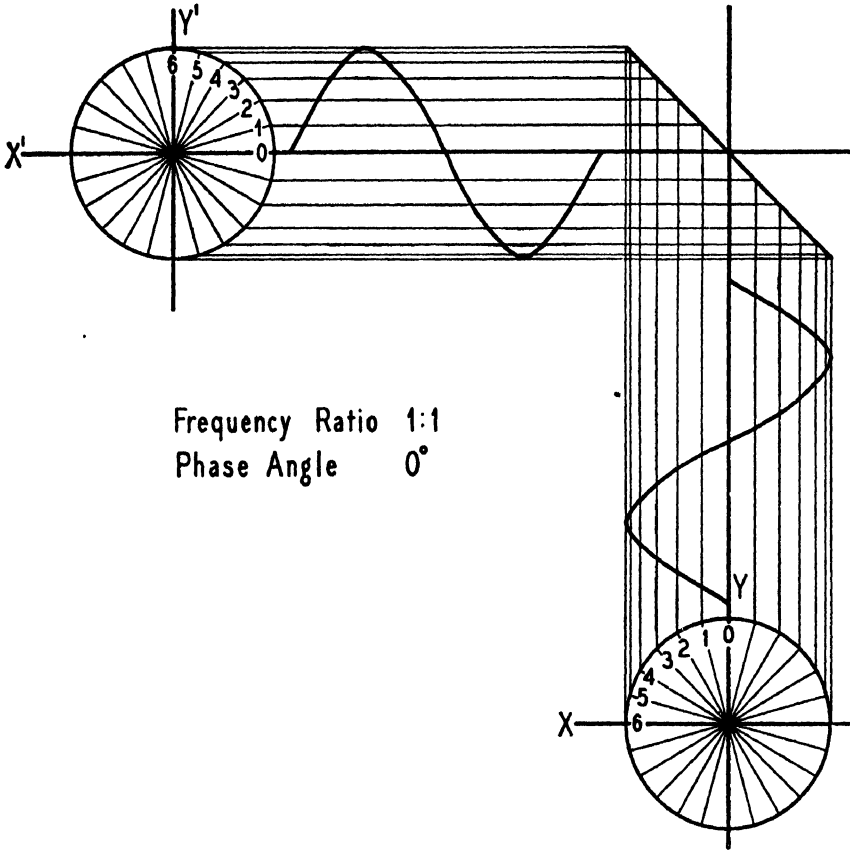


FIG. 5-33.

straight line, an ellipse, or a circle, depending upon the phase relationship between these sine functions. It will be a straight line if the phase angle is zero, an ellipse if it is between zero and 90° , and a circle if the phase angle is 90° . Figures 5-33, 5-34, 5-35 show the patterns which

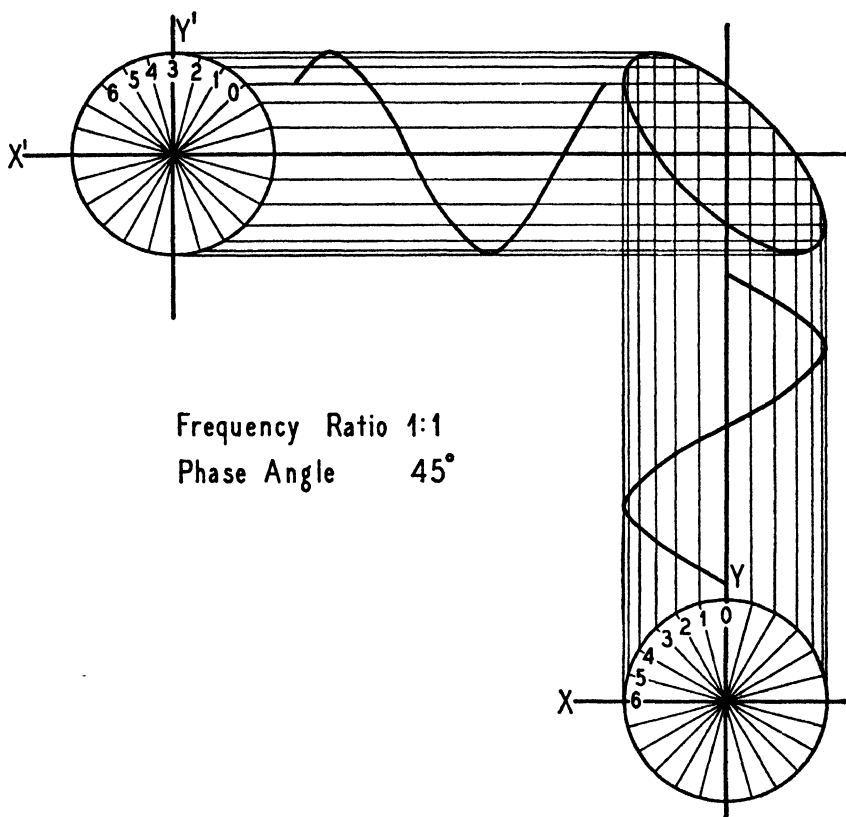


FIG. 5-34.

are obtained by projecting two sinusoidal electric voltages on the respective axes. As a further extension of plotting these trigonometric functions, the frequency of one sine function plotted along the horizontal axis may be twice that of the frequency of the sine function plotted along the vertical axis. The resultant pattern obtained is shown in Fig. 5-36. Other patterns are obtained when the frequency ratio is 3 to 1, 4 to 1, etc. Since alternating currents are usually of the sinusoidal form, this method of plotting curves or of observing them on an oscilloscope screen provides a very practical method of measuring frequency and phase angle relations.

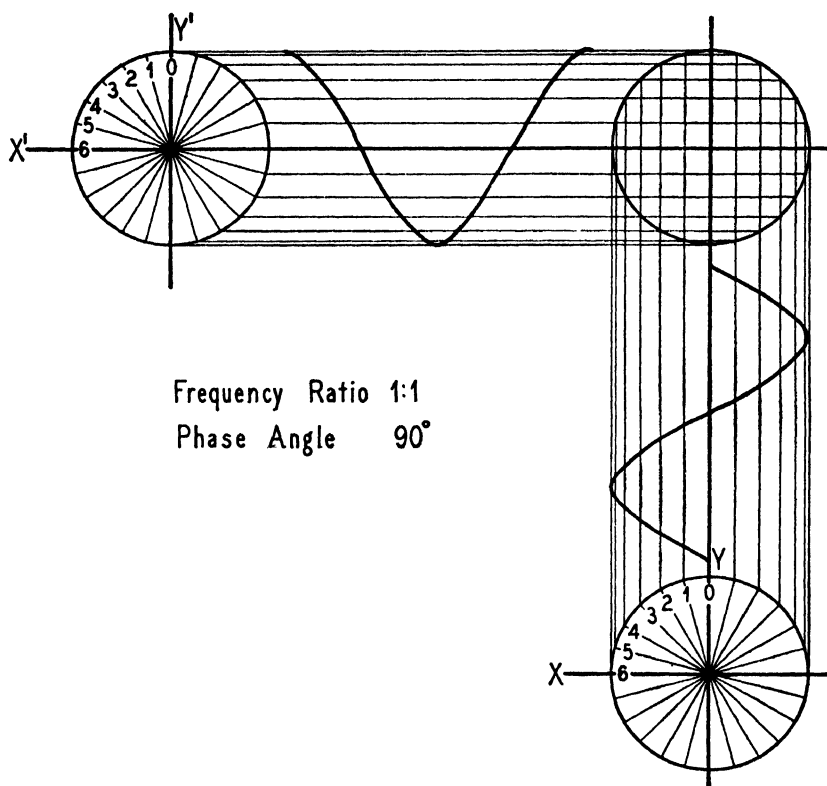


FIG. 5-35.

EXERCISES

Plot the Lissajous figures resulting from projecting the following voltages on perpendicular axes:

1. $e_1 = 10 \sin 377t$, $e_2 = 10 \sin (377t + 30^\circ)$.
2. $e_1 = 10 \sin 377t$, $e_2 = 10 \sin (754t + 45^\circ)$.
3. $e_1 = 10 \sin 377t$, $e_2 = 10 \sin (754t - 45^\circ)$.
4. $e_1 = 10 \sin 754t$, $e_2 = 10 \sin 1131t$.
5. $e_1 = 10 \sin 377t$, $e_2 = 10(754t + 90^\circ)$.

PROGRESS REPORT

In Chapter 4, the trigonometric functions were defined as certain numbers connected with an angle, and they were used to solve geometric problems. In this chapter, we disassociated the concept of trigonometric functions from their original definition, and considered, for example, the function $y = \sin x$ as a relation between the two *numbers*

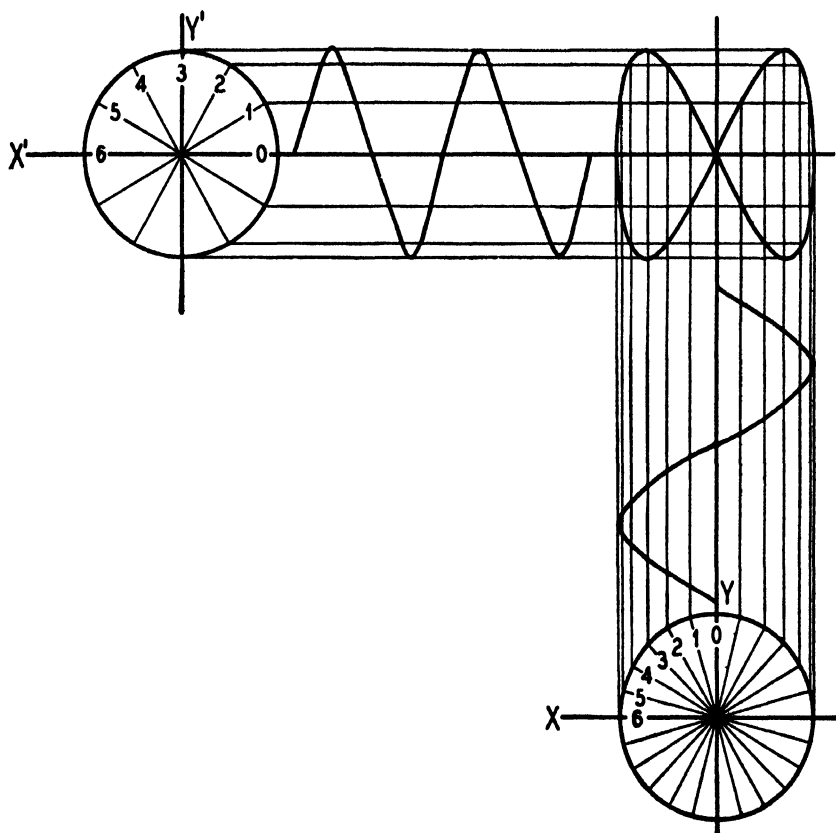


FIG. 5-36.

x and y . The trigonometric functions were studied in this chapter, their graphs were plotted, and the extremely important property of their being periodic functions was explained.

Functions defined by the formula $\theta = A \sin (\omega t + \alpha)$ were also examined. They were found to be periodic functions of period $\frac{2\pi}{\omega}$, amplitude A , and initial phase α , and their graphs were plotted. Then functions were studied which are sums of different terms of the form $A \sin (\omega t + \alpha)$.

The last few sections of this chapter were devoted to the study of mechanical and electrical phenomena which can be described by trigonometric functions. In this connection the terms frequency, phase angle, lead and lag were explained.

CHAPTER 6

SIMPLE PROPERTIES OF VECTORS

We have been interested so far in only one property of the physical quantities, namely, that of magnitude. However, many physical quantities have a second property, that of direction, which it is important to consider. For instance, not only is the speed of the wind important to an aviator but also its direction: a head wind will decrease his speed, a tail wind will increase his speed, and a cross wind will not only affect his speed but also will cause him to drift from his course. A shell fired from a gun is urged in the direction in which the gun is aimed by the force of the exploding powder; it is also urged earthward by the force of gravity, retarded by the resistance of the air, and perhaps even affected by the wind. All these forces and their directions influence the path of the projectile and consequently help to determine where it will strike. Displacement, motion, velocity, momentum, acceleration, and force are all physical quantities of which direction is an important property.

6-1. Scalars. Physical quantities, such as mass and temperature which are characterized adequately for our purposes by the property of magnitude alone, are called **scalar quantities**. The numbers which represent them are called **scalars**. In order to solve problems concerning magnitudes alone we have used numbers as a tool, together with the science of operations with numbers which is called algebra.

In order to solve problems involving physical quantities with both magnitude and direction, we shall develop the **vector** as a tool. Physical quantities with both magnitude and direction are called **vector quantities**. The science of operations with vectors is called **vector algebra**.

Let us suppose an aviator flies a certain distance at a uniform speed of 180 miles per hour. Since his speed is fully specified by its magnitude alone, it is a scalar quantity. However, suppose that we desire to specify not only his speed, but also the direction in which he is flying, by saying that he is flying eastward at a speed of 180 miles per hour. Thus, we get a quantity called **velocity**, which has both magnitude and direction. Consequently, velocity is a vector quantity and will be represented by a vector.

6-2. Vectors. A straight line segment to which a direction has been given is called a **directed line segment**. Usually the direction is indicated by an arrowhead, as shown on the segment PQ in Fig. 6-1. Obviously, there are only two possible directions which can be assigned to a given segment.

A vector is a directed line segment. The magnitude or absolute value of a vector is its (positive) length; its direction is given by its position in the plane and the direction assigned to the segment. In the plane of Fig. 6-1, the segment PQ with direction as shown by the arrowhead is a vector. It is sometimes helpful to think of a vector as being drawn in the given direction from an **initial point** to a **terminal point**. In Fig. 6-1 P is the

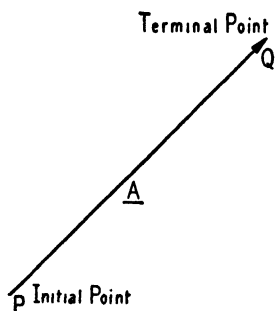


FIG. 6-1.

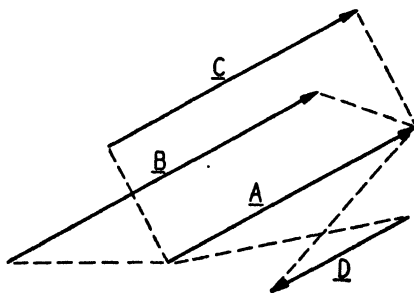


FIG. 6-2.

initial point and Q the terminal point. Thus a vector can be specified by giving its initial point and its terminal point. The distance between these points is the magnitude, and the direction of the segment is from the initial to the terminal point.

It is useful to have a notation for vectors. We shall use bold-faced capitals (in figures, underlined capitals) for this purpose, as \mathbf{A} in Fig. 6-1. Scalars will be denoted by lower-case letters such as a , b , and capitals which are not bold-faced, as A , B . The magnitude of a vector is a positive scalar. The magnitude of the vector \mathbf{A} is denoted either by the corresponding capital A not bold-faced or by parallel bars $|\mathbf{A}|$. In this chapter it is supposed that all vectors are located in one and the same plane.

Two vectors are said to have the same direction if they are parallel and such that the segment joining their initial points does not intersect the segment joining their terminal points. In Fig. 6-2, all the vectors \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} are parallel. The segment joining the initial points of \mathbf{A} and \mathbf{B} does not intersect the segment joining the terminal points, so that \mathbf{A} and \mathbf{B} have the same direction. Likewise \mathbf{A} and \mathbf{C} have the same direction, and thus \mathbf{A} , \mathbf{B} , and \mathbf{C} have the same direction. However, \mathbf{A} and \mathbf{D}

do not have the same direction, since the segment joining their initial points intersects the segment joining their terminal points. Two vectors, such as \mathbf{A} and \mathbf{D} , which are parallel but do not have the same direction are said to be **opposite in sense**.

Two vectors are equal if they have the same magnitude and the same direction. In Fig. 6-2 the vectors \mathbf{A} and \mathbf{C} are equal, written as $\mathbf{A} = \mathbf{C}$. Since through any point in the plane a line can be drawn parallel to a given line, *any point in the plane can be taken as the initial point of a vector equal to any given vector.*

In any operation with vectors, a given vector may be replaced by any other equal vector. Usually convenience dictates the choice of such a substitute vector. If \mathbf{A} is a given vector, we shall follow the practice of denoting also by \mathbf{A} any vector equal to \mathbf{A} . Any point in the plane is thus the initial point of a vector \mathbf{A} equal to a given vector \mathbf{A} .

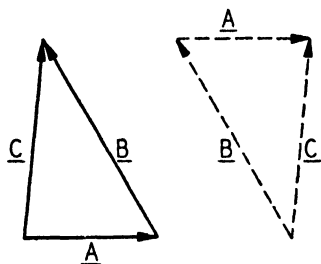


FIG. 6-3.

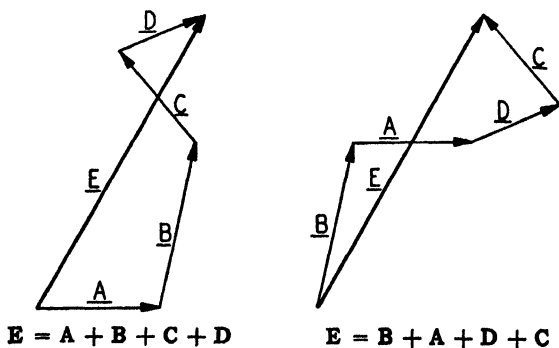
6-3. Addition and Subtraction of Vectors. The sum of two vectors is defined as follows.

The sum of \mathbf{A} and \mathbf{B} is the vector \mathbf{C} whose initial point is the initial point of \mathbf{A} and whose terminal point is the terminal point of a vector equal to \mathbf{B} drawn so that its initial point is the terminal point of \mathbf{A} . Such an addition is shown in Fig. 6-3 by the full lines.

The dotted lines show how a vector equal to \mathbf{C} can be found by performing the process in the reverse order. *The order in which vectors are added is immaterial.* To denote that \mathbf{C} is the sum of \mathbf{A} and \mathbf{B} we write

$$\mathbf{C} = \mathbf{A} + \mathbf{B}.$$

Several vectors can be added as shown in Fig. 6-4. The figure also



$$\mathbf{E} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$$

$$\mathbf{E} = \mathbf{B} + \mathbf{A} + \mathbf{D} + \mathbf{C}$$

FIG. 6-4.

shows that the order of vector addition has no effect on the result. The sum of several vectors is often called the **resultant** of the vectors.

A second definition of vector addition can be formulated which is equivalent to the one already given:

To add \mathbf{A} and \mathbf{B} , draw a vector \mathbf{B} whose initial point is the initial point of \mathbf{A} , and complete the parallelogram of which these vectors form two sides. The diagonal vector \mathbf{C} with initial point the initial point of \mathbf{A} and \mathbf{B} and terminal point at the opposite vertex of the parallelogram is the sum of \mathbf{A} and \mathbf{B} . Such an addition is shown in Fig. 6-5.

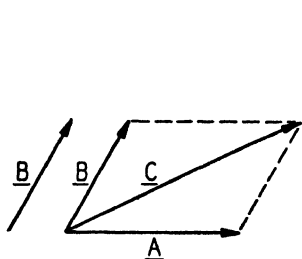


FIG. 6-5.

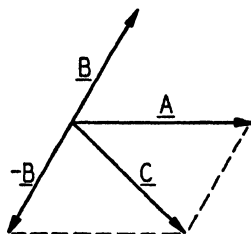


FIG. 6-6.

The vector $-\mathbf{A}$ is a vector of the same magnitude as \mathbf{A} and parallel to \mathbf{A} , but opposite in sense.

The result of subtracting \mathbf{B} from \mathbf{A} is obtained by adding \mathbf{A} and $-\mathbf{B}$. Figure 6-6 shows a vector \mathbf{B} subtracted from \mathbf{A} , which we denote by $\mathbf{A} - \mathbf{B}$.

The reader may also devise a convenient direct parallelogram rule for the subtraction of vectors. He may also verify that if several vectors are to be subtracted from a given vector the order in which they are subtracted is immaterial.

If a is a positive scalar, the vector $a \cdot \mathbf{A}$ is a vector with the same direction as \mathbf{A} and with magnitude aA . If a is a negative scalar, the vector $a \cdot \mathbf{A}$ is equal to the vector $|a| \cdot (-\mathbf{A})$.

The vector of zero magnitude is considered to have no direction. It is called the **null vector**, and denoted by 0.

The direction of a vector \mathbf{B} can be specified conveniently by giving an angle which the vector \mathbf{B} makes with a given vector \mathbf{A} . This angle is formed by choosing a vector \mathbf{A} with the same initial point as \mathbf{B} , and by letting \mathbf{A} be the initial side, \mathbf{B} the terminal side of the angle; the angle is positive if measured counterclockwise, negative if measured clockwise. Usually, as here, the vector \mathbf{A} is chosen horizontal with direction to the right. Several vectors with directions specified in this way are

shown in Fig. 6-7. We see then that a vector can be specified by giving:

- (a) Its magnitude.
- (b) The angle (as described above) which it makes with some given vector.
- (c) Its initial or terminal point.

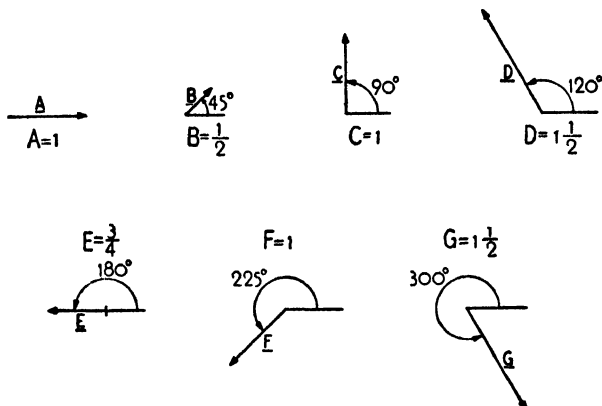


FIG. 6-7.

Two vectors, having (a) and (b) the same, are equal, and thus for most operations (c) is of no interest, since any vector can be replaced by an equal in operations. The vectors of Fig. 6-7 are given by their magnitudes, the angles they make with the horizontal, and their initial points. However, in operations with these vectors other initial points may be chosen at pleasure.

Example. Find graphically the vector $\mathbf{H} = 2\mathbf{A} + \mathbf{D} - 3\mathbf{F} + \mathbf{G}$ where \mathbf{A} , \mathbf{D} , \mathbf{F} , and \mathbf{G} are given in Fig. 6-7.

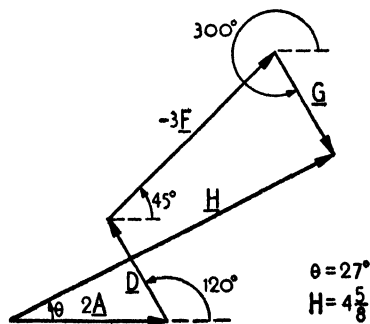


FIG. 6-8.

The figure is drawn in Fig. 6-8 by the use of a ruler and protractor. After the construction has been made, a measurement shows that the magnitude of \mathbf{H} is $4\frac{5}{8}$, and that \mathbf{H} makes an angle of 27° with the horizontal vector \mathbf{A} .

EXERCISES

Given the vectors **A**, **B**, **C**, **D**, **E**, **F**, **G** of Fig. 6-7, find graphically each of the following vectors **H**. Make your constructions carefully, using a ruler and a protractor. Find the magnitude of **H** and the positive angle between 0° and 360° which it makes with the horizontal vector **A** by measurement from your construction.

- | | |
|--|---|
| 1. $\mathbf{H} = \mathbf{A} + \mathbf{B}$. | 2. $\mathbf{H} = \mathbf{C} + \mathbf{D}$. |
| 3. $\mathbf{H} = \mathbf{D} + 2\mathbf{F}$. | 4. $\mathbf{H} = \mathbf{F} - \mathbf{G}$. |
| 5. $\mathbf{H} = \mathbf{C} - 2\mathbf{D}$. | 6. $\mathbf{H} = \mathbf{A} + \mathbf{E}$. |
| 7. $\mathbf{H} = \mathbf{E} - \mathbf{A}$. | 8. $\mathbf{H} = \mathbf{B} + \mathbf{C}$. |
| 9. $\mathbf{H} = 3\mathbf{D} + \mathbf{G}$. | 10. $\mathbf{H} = \mathbf{E} - \mathbf{F}$. |
| 11. $\mathbf{H} = \mathbf{A} + \mathbf{D} + \mathbf{G}$. | 12. $\mathbf{H} = 2\mathbf{C} + 3\mathbf{F}$. |
| 13. $\mathbf{H} = 2\mathbf{F} - \mathbf{G}$. | 14. $\mathbf{H} = 2\mathbf{C} + \mathbf{D} - 4\mathbf{B}$. |
| 15. $\mathbf{H} = \mathbf{A} - \frac{1}{2}\mathbf{D} + 3\mathbf{E}$. | 16. $\mathbf{H} = \mathbf{D} - 2\mathbf{F} + \mathbf{G}$. |
| 17. $\mathbf{H} = 3\mathbf{A} - 2\mathbf{D} + \mathbf{E}$. | 18. $\mathbf{H} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$. |
| 19. $\mathbf{H} = \mathbf{D} + \mathbf{E} + \mathbf{F} + \mathbf{G}$. | 20. $\mathbf{H} = 2\mathbf{D} - 2\mathbf{F} + \frac{1}{2}\mathbf{G}$. |
| 21. $\mathbf{H} = \mathbf{A} + \mathbf{B} - \mathbf{C} + 2\mathbf{D} - \mathbf{G}$. | 22. $\mathbf{H} = \mathbf{B} + \mathbf{C} - \mathbf{E} + 3\mathbf{G}$. |
| 23. $\mathbf{H} = 2\mathbf{B} - \mathbf{C} + \mathbf{D} + \mathbf{F}$. | 24. $\mathbf{H} = \mathbf{B} + \mathbf{C} + \mathbf{D} + \mathbf{E}$. |
| 25. $\mathbf{H} = 2\mathbf{D} - \mathbf{E} + 3\mathbf{F} + \mathbf{G}$. | |

In the following exercises solve the problem by drawing a vector diagram.

26. Velocity has both direction and magnitude, hence it is a vector quantity. A boat is propelled directly across a small stream at a speed of 4 knots. If the stream has a downstream velocity of 3 knots, determine the resulting velocity of the boat. Express the angle of the course with respect to a line parallel to the direction of the flow of water.

27. An airplane pilot points the nose of his ship toward the north. If the wind forces the plane toward the east at a speed of 20 miles per hour and the plane is cruising at a speed of 100 miles per hour, determine the magnitude and direction of the velocity of the plane.

28. A man is pulling a box of castings across the floor. If the friction between the box and the floor exerts a force of 10 lb. in a northerly direction and the man is pulling toward the south with a force of 100 lb., what is the resultant force upon the box?

29. If an object or body is standing still or moving with a constant velocity, the resultant force upon that body must be zero. A pail of tar is being pulled to the roof at a constant velocity by means of a rope and pulley system. If the pail of tar exerts a force of 50 lb. in the downward direction, then what is the tension in the rope, that is, what is the force exerted by the rope?

30. A Flying Fortress is traveling with a velocity of 250 miles per hour in a horizontal direction. A bomb dropped from the plane will have the same horizontal velocity. In addition, the bomb will have a downward velocity. If at a certain instant the downward velocity is 50 miles per hour, what will be the magnitude and direction of the resultant velocity of the bomb?

31. A man walks 10 miles toward the north and then 5 miles in a northeasterly direction. What will be his position at the end of the trip with reference to the starting point?

32. A train, in order to go from *A* to *B*, must travel around the lake upon which the cities are located. The train must travel 10 miles toward the north, 5 miles

toward the northeast, 25 miles east, 15 miles south, and finally 5 miles southwest. An airplane may go directly across the water from A to B . What would be the cost of the air trip if the cost per mile is 11 cents?

33. In a water pipe, the water in contact with the walls of the pipe exerts a frictional force in a direction opposite to the direction of water flow. A long pipe is used to supply water at a given pressure to a water turbine. If the force in the direction of water flow must be 250 lb. and the frictional force exerted by the water in contact with the walls of the pipe is 10 lb., what force must be exerted upon the water at the point of entry to the pipe?

34. A Navy dive bomber dives, making an angle of 15° with the vertical. At the instant the bomb is released, the velocity is 300 miles per hour along the 15° line. The bomb will, of course, have this velocity. In addition, however, the bomb will have a downward velocity owing to the action of gravity. If at an instant 10 seconds after the bomb release, the downward velocity is 50 miles per hour, what will be the magnitude and direction of the resultant velocity?

35. Two trains travel between the towns A and B . The first train goes directly in a straight line from A to B . The second train also goes to towns C , D , and E , which are located off the direct route. Thus, the second train travels 50 miles to the northeast to C , then 75 miles to the north to D , 5 miles to the west to E , and finally 25 miles to the northeast to town B . What distance must the first train travel in going directly from A to B ?

6-4. Resolution of a Vector into Components. If a vector A is the sum of several vectors B , C , D , and so on, then these latter vectors are said to be **components** of A . Thus in the example of Sec. 6-3, H has the components $2A$, D , $-3F$, and G .

Given a vector A , it is possible to find a set of vectors whose sum is A in many ways. In other words, A has many sets of components. Figure 6-9 shows three sets of components for a vector A . In fact, we

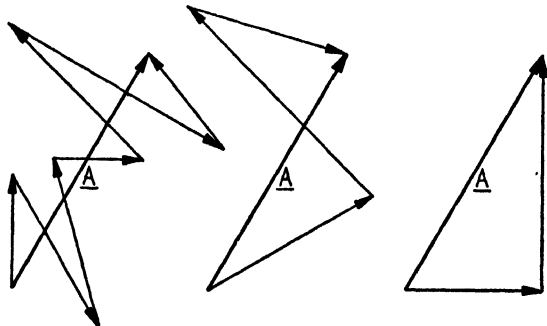


FIG. 6-9.

may draw components at random under the simple restrictions that the initial point of the first component is the initial point of A , that the initial point of each succeeding vector is the terminal point of its predecessor, and that the last component must have as its terminal point the

terminal point of **A**. Thus every component but the last is essentially at our pleasure.

The process of finding some vectors of which a given vector is the sum is called **resolution into components**.

A vector may be resolved into components parallel to two given non-parallel vectors as shown in Fig. 6-10. **A** is the given vector, and we desire components parallel to **B** and **C**. The components can be constructed easily by drawing lines through the initial and terminal points parallel to **B** and **C**. Three ways of finding these components are shown.

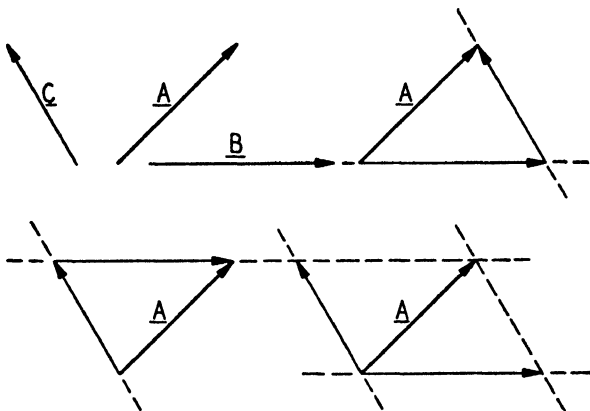


FIG. 6-10.

We shall use this process most often when two components of a given vector are desired which are perpendicular to each other. Usually one component will be horizontal and the other vertical. This case will be discussed in the next section.

EXERCISES

The vectors **A**, **B**, **C**, **D**, **E**, **F**, and **G** given below are those of Fig. 6-7. In each problem given below make your constructions carefully with a ruler and a protractor.

Resolve the vector given first into components parallel to the last two vectors.

- | | | |
|--|--|---------------------------------------|
| 1. A ; D , F . | 2. E ; F , G . | 3. 4B ; A , C . |
| 4. D ; F , G . | 5. 3E ; B , D . | 6. 2A ; F , G . |
| 7. 3C ; B , D . | 8. -2F ; E , G . | 9. G ; E , F . |
| 10. -3E ; A , C . | 11. -2B ; A , D . | 12. 3D ; D , E . |
| 13. -A ; D , E . | 14. 2C ; F , G . | 15. 2F ; C , D . |

Resolve each of the vectors given below into horizontal and vertical components.

- | | | | |
|-----------------|------------------|------------------|------------------|
| 16. 4B . | 17. -2A . | 18. D . | 19. E . |
| 20. F . | 21. G . | 22. 2C . | 23. -3D . |
| 24. 3A . | 25. -2E . | 26. -3D . | 27. -3F . |
| 28. -G . | | | |

In the following exercises solve the problem by drawing a vector diagram.

29. A magnet is capable of exerting a force. Field intensity is defined as the force upon a unit north pole and is a measure of the strength of a magnet. The earth is, of course, a magnet. If the earth's field intensity at some particular point has a magnitude of 0.5 dyne per unit pole and makes an angle of 70° with the horizontal, determine the horizontal and vertical components of the field intensity.

30. An airplane in landing comes in along a line making an angle of 30° with the horizontal. Of the many forces acting on the plane consider only the following three: the force directly downward due to the action of gravity, the force directly upward due to the action of the air and wing surfaces, and the force produced by the propeller and flaps directly along the 30° line in a direction opposite to the motion of the plane. The downward force due to the action of gravity is 4000 lb., the upward force due to the wing surfaces is 3000 lb., and the force along the 30° line has a magnitude of 1900 lb. Resolve all these forces into their horizontal and vertical components.

31. Two iron pipes of equal length, forming an isosceles triangle, are used to support a platform. The two iron pipes come together at the top making an angle at 30° ; the force exerted by the platform at this point has a magnitude of 250 lb. and is directed downward. Since the platform is stationary the two iron pipes must exert an equal and opposite force and each pipe will support one-half of the required force. The forces due to the pipes will act directly along the pipes. Determine the vertical and horizontal components of the forces acting through the pipes.

32. In a sling shot a piece of rubber band is fastened by means of its two ends to a tree fork or its equivalent. If the rubber band is pulled back until the angle between the two sides of the rubber band is 30° , and the tension directed along the rubber band on each side is 30 lb., what is then the magnitude of the force tending to hurl the stone forward?

33. A retired engineer in a canoe hooks a pike which takes off at an angle of 60° off the starboard bow. If the force exerted by the fish is 8 lb., resolve this force into two components, one in a lengthwise, the other in a crosswise, direction with the canoe.

34. A ladder is placed against the side of a building. The angle between the ladder and the building is 30° . A force of 180 lb. acts downward along the ladder. What are the vertical and horizontal components of this force?

6-5. Rectangular Coordinates in the Vector Plane. Suppose that a rectangular Cartesian coordinate system is placed in the vector plane. Since a point now can be specified by giving its coordinates, a vector can be specified by giving the coordinates of its initial and terminal points. However, it is convenient if every vector which is used in operations has its initial point at the origin of the coordinate system. *Every vector mentioned henceforth, unless specifically indicated otherwise, will have the origin as its initial point. With this agreement, a vector can be specified by giving the coordinates of its terminal point.*

Let **A** and **B** be two vectors whose initial points are at the origin and which lie along the x -axis. Since the magnitude A of the vector **A** is positive, if **A** is to the right of the origin its terminal point is $(A, 0)$. If **A** extends to the left of the origin its terminal point is $(-A, 0)$.

Thus, if \mathbf{A} and \mathbf{B} both extend to the right of the origin, the terminal point of $\mathbf{A} + \mathbf{B}$ is $(A + B, 0)$. Likewise, if \mathbf{A} and \mathbf{B} both extend to the left of the origin, the terminal point of $\mathbf{A} + \mathbf{B}$ is $(-A - B, 0)$. If \mathbf{A} is to the right of the origin, and \mathbf{B} is to the left, the terminal point of $\mathbf{A} + \mathbf{B}$ is $(A - B, 0)$.

These remarks establish the following very useful result:

Given a set of vectors each of which:

- (a) *Has initial point at the origin.*
- (b) *Extends along the x -axis.*

Then the vector which is the sum of the vectors of the set:

- (a) *Extends along the x -axis.*
- (b) *Has its initial point at the origin.*
- (c) *Has a terminal point whose x -coordinate is the sum of the x -coordinates of the terminal points of the vectors of the set.*

A similar statement holds, of course, for a sum of vectors which have initial points at the origin and which extend along the y -axis. What happens in subtraction of vectors can be easily inferred.

Let $P(x, y)$ be the terminal point of a vector \mathbf{A} whose initial point is at the origin, as shown in Fig. 6-11. By dropping perpendiculars from P to the axes it can be seen that \mathbf{A} is the sum of two components \mathbf{A}_x and \mathbf{A}_y along the x - and y -axes respectively. \mathbf{A}_x is called the **x -component** of \mathbf{A} , and \mathbf{A}_y is called the **y -component** of \mathbf{A} . We see that the terminal

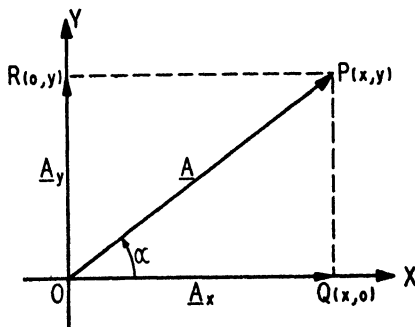


FIG. 6-11.

point of \mathbf{A}_x is $Q(x, 0)$ and that the terminal point of \mathbf{A}_y is $R(0, y)$.

Let \mathbf{A} and \mathbf{B} be two vectors with initial points at the origin and terminal points $P_A(x_A, y_A)$ and $P_B(x_B, y_B)$ respectively. We may write these vectors as the sum of their x - and y -components:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y, \quad \mathbf{B} = \mathbf{B}_x + \mathbf{B}_y.$$

Then,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{B}_x + \mathbf{B}_y = (\mathbf{A}_x + \mathbf{B}_x) + (\mathbf{A}_y + \mathbf{B}_y).$$

If we set $\mathbf{C}_x = \mathbf{A}_x + \mathbf{B}_x$ and $\mathbf{C}_y = \mathbf{A}_y + \mathbf{B}_y$, then $\mathbf{C} = \mathbf{C}_x + \mathbf{C}_y$, and by the previous results of this section, the terminal point of \mathbf{C}_x is $(x_A + x_B, 0)$, and the terminal point of \mathbf{C}_y is $(0, y_A + y_B)$. Then, obviously,

the terminal point of $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is $P(x_A + x_B, y_A + y_B)$. A similar result can be established for subtraction.

These facts may be summarized as follows.

Given the vectors with initial points at the origin and terminal points as follows:

$$\mathbf{A}, P_A(x_A, y_A),$$

$$\mathbf{B}, P_B(x_B, y_B),$$

$$\mathbf{C}, P_C(x_C, y_C).$$

Then $\mathbf{A} + \mathbf{B} - \mathbf{C}$ is a vector with initial point at the origin and terminal point $P(x_A + x_B - x_C, y_A + y_B - y_C)$.

Example 1. Given \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} with initial points at the origin and terminal points respectively as follows: $P_A(2, 5)$, $P_B(5, 3)$, $P_C(-3, 2)$, $P_D(-4, -2)$. Find the vector $\mathbf{E} = \mathbf{A} + \mathbf{B} + \mathbf{C} - \mathbf{D}$.

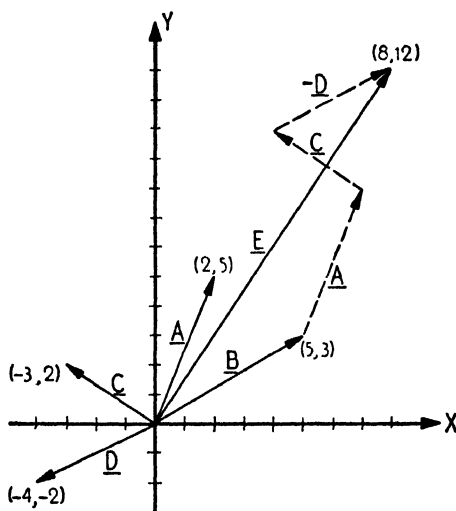


FIG. 6-12.

The terminal point of \mathbf{E} has x -coordinate $2 + 5 - 3 - (-4) = 8$ and y -coordinate $5 + 3 + 2 - (-2) = 12$. Thus the terminal point of \mathbf{E} is $P_E(8, 12)$. This result is verified in Fig. 6-12 by constructing the geometrical vector addition by dotted lines.

It was seen in Sec. 6-3 that a vector could be specified by giving its magnitude, its initial point, and the angle which it makes with a given vector. Let \mathbf{A} be a vector with initial point at the origin, magnitude A , and let \mathbf{A} form the terminal side of an angle α in standard position. The angle α is called the **direction angle** of \mathbf{A} . If the terminal point

of \mathbf{A} is denoted by $P(x, y)$, the values of x and y can be found as follows:

$$\sin \alpha = \frac{y}{A}, \quad \cos \alpha = \frac{x}{A}$$

from which

$$(1) \quad x = A \cos \alpha, \quad y = A \sin \alpha.$$

Suppose that $P(x, y)$ is given and the values of A and α are desired (Fig. 6-11). From the Pythagorean theorem

$$(2) \quad A = \sqrt{x^2 + y^2},$$

and from the definitions of the trigonometric functions

$$(3) \quad \tan \alpha = \frac{y}{x}.$$

Of course, (3) does not determine α uniquely, but the quadrant in which P lies determines the quadrant in which the terminal side of α should fall.

If the x - and y -components of \mathbf{A} are \mathbf{A}_x and \mathbf{A}_y respectively, we see that $A_x = |x|$ and $A_y = |y|$, whence $A_x^2 = x^2$ and $A_y^2 = y^2$. Then from (2),

$$(4) \quad A = \sqrt{A_x^2 + A_y^2}.$$

Obviously, the terminal point of the vector $-\mathbf{A}$ is $(-x, -y)$.

Let a be a positive scalar. Then the vector $a\mathbf{A}$ has magnitude aA and the same direction angle α as \mathbf{A} . $P(x, y)$ is the terminal point of \mathbf{A} , and if we let $P_a(x_a, y_a)$ be the terminal point of $a\mathbf{A}$, we have by (1),

$$x = A \cos \alpha, \quad y = A \sin \alpha;$$

$$x_a = aA \cos \alpha, \quad y_a = aA \sin \alpha.$$

Solving for x_a and y_a , we obtain:

$$x_a = ax, \quad y_a = ay.$$

Thus the terminal point of $a\mathbf{A}$ is (ax, ay) . Since, if a is a negative scalar, $a\mathbf{A} = |a|(-\mathbf{A})$, the terminal point of $a\mathbf{A}$ is $(|a|[-x], |a|[-y])$, which can be written (ax, ay) . Hence, if a is any scalar and $P(x, y)$ is the terminal point of a vector \mathbf{A} with initial point at the origin, the terminal point of $a\mathbf{A}$ is (ax, ay) .

Example 2. Given the vectors **A**, **B**, and **C** with the terminal points $P_A(4, 1)$, $P_B(-3, 2)$, and $P_C(1, 3)$ respectively. Find the terminal point of $\mathbf{D} = 2\mathbf{A} + 3\mathbf{B} - 4\mathbf{C}$ and verify this result by adding the vectors graphically.

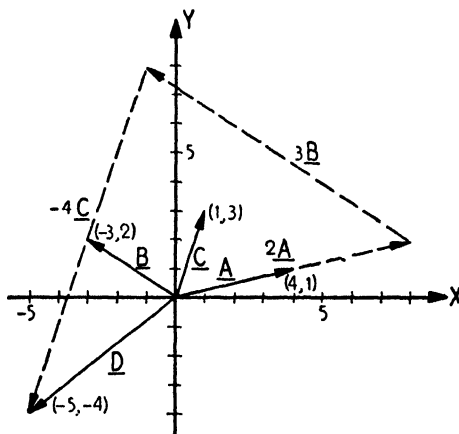


FIG. 6-13.

The x -coordinate of the terminal point of **D** is $2 \cdot 4 + 3(-3) - 4 \cdot 1 = -5$, and the y -coordinate is $2 \cdot 1 + 3 \cdot 2 - 4 \cdot 3 = -4$. Thus the terminal point of **D** is $(-5, -4)$. The graphical addition is shown in Fig. 6-13.

Example 3. The vectors **A** and **B** have initial points at the origin and direction angles $\alpha = 33.2^\circ$ and $\beta = 119.3^\circ$, respectively. If $A = 25.6$ and $B = 14.2$, find the magnitude of $\mathbf{C} = \mathbf{A} + \mathbf{B}$, and its direction angle, using the slide rule and obtaining results accurate to three significant figures. Verify the result by a graphical construction.

Let the terminal points of **A**, **B**, and **C** be denoted by $P_A(x_A, y_A)$, $P_B(x_B, y_B)$, and $P_C(x_C, y_C)$, respectively. Then

$$x_A = 25.6 \cos 33.2^\circ = 25.6 (0.837) = 21.4,$$

$$y_A = 25.6 \sin 33.2^\circ = 25.6 (0.548) = 14.03,$$

$$x_B = 14.2 \cos 119.3^\circ = 14.2 (-0.489) = -6.94,$$

$$y_B = 14.2 \sin 119.3^\circ = 14.2 (0.872) = 12.38.$$

It follows that

$$x_C = 21.4 - 6.94 = 14.46 = 14.5,$$

$$y_C = 14.04 + 12.38 = 26.42 = 26.4,$$

and that

$$C = \sqrt{(14.5)^2 + (26.4)^2} = \sqrt{210 + 697}$$

$$= \sqrt{907} = 30.1.$$

If γ is the direction angle of \mathbf{C} ,

$$\tan \gamma = \frac{26.4}{14.5} = 1.82,$$

and

$$\gamma = 61.2^\circ.$$

The graphical construction is shown in Fig. 6-14.

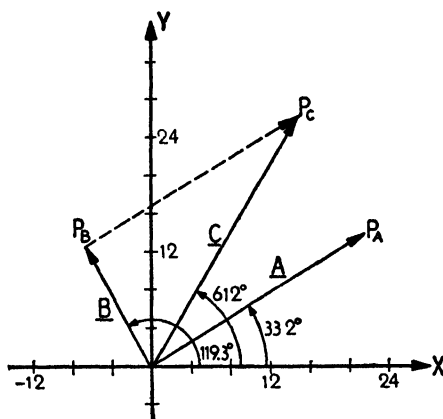


FIG. 6-14.

EXERCISES

Each of the given vectors has initial point at the origin and is given by its terminal point. Given \mathbf{A} : (3, 2); \mathbf{B} : (1, 4); \mathbf{C} : (0, 3); \mathbf{D} : (-2, -2); \mathbf{E} : (-5, 0); \mathbf{F} : (-3, -2); \mathbf{G} : (-1, -4); \mathbf{H} : (2, -3); and \mathbf{I} : (4, -1). Find the coordinates of the terminal point of each of the following vectors \mathbf{J} , and verify your result by performing the operations graphically.

1. $\mathbf{J} = \mathbf{A} + \mathbf{B} + \mathbf{C}$.
2. $\mathbf{J} = \mathbf{B} + \mathbf{C} + \mathbf{D}$.
3. $\mathbf{J} = \mathbf{D} + \mathbf{E} + \mathbf{F} + \mathbf{G}$.
4. $\mathbf{J} = \mathbf{A} + \mathbf{B} + \mathbf{H} + \mathbf{I}$.
5. $\mathbf{J} = 2\mathbf{E} + 2\mathbf{H}$.
6. $\mathbf{J} = 2\mathbf{A} + 3\mathbf{B} + 4\mathbf{G}$.
7. $\mathbf{J} = 3\mathbf{A} - 2\mathbf{B} + \mathbf{F}$.
8. $\mathbf{J} = \mathbf{B} - 2\mathbf{D} + 3\mathbf{F}$.
9. $\mathbf{J} = \mathbf{C} - 2\mathbf{D} + 3\mathbf{G}$.
10. $\mathbf{J} = \mathbf{A} + 2\mathbf{D} - \mathbf{H}$.
11. $\mathbf{J} = \mathbf{G} - \mathbf{H} + 2\mathbf{A}$.
12. $\mathbf{J} = \mathbf{B} - \mathbf{C} + \mathbf{F}$.
13. $\mathbf{J} = 2\mathbf{E} - 3\mathbf{F} + \mathbf{G}$.
14. $\mathbf{J} = \mathbf{C} + \mathbf{D} + \mathbf{E} + \mathbf{F} + \mathbf{G}$.
15. $\mathbf{J} = \mathbf{B} - 2\mathbf{F} + 3\mathbf{I}$.
16. $\mathbf{J} = 2\mathbf{F} - \mathbf{H} + 3\mathbf{A}$.
17. $\mathbf{J} = \mathbf{A} + \mathbf{C} - \mathbf{E} + 3\mathbf{G}$.
18. $\mathbf{J} = 2\mathbf{C} + \mathbf{A} + \mathbf{G}$.
19. $\mathbf{J} = \mathbf{E} + 3\mathbf{D} - 4\mathbf{F}$.
20. $\mathbf{J} = \mathbf{A} + 2\mathbf{B} + \mathbf{F} + \mathbf{G}$.

Each of the points given below is the terminal point of a vector with initial point at the origin. Find the magnitude of the vector correct to three significant figures, and the direction angle which is between 0° and 360° correct to the nearest tenth of a degree. Use the slide rule in all computations.

- | | | |
|-------------|-------------|--------------|
| 21. (2, 3). | 22. (3, 4). | 23. (6, 0). |
| 24. (5, 6). | 25. (0, 8). | 26. (-1, 6). |

27. $(-2, 5)$.

28. $(-3, 4)$.

29. $(-5, 5)$.

30. $(-7, 0)$.

31. $(-8, -2)$.

32. $(-7, -5)$.

33. $(-14, -8)$.

34. $(0, -5)$.

35. $(5, -3)$.

36. $(3, -4)$.

37. $(8, -3)$.

38. $(14, -6)$.

39. $(6, -2)$.

40. $(10, -10)$.

In each exercise below the magnitude and direction angle of a vector with initial point at the origin is given. Find the coordinates of the terminal point of each vector correct to three significant figures, using the slide rule in computations.

41. 10.0, 30.2° .

42. 36.1, 47.2° .

43. 51.2, 69.5° .

44. 47.8, 82.3° .

45. 15.2, 41.5° .

46. 95.3, 102.3° .

47. 159, 136.2° .

48. 67.3, 170.1° .

49. 86.2, 156.3° .

50. 19.5, 195.2° .

51. 37.3, 198.2° .

52. 18.4, 243.5° .

53. 17.6, 282.3° .

54. 35.2, -46.2° .

55. 48.3, -51.7° .

56. 86.4, -98.2° .

57. 76.2, -48.3° .

58. 596, -125.3° .

59. 1590, 351.2° .

60. 2040, 347.2° .

Each vector given below has its initial point at the origin. The direction angle is denoted by the Greek letter corresponding to its name, as A, α . Find the magnitude and direction angle of the sum of the vectors given in each exercise. Use the slide rule and obtain results accurate to three significant figures and the nearest tenth of a degree.

61. A: $A = 46.2$, $\alpha = 36.1^\circ$.

B: $B = 35.1$, $\beta = 99.3^\circ$.

63. A: $A = 0.823$, $\alpha = 136.2^\circ$.

B: $B = 0.436$, $\beta = 218.1^\circ$.

65. A: $A = 859$, $\alpha = 256.3^\circ$.

B: $B = 642$, $\beta = 328.4^\circ$.

67. A: $A = 6.73$, $\alpha = 138.3^\circ$.

B: $B = 5.94$, $\beta = 285.2^\circ$.

69. A: $A = 2.86$, $\alpha = 46.0^\circ$.

B: $B = 3.79$, $\beta = -35.8^\circ$.

71. A: $A = 49.2$, $\alpha = 51.8^\circ$.

B: $B = 51.3$, $\beta = 212.4^\circ$.

C: $C = 65.8$, $\gamma = 125.2^\circ$.

73. A: $A = 6.73$, $\alpha = 59.2^\circ$.

B: $B = 5.08$, $\beta = -36.7^\circ$.

C: $C = 4.75$, $\gamma = 138.2^\circ$.

75. A: $A = 0.525$, $\alpha = 98.2^\circ$.

B: $B = 0.673$, $\beta = 137.5^\circ$.

C: $C = 0.896$, $\gamma = -42.5^\circ$.

62. A: $A = 8.97$, $\alpha = 61.5^\circ$.

B: $B = 6.32$, $\beta = 125.2^\circ$.

64. A: $A = 0.0437$, $\alpha = 215.2^\circ$.

B: $B = 0.0126$, $\beta = 300.4^\circ$.

66. A: $A = 4950$, $\alpha = -46.2^\circ$.

B: $B = 3840$, $\beta = -108.3^\circ$.

68. A: $A = 8.76$, $\alpha = 196.0^\circ$.

B: $B = 9.92$, $\beta = 78.3^\circ$.

70. A: $A = 46.8$, $\alpha = 51.2^\circ$.

B: $B = 59.2$, $\beta = -95.2^\circ$.

72. A: $A = 519$, $\alpha = 156.2^\circ$.

B: $B = 643$, $\beta = 235.8^\circ$.

C: $C = 859$, $\gamma = 46.9^\circ$.

74. A: $A = 0.876$, $\alpha = 45.9^\circ$.

B: $B = 0.502$, $\beta = 67.3^\circ$.

C: $C = 0.617$, $\gamma = 169.4^\circ$.

76. A: $A = 6970$, $\alpha = 67.4^\circ$.

B: $B = 4230$, $\beta = 89.5^\circ$.

C: $C = 1576$, $\gamma = 235.4^\circ$.

6-6. Application to Engineering Problems: The Inclined Plane. Force is one of the most important vector quantities with which the engineer deals. In order to solve problems involving the application of forces, it is necessary to understand several fundamental physical principles. One of the first laws of mechanics states that *when a body is at rest or moving with a constant velocity, the resultant force upon that body is zero*. Thus, when a book is lying on a table, the action of gravity exerts a

force downward, and the table must exert an equal and opposite force upon the book in order that the resultant force upon the book will be equal to zero. It is an experimental fact that *the force due to the action of gravity acts vertically downward*.

With these two fundamental principles of mechanics and the technique of vector manipulation, the student is equipped to consider the problem of the inclined plane. Wedges, factory chutes, and V-type automobile engines involve these basic principles.

Example. A loading platform is 4 ft. high, and a board is placed from the platform to the ground, thus becoming an inclined plane. The board rests on the ground at a distance of 4 ft. measured horizontally from the edge of the platform as shown in Fig. 6-15. If a box weighing 100 lb. is being pushed up the board with a constant velocity, and a frictional force of 10 lb. acts in a direction opposite to that of the motion of the box, what is the magnitude and direction of the force being applied to the box?

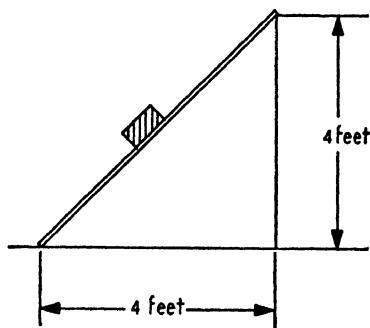


FIG. 6-15.

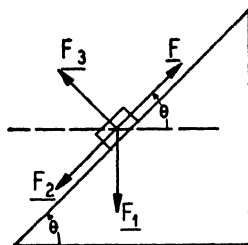


FIG. 6-16.

The vector diagram is drawn in Fig. 6-16. There is a downward force of 100 lb. due to the action of gravity. This is represented vertically downward in the figure by the vector F_1 . As the box is being pushed up the incline, the frictional force acting in a direction opposite to that of the motion of the box will be directed as indicated by the force F_2 .

The box is being pushed up the plane and hence a force F will have to act upward along the inclined plane as indicated in the figure. This force F is the applied force which is to be determined.

There is one additional force, namely, that of the board pushing upward with a force F_3 against the bottom of the box as indicated in Fig. 6-16. This force must act in this direction since the box is supported by the board. The magnitude of this force is not given.

Since the box is moving with a constant velocity, we may apply the law of mechanics which states that the resultant force acting on the box is zero.

The forces acting on the box may be set up as indicated in Fig. 6-17. Since these four forces are the only forces acting on the box, the sum of them must be zero.

From Fig. 6-15 it is seen that the angle θ which the board makes with the horizontal and likewise with the vertical is 45° .

It is evident then that the angle between the force F which lies along the inclined plane and a horizontal line as indicated in Fig. 6-17 is also the angle θ and, therefore, 45° .

Since the opposite angles of two intersecting lines are equal, the same figure shows that the angle between the frictional force F_2 and the horizontal is also equal to $\theta = 45^\circ$.

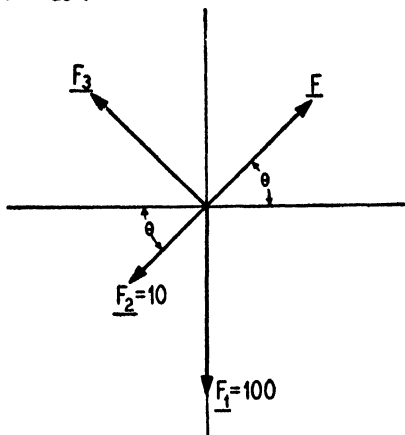


FIG. 6-17.

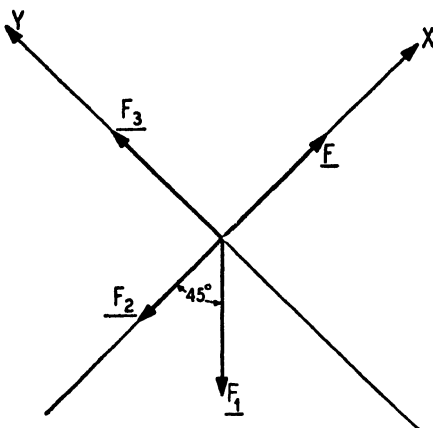


FIG. 6-18.

In this case it is convenient to resolve the forces into components along the inclined plane and perpendicular to the inclined plane. We, therefore, place the x -axis along the inclined plane as shown in Fig. 6-18. Thus the force F_3 is perpendicular to the inclined plane and hence is along the y -axis. The force F_1 makes an angle of 45° with the inclined plane and hence makes an angle of 45° with the x -axis. Thus the vector F_1 will have terminal point:

$$x_1 = -100 \sin 45^\circ = -70.7 \text{ lb.},$$

$$y_1 = -100 \cos 45^\circ = -70.7 \text{ lb.}$$

The terminal point of the frictional force vector F_2 will be given by

$$x_2 = -10 \text{ lb.},$$

$$y_2 = 0.$$

The terminal point of F_3 , the force due to the action of the inclined plane, will be

$$x_3 = 0,$$

$$y_3 = +F_3 \text{ lb.}$$

The applied force F will have terminal point:

$$x_4 = +F \text{ lb.},$$

$$y_4 = 0.$$

We are asked to solve for the applied force F . The condition which must be satisfied is that the resultant of the four forces must be equal to zero. If a force

is equal to zero its x - and y -components must also be equal to zero. Therefore, the resultant x - and y -components of the forces of the problem must be equal to zero, to satisfy the required conditions. Solving for these components in terms of the two unknowns F and F_3 , two equations in two unknowns are obtained.

Adding the x -coordinates algebraically and equating to zero, we obtain

$$(1) \quad x_1 + x_2 + x_3 + x = -70.7 - 10 + 0 + F = 0$$

$$(2) \quad y_1 + y_2 + y_3 + y = -70.7 + 0 + F_3 + 0 = 0$$

Solving equation (1) for F :

$$(3) \quad F = 80.7 \text{ lb.}$$

Solving equation (2) for F_3 :

$$F_3 = 70.7 \text{ lb.}$$

The problem asked for the applied force F and, hence, the answer is that $F = 80.7 \text{ lb.}$, *directed upward along the inclined plane.*

6-7. The Engine Speed Governor. Another force associated particularly with rotating machinery is **centrifugal force**. This force occurs whenever there is rotation of any type and is always directed outward from the axis of rotation. If a stone is fastened to the end of a string and whirled about, the stone tends to fly outward and eventually, if whirled fast enough, would rotate in a horizontal plane. This tendency to fly outward is due to the action of the centrifugal force. The inward constraining force exerted by the string on the stone keeps the latter in a circular path. Another example of this force is clearly illustrated when a pail of water is whirled around in a vertical plane. The water will be forced to the bottom of the pail and will not spill even when the pail is upside down. Centrifugal force depends on velocity and increases with it. An engine flywheel may fracture if rotated at too high a speed, because of the difference in the action of centrifugal force upon various parts of the wheel, and this fact consequently must be considered in the design.

The engine speed governor is one of the many engineering applications which utilize this force. Since the magnitude of the centrifugal force is a function of the speed, this force may be used to open or close a valve. Whenever the speed of the engine becomes higher than a predetermined value, the governor closes the valve in the steam supply, causing the engine to slow down. Exactly the opposite action occurs when an engine begins to run too slowly. A similar governing device coupled with frictional means is used to maintain constant speed of phonograph turntables.

Consider a simple type where the governor mechanism consists of two metallic cylinders located directly opposite one another and which are rotated about an axis which cuts through the center and is perpendicular

to a line drawn between the two cylinders. The two cylinders are connected to the top of the shaft about which they rotate by means of rods from the top of the shaft to the cylinder as indicated in Fig. 6-19.

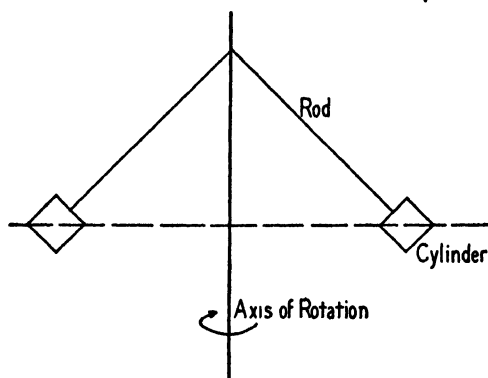


FIG. 6-19.

The connection at the top of the shaft is flexible, in order that the cylinders may respond freely to the action of gravity and to the action of the centrifugal force when they are rotating. A lever arrangement not shown in the figure transmits the movement of the cylinders to the valve of the steam engine.

Example. The centrifugal force exerted by each brass cylinder of a steam engine governor is 100 lb., and the force due to the action of gravity is 20 lb. on each cylinder. The governor is perfectly symmetrical. What will be the angle between the rod holding one of the cylinders and the shaft?

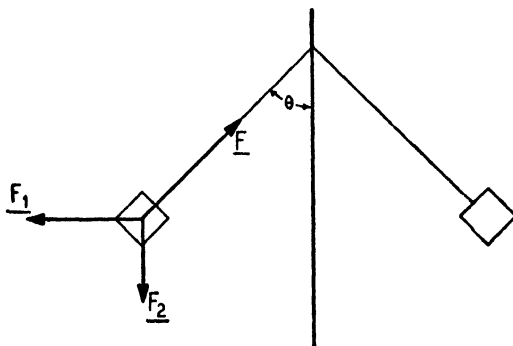


FIG. 6-20.

We first construct a vector diagram indicating all forces acting upon a cylinder, as shown in Fig. 6-20. The centrifugal force F_1 of magnitude 100 lb. will act directly outward from the axis of rotation as indicated in the figure, and a force F_2

of magnitude 20 lb. will act vertically downward due to the action of gravity. There will be a force F acting along the rod joining the cylinder to the shaft as shown. The angle θ , which we wish to find, then is equal to the angle which the force F makes with the shaft.

The forces acting on the cylinders are redrawn on a coordinate system as shown in Fig. 6-21. If the cylinder is not moving outward or inward and not moving up-

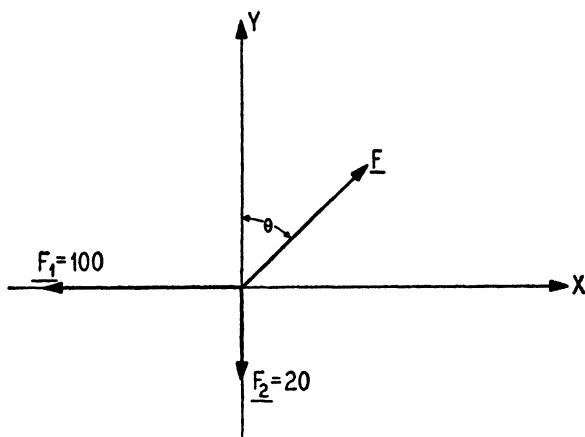


FIG. 6-21.

ward or downward, the resultant forces acting in these directions must be zero, according to the previously given fundamental law of mechanics.

The terminal point of F_1 is given by

$$x_1 = -100 \text{ lb.},$$

$$y_1 = 0.$$

The terminal point of F_2 is given by

$$x_2 = 0,$$

$$y_2 = -20 \text{ lb.}$$

Let θ be the angle between the force F and the y -axis, as indicated in Fig. 6-21. The terminal point of force F will then be given by

$$x = F \sin \theta,$$

$$y = F \cos \theta.$$

Since the resultant of the three forces in Fig. 6-21 must be equal to zero, the resultant of the vertical and horizontal components of these three forces must also be equal to zero.

Adding the respective coordinates we have:

$$(1) \quad x_1 + x_2 + x = -100 + 0 + F \sin \theta = 0,$$

$$(2) \quad y_1 + y_2 + y = 0 - 20 + F \cos \theta = 0.$$

Rewriting equations (1) and (2):

$$(3) \quad F \sin \theta = 100,$$

$$(4) \quad F \cos \theta = 20.$$

Dividing equation (3) by equation (4)

$$\frac{\sin \theta}{\cos \theta} = \tan \theta = 5.$$

Therefore

$$\theta = 79^\circ.$$

6-8. Vector Representation of Alternating Currents and Voltages.

One of the important engineering applications of vectors occurs in the solution of problems dealing with an alternating current. It may be shown that whenever some quantity varies sinusoidally with respect to time, the value at any instant may be represented by a vector. Modern alternating-current generators produce a sinusoidal variation of voltage with respect to time, and therefore the voltage at any instant may be represented by a vector. Ordinarily the current which flows through a circuit because of a sinusoidally applied voltage will also be sinusoidal.

In order to consider a typical application of vector algebra to the solution of an alternating-current problem, it is necessary to introduce several facts with regard to these currents. It may be shown experimentally that there are three fundamental properties, resistance, capacitance, and inductance, which a circuit may possess. The circuit may possess each property individually or in any combination. It can be shown that:

1. The voltage drop across a resistor may be represented by a vector which has the same direction as the vector representing the current flowing through the resistor.

2. The voltage drop across a condenser may be represented by a vector which makes an angle of 270° or -90° with the vector representing the current flowing through the condenser.

3. The voltage drop across an inductor may be represented by a vector which makes an angle of 90° with the vector representing the current flowing through the inductance.

4. The vector sum of all the voltage drops in a series circuit must be equal to the applied voltage.

Voltage drop is the name given to the quantity which a voltmeter measures when placed across a resistor, a condenser, an inductor, or any combination of these elements. In Fig. 6-22 a simple series circuit is shown, indicating the symbols used to represent a condenser, resistor, an inductor, and the generator. A series circuit is one in which the same current flows through every element of the circuit.

The straight lines represent wires; current will flow through the wires

and must flow out one side of the generator and in the other. From Fig. 6-22 it is apparent that the same current will flow through each

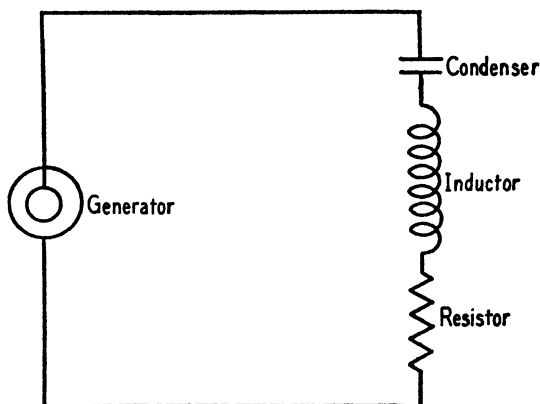


FIG. 6-22.

element of the circuit, and so the figure constitutes a diagram of a series circuit. The generator voltage in the circuit of Fig. 6-22 is the applied voltage.

Consider a problem utilizing the vector representation of voltage and current.

Example. In a simple series circuit of inductance, capacitance, and resistance, the voltage drop across the condenser is 10 volts, that across the inductance 50 volts, and that across the resistance 30 volts. Determine the magnitude and the direction of the applied voltage.

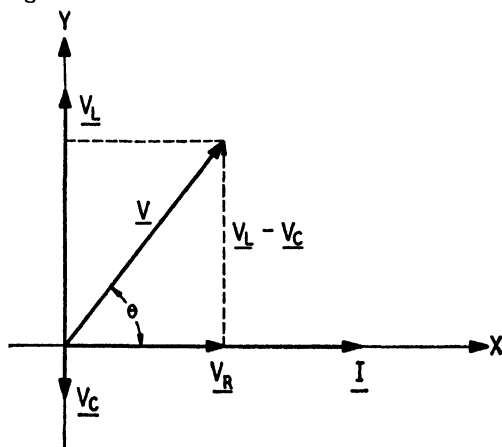


FIG. 6-23.

The vector diagram for this circuit is drawn in Fig. 6-23. In a series circuit the same current flows through each element. Denote the vector representing this cur-

rent by I and let this vector be drawn in a horizontal direction as indicated in Fig. 6-23. It is necessary to choose arbitrarily this vector since the angle of all the vectors representing the voltage drops are given with respect to this current. After choosing the direction of the vector I , the relative positions of the vectors representing the voltage drops are determined. From the fundamental principles stated above, the vector V_L , representing the voltage drop across the inductance, will make an angle of 90° with the vector I as shown in Fig. 6-23. The vector V_C , representing the drop across the condenser, will make an angle of 270° or -90° with the current vector as shown. Finally, the vector V_R , representing the voltage drop across the resistor, will make an angle of 0° with I .

According to the introductory Principle 4 stated above, the vector sum of all these voltage drops will be equal to the applied voltage.

Now

$$V_L = 50, \quad V_C = 10, \quad V_R = 30.$$

Let the vector I lie along the x -axis. Using the conventions of the preceding two examples,

$$x_L = 0, \quad x_C = 0, \quad x_R = 30;$$

$$y_L = 50, \quad y_C = -10, \quad y_R = 0.$$

Adding the x - and y -coordinates respectively:

$$x = x_L + x_C + x_R = 0 + 0 + 30 = 30,$$

$$y = y_L + y_C + y_R = 50 + (-10) + 0 = 40.$$

The magnitude of the applied voltage will be given by

$$V = \sqrt{x^2 + y^2} = \sqrt{(30)^2 + (40)^2} = 50 \text{ volts.}$$

The angle θ which the vector representing the applied voltage makes with the vector I is found from $\tan \theta = \frac{y}{x} = \frac{4}{3}$ to be 53.1° .

EXERCISES

1. An automobile in going around a curve at a constant speed has three forces acting upon it: the force of gravity vertically downward, the force due to the friction between the tires and the road directed toward the center of the curve and parallel to the road, and the centrifugal force directed outward from the center of the curve in a horizontal direction. If the pavement around the curve is level, only the force of friction counteracts the centrifugal force. However, if the road is banked properly, the force due to the action of gravity assists the friction; the effects are similar to those of an inclined plane. If the road is banked at an angle of 30° with the horizontal, the force due to the action of gravity is 3000 lb. and the centrifugal force is 1000 lb. What will be the required magnitude of the frictional force between the tires and the road?

2. A man is pulling a heavy box up an inclined plane. A rope is attached to the center of the box, the man pulls on this rope which makes an angle 30° with the inclined plane. The force which the man exerts is directed along the rope. If the inclined plane makes an angle of 45° with the horizontal, the force due to friction is 10 lb. and the force due to the action of gravity is 500 lb. directed vertically downward.

What force must be exerted by the man to move the box up the plane at a constant velocity?

3. A box, standing on a board, weighs 200 lb. and slides down with a constant speed if the board makes an angle of 35° with the horizontal direction. Find the force of friction between the box and the board.

4. An object of weight 500 lb. is on an inclined plane and it is known that the friction between the object and the plane is one-tenth of the force which works perpendicular to the inclined plane between this plane and the body. What is the angle between the inclined plane and the horizontal plane if a force of 50 lb. is sufficient to push up the object with a constant speed?

5. A machine part is hung from the inclined roof of a small building. The force due to the action of gravity upon the machine part is 800 lb., acting vertically downward. The roof makes an angle of 15° with the horizontal, and therefore the gravitational force upon the machine part has a line of action which makes an angle of 75° with the roof. Determine the force acting along the roof.

6. Consider a governor such as described in Sec. 6-7 in which three forces are acting: the force due to gravity, the centrifugal force and the force exerted by the rod which connects the cylinders to the top of the shaft about which the cylinders rotate. If the force due to the action of gravity on each cylinder is 10 lb., what must be the centrifugal force in order that the angle between the shaft and the rod holding the cylinder be 30° ?

7. Assume that the centrifugal force working on each cylinder of a governor of the kind described in Sec. 6-7, is 80 lb. What is the weight of each cylinder in order that the angle between the shaft and the rod which holds the cylinder be 25° ?

8. The rafters of a gable roof of a building come to a point at the top framing a V-shaped structure. Consider two of these rafters. The width of the building is 20 ft. Two timbers are fastened together at the top to form the V-structure. The point of the V is located directly in the center of the building. If the force due to the action of gravity on the roof structure is 500 lb. and acts vertically downward from the point of the V, and the vertical height of the gable is 10 ft., what will be the force acting along each timber?

9. A trolley wire of a street car system is supported by a wire stretched between two poles, separated by a distance of 50 ft. The supporting wire sags 4 in. at the center due to the weight of the trolley wire. The force causing the sag in the wire has a magnitude of 500 lb. at the central point and is directed downward. What will be the force in each direction along the supporting wire at a point midway between the two poles?

10. An alternating-current circuit is composed of a condenser and an inductance in series. If the voltage drop across the inductance is 25 volts and that across the condenser is 80 volts, what is the magnitude of the applied voltage, and what angle does the vector representing the applied voltage make with the vector representing the current flowing through the circuit?

11. A series combination of inductance, capacitance, and resistance is connected across an alternating voltage. The vector representing this applied voltage makes an angle of 30° with the vector representing the current. If the voltage drop across the inductance is 80 volts and that across the condenser 20 volts, what will be the magnitude and direction of the vector representing the voltage drop across the resistor?

12. The winding of an electromagnet is connected across an alternating-current supply line. The magnet winding may be considered to consist of an inductive and

a resistive portion in series. The voltage drop across the inductive portion of the winding is 180 volts. If the vector representing the applied voltage makes an angle of 75° with the current vector, what will be the magnitude and the direction of the vector representing the resistance voltage drop?

13. A condenser in a radio amplifier is connected in series with a resistor. If the voltage drop across the condenser is 20 millivolts and the voltage drop across the resistor is 100 millivolts, what will be the magnitude and direction of the vector representing the applied voltage?

14. An alternating-current circuit consists of a condenser, an inductor, and a resistor in series. The voltage drop across the inductance is 90 volts. What are the magnitudes of the voltage drops across the condenser and the resistor, and what are the angles between the vectors of these voltages and the current vector if the total alternating voltage is 110 volts and the corresponding vector makes an angle of 45° with the current vector?

PROGRESS REPORT

A number of problems were enumerated in the introduction of this chapter, which deal with physical quantities that have not only a magnitude but also a direction. In order to have a mathematical concept which could be used in dealing with such problems, the idea of a vector as a directed line segment was introduced. The operations with vectors, including addition, subtraction, and multiplication by a scalar factor, were defined, and thus a vector algebra was constructed. A special problem which is of great importance in many applications is the resolution of a vector into two components with given directions, especially into two components parallel to the x - and y -axis. This resolution of a vector into components was carried out graphically and numerically, and it was used to find the resultant of a system of forces, or, generally, to find by computation the sum of any number of vectors.

Several applications to engineering problems, including the inclined plane, the engine governor, and the alternating circuits, showed how the solution of such problems is simplified by introducing the use of vectors.

CHAPTER 7

ALGEBRAIC OPERATIONS

In the previous chapters we considered certain simple algebraic expressions and attained a skill in manipulating them. However, in both theoretical and applied mathematics much more complex algebraic relations are encountered. Therefore, if we are to be able to solve a wide variety of practical and theoretical problems, it is necessary that we possess skill in manipulating these more complex algebraic relations. The purpose of this chapter is to develop this skill.

7-1. Special Products. Certain special products were given in Sec. 2-7. These are repeated here together with other new forms which are very useful in mathematics and its applications. They can all be verified by actual multiplication.

$$(1) \quad (a + b)(a - b) = a^2 - b^2.$$

$$(2) \quad (a + b)^2 = a^2 + 2ab + b^2.$$

$$(3) \quad (a - b)^2 = a^2 - 2ab + b^2.$$

$$(4) \quad a(x + y - z) = ax + ay - az.$$

$$(5) \quad (a + b)(x - y) = ax + bx - ay - by.$$

$$(6) \quad (x + a)(x + b) = x^2 + (a + b)x + ab.$$

$$(7) \quad (ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

$$(8) \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

$$(9) \quad (a + b)(a^2 - ab + b^2) = a^3 + b^3.$$

$$(10) \quad (a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

$$(11) \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(12) \quad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$(13) \quad (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(14) \quad (a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

The reader should satisfy himself of the correctness of the above forms by actual multiplication.

The above products can be described in words. For example, formula (8) states that *the square of a trinomial equals the sum of the squares of all the terms of the trinomial plus twice the product of each term by every term that follows it*. Similar statements may be supplied by the reader for the other formulas.

Algebraic computations may be shortened by the use of the above forms, as illustrated in the following examples.

Example 1. Find the product $(2L - 3)(4L + 5)$.

Using (7), where we substitute L for x , 2 for a , -3 for b , 4 for c , and 5 for d , we obtain:

$$(2L - 3)(4L + 5) = 8L^2 - 2L - 15.$$

Example 2. Expand the expression $(Rx + Py + Qz)^2$.

Substituting Rx for a , Py for b , and Qz for c in (8) we obtain:

$$(Rx + Py + Qz)^2 = R^2x^2 + P^2y^2 + Q^2z^2 + 2PRxy + 2QRxz + 2PQyz.$$

Example 3. Find the product $(x - 2y)(x^2 + 2xy + 4y^2)$.

Substituting x for a and $2y$ for b in (10) we obtain:

$$(x - 2y)(x^2 + 2xy + 4y^2) = x^3 - 8y^3.$$

Example 4. Expand $(z + 2)^3$.

Substituting z for a and 2 for b in (11), we obtain:

$$(z + 2)^3 = z^3 + 6z^2 + 12z + 8.$$

Example 5. Using formula (14) find $(0.99)^4$.

Substituting 1 for a and 0.01 for b in (14) we obtain:

$$\begin{aligned} (0.99)^4 &= (1 - 0.01)^4 \\ &= 1^4 - 4 \cdot 1^3 \cdot 0.01 + 6 \cdot 1^2 \cdot (0.01)^2 - 4 \cdot 1 \cdot (0.01)^3 + (0.01)^4 = 0.96. \end{aligned}$$

Formulas 2, 3, 11, 12, 13, and 14 are particular cases of a general form called the **binomial expansion**, given by the following:

$$\begin{aligned} (a + b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 \\ (15) \quad &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \dots + nab^{n-1} + b^n. \end{aligned}$$

EXERCISES

Perform the indicated operations.

1. $(6K + 5)(6K - 5)$.

3. $(3P + 4)^2$.

5. $(y - 3)^2$.

7. $4(K + N - S)^2$.

2. $(2x + y)^2$.

4. $(-6xy + 7)(6xy + 7)$.

6. $(2r^2 - 3s^2)^2$.

8. $(a + c)(Q_1 + Q_2)$.

9. $(2C + L)(2n + k)$.
11. $(4Q + 2)(2Q - 1)$.
13. $(x - y + z)^2$.
15. $(4x^2 - 6x + 9)(2x + 3)$.
17. $(9a^2 + 12a + 16)(3a - 4)$.
19. $(7Q^n + 2Q^{n-1})^3$.
21. $(h - k)^3$.
23. $(n^2 + d)^4$.
25. $(Z^{n-1} - I)^4$.
27. $(7p + 3)(3p - 7)$.
29. $(2 + z)(2 + y)$.
31. $(6a^2 + 5)(3 - a^2)$.
33. $(x + y + 2)^2$.
35. $(3 - L - L^2)^2$.
37. $(2X_1 - X_2 - 2)^2$.
39. $(w - 5x + 3a)^2$.
41. $(4x^2 - 3xy^2 - 5x^3)^2$.
43. $(5 - ei)^4$.
45. $(4a^n + 5b^m)(12a^n - 15b^m)$.
47. $(b^{2n} - 1)(b^{4n} + b^{2n} + 1)$.
49. $(2P - 1)^5$.
51. $(r - 3s)^6$.
53. $(c^6 + 7c^3 + 49)(c^3 - 7)$.
10. $(4a + 8y)(a - 2y)$.
12. $(a + 2b + c)^2$.
14. $(x - y - z)^2$.
16. $(a^2 + g^2)(a^4 - a^2g^2 + g^4)$.
18. $(EI + P_1)^3$.
20. $(3 - 4T)^3$.
22. $(1 + x)^4$.
24. $\left(x^3 - \frac{1}{x}\right)^4$.
26. $(a - c)(w_1 + w_2)$.
28. $(3y - 8)(3y - 2)$.
30. $(y + 7)(y - 4)$.
32. $(x + y - 3)^2$.
34. $(7 - l)(-5 - l)$.
36. $(1 - a)(1 + a + a^2)$.
38. $(x^2 + xy + y^2)(x - y)$.
40. $(2h + w)^3$.
42. $\left(\frac{1}{3} + xy\right)\left(\frac{1}{9} - \frac{xy}{3} + x^2y^2\right)$.
44. $(2ym^3 + 3y^2m^2)^3$.
46. $(3y^{2m-1} - z^{m+1})(3y^{2m-1} + 2z^{m+1})$.
48. $(2x^{m+n} - 3x^{m-n})^2$.
50. $(3s - \frac{1}{2})^6$.
52. $(2c^m - 3c^n)^2$.

Verify the following identities:

54. $(x^2 + y^2)(r^2 + s^2) - (xr + ys)^2 = (xs - yr)^2$.
55. $(x + y + c)^2 + (x - y)^2 + (x - c)^2 + (y - c)^2 = 3(x^2 + y^2 + c^2)$.
56. $(I_1 - I_2)^3 + 3(I_1 - I_2)^2(I_1 + I_2) + (I_1 + I_2)^3 + 3(I_1 - I_2)(I_1 + I_2)^2 = 8I_1^3$.

7-2. Factoring. In Sec. 7-1 we learned from formulas 1 through 15 how to write by inspection the product of certain special types of forms. As in Sec. 2-8, we may now interpret formulas 1 through 14 of the preceding section as working from right to left and thus obtain the process of factoring. In the problems of the present section we shall be given the product and we shall have to discover its factors. The products that we shall learn to factor are special products of the type given in Sec. 7-1. To follow the work of the present article, the reader must be well acquainted with the formulas of the preceding section. The ability to factor an expression depends on one's ability to identify it with one or more forms of these special products. The reader's efficiency in factoring will increase with careful reading and working of the illustrative examples and exercises which follow.

Formula 7 of the preceding section enables us to factor trinomials like $mx^2 + px + q$. Such expressions can sometimes be factored by inspection into two binomials $(ax + b)(cx + d)$, where $ac = m$, $bd = q$, and $ad + bc = p$. As possibilities for a and c we take the various pairs of factors of m (the number which multiplies x^2); as possibilities for b and d we take the pairs of factors of q (the number free of x). We then try these various possibilities to give $ad + bc = q$. To illustrate this method consider:

Example 1. Factor $5x^2 + 16x + 3$.

The pairs of factors of 5 are 1 and 5. Pairs of factors of 3 are 1 and 3. By trial and error we find that $(1 \cdot 1) + (3 \cdot 5) = 16$ and hence

$$5x^2 + 16x + 3 = (x + 3)(5x + 1).$$

Example 2. Factor $21x^3 - 14x^2y - 56xy^2$.

Using formula (4) of Sec. 7-1 we have

$$21x^3 - 14x^2y - 56xy^2 = 7x(3x^2 - 2xy - 8y^2).$$

To factor the trinomial $3x^2 - 2xy - 8y^2$, we find that the pairs of factors of 3 are 1 and 3, and that of -8 are ± 1 and ∓ 8 or ± 2 and ∓ 4 . By trial and error we find that $(1 \cdot 4) - (2 \cdot 3) = -2$ and hence

$$3x^2 - 2xy - 8y^2 = (3x + 4y)(x - 2y).$$

Finally we have the result

$$21x^3 - 14x^2y - 56xy^2 = 7x(3x + 4y)(x - 2y).$$

Obviously not every trinomial of this form can be factored in this manner. Thus, for example, in $6R^2 + 5R + 4$ no such pair of binomials can be found.

Example 3. Factor $x^3 - 27y^3$.

Using formula (10) of Sec. 7-1 we have:

$$x^3 - 27y^3 = x^3 - (3y)^3 = (x - 3y)(x^2 + 3xy + 9y^2).$$

Example 4. Factor $9R^2 + 16R^2I^2 + 25 - 24R^2I + 30R - 40RI$.

Using formula (8) of Sec. 7-1 we have:

$$\begin{aligned} 9R^2 + 16R^2I^2 + 25 - 24R^2I + 30R - 40RI \\ &= (3R)^2 + (-4RI)^2 + 5^2 + 2(3R)(-4RI) + 2(3R)(5) + 2(-4RI)(5) \\ &= (3R - 4RI + 5)^2. \end{aligned}$$

Example 5. Factor $12a^2 - 8a^3 - 6a + 1$.

Rearranging the order of the terms and applying formula (12) of Sec. 7-1 we obtain:

$$\begin{aligned} 1 - 6a + 12a^2 - 8a^3 &= 1^3 - 3(1)^2(2a) + 3 \cdot 1(2a)^2 - (2a)^3 \\ &= (1 - 2a)^3. \end{aligned}$$

EXERCISES

Factor the following:

1. $(2a + 3)^2 - (a + 3)^2$.
2. $25m^2 - (3r + 2s)^2$.
3. $3x^2 + 7x + 2$.
4. $6x^2 - 3x - 3$.
5. $9y^2 + 6xy + x^2$.
6. $9x^2 - 12xyz + 4y^2z^2$.
7. $-x^4 + 2a^2x^3 - a^4x^2$.
8. $k^2 - l^2 + u^2 - 4 - 2uk + 4l$.
9. $x^2 + z^2 - 4yz + 2xz + 4y^2 - 4xy$.
10. $M^6 - 3M^4X^2 + 3M^2X^4 - X^6$.
11. $8R_1^3 - 12R_1^2R_2 + 6R_1R_2^2 - R_2^3$.
12. $8C^3 - 12C^2M + 6CM^2 - M^3$.
13. $6p^2q^2 + 4pq^3 + 4p^3q + q^4 + p^4$.
14. $-216\mu\beta^3 + 216\mu^2\beta^2 + 81\beta^4 - 96\mu^3\beta + 16\mu^4$.
15. $I_1^3 - 8I_2^3$.
16. $8p^3 - 125Q^3$.
17. $27m^3 + 8n^3$.
18. $125y^3 + 512z^3$.
19. $5x^2 + 4x - 12$.
20. $2y^2 + 5y + 2$.
21. $81M^4 - 256N^4$.
22. $49a^4 + 64c^6 - 112a^2c^3$.
23. $25 + 64x^6 + 80x^3$.
24. $3R^2 - 5R - 2$.
25. $7E^2 - E - 6$.
26. $\mu^2 - X^2 + 4\beta^2 - 4Y^2 - 4\mu\beta - 4YX$.
27. $L^3 - T^3$.
28. $\frac{27}{L^3} - \frac{T^3}{8}$.
29. $X^6 + Y^6$.
30. $K^3 - L^3 + 3KL^2 - 3K^2L$.
31. $3c^2 - 5cd - 2d^2$.
32. $P^2 + 4PR + 4R^2 + 4RT + 2PT + T^2$.
33. $4a^2 - ab - 3b^2$.
34. $1 - x^3$.
35. $27a^3 - b^3 - 27a^2b + 9ab^2$.
36. $75I_1I_2^2 - 15I_1^2I_2 - 125I_1^3 + I_2^3$.
37. $-10 - 29x + 21x^2$.
38. $10x^2 - x - 3$.
39. $(x + y + z)^2 - 64$.
40. $196\mu^2\beta^2X^2 + 112\mu\beta^2X^2Y$.
41. $K^2 - 2CL - 2CK + C^2 + 2KL + L^2$.
42. $9x^2 + 4z^2 - 12xz + 16yz - 24xy + 16y^2$.
43. $6Z^2 + 10Z - 4$.
44. $21y^2 + 28y - 28$.
45. $p^3 + q^3$.
46. $x^6 - 64y^6$.
47. $R_1^3S_1^3 + R_2^3S_2^3$.
48. $1 + 729E^3$.
49. $a^2 + 2ab - d^2 + b^2$.
50. $18R_1^2 - 3R_1R_2 - 45R_2^2$.
51. $10X_1^2 - 11X_1X_2 - 6X_2^2$.
52. $a^2 - 1 + b^2 + 2ab$.
53. $8 - (x + y + z)^3$.
54. $a^3 - 27c^3$.
55. $a^2 - 11a - 60$.
56. $1 - X_1^2 - 2X_1X_2 - X_2^2$.
57. $8M^2 - 46M - 12$.
58. $6x^2y^2 - 4xy - 42$.
59. $w^3x^6 + x^9a^3$.
60. $512(z + 1)^3 + 1$.
61. $a^3 + (b - c)^3$.
62. $4(a + b)^2 + 33(a + b) - 70$.
63. $-7X^n + 5X^{2n} - 6$.
64. $(x + y)^3 + z^3$.
65. $a^3 - (b^2 - c^2)^3$.
66. $16 - 16y^3 - n^4 + n^4y^3$.
67. $(x_0 + 1)^3 + (x_0 - 1)^3$.
68. $(2p + q)^3 - (p - q)^3$.
69. $x^3 - x^2y - x^2 - 6x + 6y + 6$.
70. $(2 + S)^3 - (2 - S)^3$.
71. $(x + 1)^3 - (x - 1)^3$.
72. $9x^2 + 6x - 8$.
73. $36m^2n + 27n^3 + 54mn^2 + 8m^3$.
74. $x^6 - y^{12}$.
75. $a^9 + b^3$.
76. $r^3 + 6r^4s - 91s^2$.
77. $1 + a^9$.
78. $64d^3 + 27\beta^3$.

79. $\mu^3 + 2X_a X_b^2 + 4\beta^2 - X_a^2 + 4\mu\beta - X_b^4.$

80. $16p^4q^2 - (4p^2 + 9q^2 + 12pq).$

81. $x^2 + y^2 - 2xy - z^2.$

82. $a^7b - ab^7.$

83. $4x^2 + 9y^2 + z^2 - 12xy - 4xz + 6yz.$

84. $x^4 - 3x^2 + 1.$

85. $4E^2 - (4E_1^2 + 12E_1E_2 + 9E_2^2).$

86. $27b^6 + 26a^3b^3 - a^6.$

87. $4K^4 + 25 - 29K^2.$

88. $x^2 + a + x - a^2.$

89. $-7s^4 + s^3 + 12.$

90. $-2\beta^2 + \beta^4 + 49.$

91. $Q^4 + 9 + 2Q^2.$

92. $B + B^3 - B^2 - 1.$

93. $3a^3b + 10c^2 - 5abc - 6a^2c.$

94. $1 + X^4 - 23X^2.$

95. $4\mu^4 - 29\mu^2 + 49.$

96. $16p^4 - 81p^2q^2 + 9q^4.$

97. $S^3 - S + r - r^3.$

98. $6T^2 + 4T + T^4 + 4T^3 + 1.$

99. $t^2 - 4tv + 4v^2 + 6vx - 3tx.$

100. $8a^3 + a^3b^3 + b^2a^2.$

7-3. Operations with Fractions. We can now add, subtract, multiply, and divide fractions whose denominators require factoring of the type explained in this chapter. The work of this present section is an extension to more complicated fractions of what was accomplished in Sec. 2-10 and Sec. 2-11. The reader should therefore review these sections before proceeding with the following illustrative examples and exercises.

Example 1. Simplifying,

$$\begin{aligned} & \frac{1}{r+R} + \frac{r-R}{r^2-rR+R^2} - \frac{r^2-rR}{r^3+R^3} \\ &= \frac{1}{r+R} + \frac{r-R}{r^2-rR+R^2} - \frac{r^2-rR}{(r+R)(r^2-rR+R^2)} \\ &= \frac{r^2-rR+R^2}{(r+R)(r^2-rR+R^2)} + \frac{(r-R)(r+R)}{(r+R)(r^2-rR+R^2)} - \frac{r^2-rR}{(r+R)(r^2-rR+R^2)} \\ &= \frac{r^2-rR+R^2+r^2-R^2-r^2+R^2}{(r+R)(r^2-rR+R^2)} \\ &= \frac{r^2}{r^3+R^3}. \end{aligned}$$

Example 2. Simplifying,

$$\begin{aligned} & \frac{a^3-b^3}{a(a+b)} \cdot \frac{b^2-a^2}{a^2b-a^4} \cdot \frac{a^4}{a^2-b^2} \\ &= \frac{(a-b)(a^2+ab+b^2)}{a(a+b)} \cdot \frac{(b-a)(b+a)}{a^2(b-a)} \cdot \frac{a^4}{(a-b)(a+b)} \\ &= \frac{a^2+ab+b^2}{a+b}. \end{aligned}$$

Example 3. Simplifying,

$$\begin{aligned} \frac{6x^2 + 13xy - 5y^2}{2x^2 + 9xy + 9y^2} \div \frac{2x^2 - xy - 15y^2}{6x^2 + 13xy + 6y^2} &= \frac{6x^2 + 13xy - 5y^2}{2x^2 + 9xy + 9y^2} \cdot \frac{6x^2 + 13xy + 6y^2}{2x^2 - xy - 15y^2} \\ &= \frac{(2x-5y)(3x-y)}{(x+3y)(2x+3y)} \cdot \frac{(2x+3y)(3x+2y)}{(x-3y)(2x+5y)} \\ &= \frac{(3x-y)(3x+2y)}{(x+3y)(x-3y)}. \end{aligned}$$

EXERCISES

Add:

- $\frac{9-3p}{16q} + \frac{3+5p}{20q^2}.$
- $\frac{X_1}{X_2} + \frac{2X_1^2 + X_2^2}{X_1X_2} + \frac{3X_1X_2^2 - X_1^3 - X_2^3}{X_1^2X_2} - \frac{4X_1X_2^3 - 2X_1^2X_2^2 - X_2^4}{X_1^2X_2^2}.$
- $\frac{11xy+2}{x^2y^2} - \frac{5y^2-3}{xy^3} - \frac{6x^2-5}{x^3y}.$
- $\frac{I_1^2 + I_1I_2 + I_2^2}{I_1 + I_2} - \frac{I_1^2 - I_1I_2 + I_2^2}{I_1 - I_2} + \frac{2I_2^3 - I_2^2 + I_1^2}{I_1^2 - I_2^2}.$
- $\frac{l-c}{l+c} + \frac{l+c}{l-c} + \frac{l^2+c^2}{l^2-c^2}.$
- $\frac{e}{e_1^3 + e_2^3} - \frac{e_2}{e_1^3 - e_2^3} + \frac{e_1^3e_2 + e_1e_2^3}{e_1^6 - e_2^6}.$
- $\frac{3}{i-r} + \frac{4r}{(i-r)^2} - \frac{5r^2}{(i-r)^3}.$
- $\frac{2a-5}{a^2+7a+10} + \frac{3}{a^2+a-2} - \frac{2(a+1)}{a^2+4a-5}.$
- $\frac{T+5}{T^2-4T+3} - \frac{T-2}{T^2-8T+15} + \frac{T+1}{T^2-6T+5}.$
- $\frac{5(I_1-3I_2)}{I_1-2I_2} + \frac{8}{I_1^2-5I_1I_2+6I_2^2} - \frac{2(I_1-2I_2)}{I_1-3I_2}.$
- $\frac{e^2-(w-t)^2}{(t+e)^2-w^2} + \frac{w^2-(t-e)^2}{(e+w)^2-t^2} + \frac{t^2-(e-w)^2}{(w+t)^2-e^2}.$
- $-\frac{2}{n^3+n^2+n+1} + \frac{3}{n^3-n^2+n-1}.$
- $\frac{\mu+1}{\mu^2-3\mu+9} - \frac{\mu^2+\mu+1}{\mu^3+27}.$
- $\frac{1}{a-3c} - \frac{(a-3c)^2}{a^3-27c^3}.$
- $\frac{1}{l-2c} + \frac{c^2}{l^3-8c^3} - \frac{l+c}{l^2+2cl+4c^2}.$

$$16. \frac{E^2 - 2E + 3}{E^3 + 1} + \frac{E - 2}{E^2 - E + 1} - \frac{1}{E + 1}.$$

$$17. \frac{1}{t - 3} + \frac{t - 1}{t^2 + 3t + 9} + \frac{t^2 + t - 3}{t^3 - 27}.$$

$$18. \frac{1}{r - s} + \frac{r - s}{r^2 + rs + s^2} - \frac{rs - 2r^2}{s^3 - r^3}.$$

$$19. \frac{xy}{x^3 - y^3} + \frac{2x}{x^2 + xy + y^2}.$$

$$20. \frac{1}{x + 3} + \frac{x + 1}{x^2 - 3x + 9} + \frac{x^2 + x + 1}{x^3 + 27}.$$

$$21. \frac{T}{a^2 - aT + T^2} + \frac{1}{a + T} + \frac{a^2}{a^3 + T^3}.$$

$$22. \frac{P^2 - P - 1}{P^3 - 1} + \frac{1}{P - 1}.$$

$$23. \frac{1}{x + y} + \frac{x - y}{x^2 - xy + y^2} - \frac{x^2 - xy}{x^3 + y^3}.$$

$$24. \frac{e^2 + 3}{2(e^3 - 8)} - \frac{2e - 1}{3(e^2 + 2e + 4)} - \frac{1}{6(2 - e)}.$$

$$25. \frac{s + t}{s^2 + st + t^2} + \frac{s - t}{s^2 - st + t^2} + \frac{2s^2t^2}{s^6 - t^6}.$$

Perform the indicated operations:

$$26. \frac{2\theta^2 - 3\theta + 1}{\theta^2 - 4} \cdot \frac{\theta^2 - \theta - 6}{3\theta^2 - \theta - 2} \cdot \frac{3\theta^2 - 4\theta - 4}{2\theta^2 - 7\theta + 3}.$$

$$27. \frac{2\pi^2 - 5\pi - 3}{\pi^2 - \pi - 2} \cdot \frac{(\pi - 1)^2}{4\pi^2 - 1} \cdot \frac{2\pi^2 - 3\pi - 2}{\pi^2 - 4\pi + 3}.$$

$$28. \frac{2S^2 + 5S + 2}{6S^2 + 5S + 1} \cdot \frac{9S^2 + 15S + 4}{5S^2 + 12S + 4}.$$

$$29. \frac{4R^2 - 6R + 2}{4R^2 - 8R + 3} \cdot \frac{4R^2 - 10R + 6}{4(R^2 - 2R + 1)}.$$

$$30. \frac{10T^2 + 17T + 3}{4T - 1} \cdot \frac{20T^2 - T - 1}{2T + 3}.$$

$$31. \frac{12P^2 + 5P - 2}{6P^2 + 13P + 6} \cdot \frac{8P^2 + 10P - 3}{4P - 1}.$$

$$32. \frac{(1 - X_1)^2 - X_2^2}{(X_1 - X_2)^2 - 1} \cdot \frac{1 - (X_1 + X_2)^2}{X_1^2 - (X_2 + 1)^2}.$$

$$33. \frac{E_1^4 - E_2^4}{E_1^3 + E_2^3} \cdot \left[1 + \frac{E_1 E_2}{(E_1 - E_2)^2} \right] \div \left[\frac{1}{E_1^2} + \frac{1}{E_2^2} \right].$$

$$34. \frac{(m - r)^2}{m^3 - r^3} \cdot \frac{m^6 - r^6}{m^2 - r^2} \cdot \frac{m^2 + mr + r^2}{m^4 + m^2 r^2 + r^4}.$$

$$35. \frac{p^2 - q^2}{p^3 + q^3} \cdot \left(\frac{1}{p^2} - \frac{1}{pq} + \frac{1}{q^2} \right) \cdot \frac{p^3 q^3}{(p - q)^2} - \frac{p^2}{p - q}.$$

$$36. \frac{\mu^4 - \beta^4}{(\mu - \beta)^2} \div \frac{\mu^2 \beta + \beta^3}{\mu^3 - \beta^3}.$$

$$37. (E_g^2 - E_p^2) \div \frac{E_g^3 - E_p^3}{E_g + E_p}.$$

$$38. \frac{I_1^6 - I_2^6}{I_1^4 - I_2^4} \div \frac{I_1^3 - I_2^3}{I_1 - I_2}.$$

$$39. \frac{a^6 + b^6}{a^6 - b^6} \cdot \frac{a - b}{a + b} \div \frac{a^4 - a^2 b^2 + b^4}{a^4 + a^2 b^2 + b^4}.$$

$$40. \frac{Y - 1}{XY + X} \cdot \frac{Y^3 + 1}{Y^3 - 1}.$$

$$41. \frac{P^3 + 27}{P^5 - 9P} \div \frac{P + 3}{P^3 + 3P}.$$

$$42. \frac{c^2 - l^2}{c^3 + l^3} \cdot \frac{c^3 - l^3}{c^2 + l^2} \cdot \frac{c + l}{c - l} \cdot \frac{c^4 + c^2 l^2}{c^2 + cl + l^2}.$$

$$43. \frac{K^4 - 1}{K^3 - 1} \cdot \left[1 - \frac{K}{(K + 1)^2} \right] \div \left[\frac{1}{K} + K \right].$$

$$44. \frac{x^3 + a^3}{x^2 - 9a^2} \cdot \frac{x + 3a}{x + a}.$$

$$45. \frac{t(s^3 - t^3)}{s(s + t)} \cdot \frac{(s^2 - t^2)^2}{s^2 + st + t^2} \cdot \frac{(s + t)^2}{(s - t)^2}.$$

$$46. \left(\frac{8X^3}{Y^3} - 1 \right) \left(\frac{4X^2 + 2XY}{4X^2 + 2XY + Y^2} - 1 \right).$$

$$47. \frac{I_1^3 - I_2^3}{I_1^3 + I_2^3} \cdot \frac{I_1^2 - I_1 I_2 + I_2^2}{I_1 - I_2}.$$

$$48. \frac{(a + b)^2 - c^2}{a^2 + ab - ac} \cdot \frac{a}{(a + c)^2 - b^2} \cdot \frac{(a - b)^2 - c^2}{ab - b^2 - bc}.$$

$$49. \frac{p^3 + 3p^2 q + 3pq^2 + q^3}{p^2 - 2pq + q^2} \div \frac{p^2 + 2pq + q^2}{p^3 - 3p^2 q + 3pq^2 - q^3}.$$

$$50. \frac{E^3 + 6E^2 + 12E + 8}{E^2 - 4E + 4} \div \frac{E^2 + 4E + 4}{E^3 - 6E^2 + 12E - 8}.$$

$$51. \frac{8\pi^3 + 1}{2\pi - 1} \cdot \frac{8\pi^3 - 1}{4\pi^2 + 2\pi + 1} \cdot \frac{\pi}{2\pi + 1}.$$

$$52. \frac{E}{E^3 - 1} + \frac{E}{E^3 + 1} - \frac{E - 1}{E + 1}.$$

$$53. \frac{1}{1 + R} + \frac{1 + R^2}{(1 + R)^3} - \frac{6R^2(1 - R)}{(1 + R)^4} - \frac{2(1 - R)}{(1 + R)^3}.$$

$$54. \frac{a+r}{a^2+ar+r^2} + \frac{a-r}{a^2-ar+r^2}.$$

$$55. \frac{3(s+t)}{s^2-2st+t^2} - \frac{4(s-t)}{s^2+2st+t^2} + \frac{5s}{s^2-t^2}.$$

$$56. \frac{\beta^3+\beta^2+\beta+1}{\beta^2-\beta+1} - \frac{3}{\beta-1} - 1.$$

$$57. \frac{I_1^4 - I_1^2 I_2^2 + I_2^4}{I_1^6 - I_2^6} + \frac{I_1 + I_2}{I_1^3 - I_2^3} - \frac{I_1 - I_2}{I_1^3 + I_2^3} + \frac{1}{I_2^2 - I_1^2}.$$

$$58. \frac{pq}{p^3+q^3} - \frac{p}{p^2-q^2} + \frac{1}{p+q}.$$

7-4. Complex Fractions. A fraction whose numerator or denominator, or both, are fractions, is called a **complex fraction**. A **simple fraction** is one without a fraction in its numerator or denominator. A complex fraction, like a simple one, is of course a quotient, but it usually involves some other operations. Performing these operations is spoken of as **simplifying the fraction**.

In simplifying a complex fraction, first perform the indicated additions and subtractions; second, perform the multiplications and divisions.

Example 1. Simplify

$$\frac{x - \frac{1}{x^2}}{1 - \frac{1}{x}}.$$

First perform the subtraction indicated in the numerator and the subtraction indicated in the denominator. Thus we obtain

$$\frac{x - \frac{1}{x^2}}{1 - \frac{1}{x}} = \frac{\frac{x^3 - 1}{x^2}}{\frac{x - 1}{x}}.$$

Second perform the indicated division and simplify by factoring. Thus

$$\begin{aligned} \frac{\frac{x^3 - 1}{x^2}}{\frac{x - 1}{x}} &= \frac{x^3 - 1}{x^2} \cdot \frac{x}{x - 1} \\ &= \frac{(x - 1)(x^2 + x + 1)}{x^2} \cdot \frac{x}{x - 1} \\ &= \frac{x^2 + x + 1}{x}. \end{aligned}$$

Simplifying complex fractions may involve several separate steps. The above procedure can be repeated until a simple fraction is obtained.

Example 2. Simplify

$$\frac{E+1}{E+1 + \frac{1}{E-1 + \frac{1}{E+1}}}$$

Adding the last fraction makes the given expression equal to

$$\frac{E+1}{E+1 + \frac{1}{\frac{E^2-1+1}{E+1}}} = \frac{E+1}{E+1 + \frac{1}{E^2}}$$

Dividing the last fraction reduces the last expression to

$$\frac{E+1}{E+1 + \frac{E+1}{E^2}}$$

Proceeding as in Example 1 we reduce the last complex fraction to

$$\begin{aligned} \frac{\frac{E+1}{E^3 + E^2 + E + 1}}{E^2} &= \frac{E^2(E+1)}{E^3 + E^2 + E + 1} \\ &= \frac{E^2(E+1)}{E^2(E+1) + (E+1)} \\ &= \frac{E^2(E+1)}{(E+1)(E^2+1)} = \frac{E^2}{E^2+1}, \end{aligned}$$

which is a simple fraction.

The **reciprocal of a number** is 1 divided by that number. Thus the reciprocal of any number N is $\frac{1}{N}$. Every number, with the exception of zero, has a reciprocal.

By taking the reciprocal of a fraction we obtain a complex fraction. Since $\frac{1}{\frac{a}{b}} = \frac{b}{a}$, *the reciprocal of a fraction is the fraction inverted* (see Sec. 1-10).

Example 3. Simplify

$$\frac{1}{E + \frac{1}{I + \frac{1}{R}}}$$

Proceeding as in Example 2 we obtain:

$$\begin{aligned}
 \frac{1}{E + \frac{1}{I + \frac{1}{R}}} &= \frac{1}{E + \frac{1}{\frac{IR + 1}{R}}} \\
 &= \frac{1}{E + \frac{R}{IR + 1}} = \frac{1}{\frac{EIR + E + R}{IR + 1}} \\
 &= \frac{IR + 1}{EIR + E + R}.
 \end{aligned}$$

EXERCISES

Simplify the following:

$$1. \frac{\frac{E}{I} - \frac{R}{T}}{\frac{E}{I} + \frac{R}{T}}.$$

$$2. \frac{1 + \frac{1}{2n}}{n - \frac{1}{4n}}.$$

$$3. \frac{\frac{2p}{3q} - 2 + \frac{3q}{2p}}{\frac{2}{q} - \frac{3}{p}}.$$

$$4. \frac{\frac{e}{e+i} - \frac{e-i}{e}}{\frac{e}{e-i} - \frac{e+i}{e}}.$$

$$5. \frac{\frac{x^2}{y^2} + \frac{8y}{x}}{\frac{x}{y} - 2 + \frac{4y}{x}}.$$

$$6. \frac{E^2 + E + 1 + \frac{2}{E-1}}{E + \frac{1}{E-1}}.$$

$$7. 1 - \frac{1}{3 - \frac{1}{2 - \frac{x}{1-x}}}.$$

$$8. \frac{\frac{1-T^2}{1+T^2} - \frac{1+T^2}{1-T^2}}{\frac{1-T}{1+T} - \frac{1+T}{1-T}}.$$

$$9. \frac{\frac{1}{e} + \frac{1}{i} + \frac{1}{r}}{\frac{e}{i} + \frac{i}{r} + \frac{r}{e}}.$$

$$10. \frac{1}{C - \frac{1}{C + \frac{1}{C}}}.$$

$$11. \frac{\frac{1}{p-q} - \frac{p}{p^2 - q^2}}{\frac{p}{pq + q^2} - \frac{q}{p^2 + pq}}.$$

$$12. \frac{1 + \frac{\mu}{\beta - \mu}}{1 - \frac{\mu}{\beta + \mu}}.$$

$$13. \frac{\frac{I^3 + R^3}{I^2 - R^2}}{\frac{I^2 - IR + R^2}{I - R}}.$$

$$14. \frac{1}{K + \frac{1}{1 + \frac{K+1}{3-K}}}.$$

$$15. \frac{\frac{\frac{x}{y} + 2}{\frac{x}{y} + 1}}{\frac{x}{y}}.$$

$$16. \frac{\frac{x}{y} + 2 - \frac{y}{\frac{x}{y} + 1}}{\frac{x}{y}}.$$

$$15. \frac{n}{1 - \frac{n}{1 + n + \frac{n}{1 - n + n^2}}}.$$

$$17. \frac{1}{x + 1 + \frac{2}{x + 1 + \frac{3}{x + 1}}}.$$

$$18. 1 + \frac{T}{1 - \frac{T}{T+2}} - \frac{1}{1 + \frac{T}{1 - T + \frac{T}{T+2}}}.$$

$$19. \frac{\frac{1 - P^3}{1 - P} + \frac{1 + P^3}{1 + P}}{\frac{1 - P^3}{1 - P} - \frac{1 + P^3}{1 + P}}.$$

$$20. \frac{\frac{1}{e^4 + e^2 i^2 + i^4}}{\frac{1}{e^3 + i^3} \cdot (e + i)}.$$

$$21. \frac{\frac{P+1}{P-1} + \frac{P-1}{P+1}}{\frac{P-1}{P+1} - \frac{P+1}{P-1}}.$$

$$22. \frac{\frac{X_C^2 - X_L^2}{X_C}}{\frac{X_C + X_L}{X_L}}.$$

$$23. \frac{\frac{\frac{1}{R_1} - \frac{1}{R_2}}{\frac{R_1^2}{R_2} - \frac{R_2^2}{R_1}}}{\frac{R_2}{R_1} - \frac{R_1}{R_2}}.$$

$$24. \frac{1}{C_1 - \frac{C_1^2 - 1}{C_1 + \frac{1}{C_1 - 1}}}.$$

$$25. z^3 + \frac{z^2}{z^2 + \frac{1}{z^3 - \frac{z^3 + z^3 - 1}{z^5}}}.$$

$$26. \frac{1}{p - \frac{p^2 - 1}{p + \frac{1}{p - 1}}}.$$

$$27. p + q + \frac{1}{p + q + \frac{1}{p + q - \frac{1}{p + q}}}.$$

$$28. \frac{A-2-\frac{1}{A-2}}{A-2-\frac{4}{A-5}} \cdot \frac{A-4-\frac{4}{A-4}}{A-4-\frac{1}{A-4}}.$$

$$29. \frac{\frac{p}{p-q} - \frac{p}{p+q}}{\frac{q}{p-q} + \frac{p}{p+q}}.$$

$$30. \frac{\frac{m^2+r^2}{m^2-r^2} - \frac{m^2-r^2}{m^2+r^2}}{\frac{m+r}{m-r} - \frac{m-r}{m+r}}.$$

7-5. Linear Equations in One Unknown. Equations were introduced in Chapter 2. Before proceeding, the reader should review Sec. 2-12.

An equation is a statement of the equality of two expressions. If the two members of the equation are equal for every value of the symbols or variables involved for which the members have a meaning, the equation is called an **identical equation** or an **identity**. For example, all formulas (1) to (14) in Sec. 1 of this chapter are identities since they hold true for every value of the letters involved. At times, to emphasize that some equation is an identity, the symbol \equiv is used instead of $=$. We may write

$$a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2).$$

An equation whose members are not equal for all values of the letters is called a **conditional equation** or, briefly, an **equation**. Examples of equations are:

$$3x - 2 = 10, \quad \text{true only if } x = 4.$$

$$x^2 - 4x + 3 = 0, \quad \text{true only if } x = 1 \text{ or } x = 3.$$

$$\cos x = 1, \quad \text{true only if } x = 0^\circ, 360^\circ, 720^\circ, \dots$$

$$x + 2 = x, \quad \text{not true for any value of } x.$$

These examples illustrate the fact that a conditional equation may be true for one value, two values, unlimited number of values, or for no value of the unknown.

When an equation is given, the object is to solve it, i.e., to find values of the unknown that make the two members equal. Such values of the unknown are called **roots** or **solutions** of the equation. In the process of solving an equation we have to make certain changes and transformations. In the process of solving, it is important that we get for roots **only** those numbers which satisfy the original equation. These changes and transformations, when applied to an equation, result in a new equation. This new **derived equation** is *equivalent to the original equation if it contains all the roots of that equation and no others*. In Sec. 2-12 we

indicated the operations on an equation which yield an equivalent equation. These operations are:

1. *Addition or subtraction of the same expression to both members of the equation.*

2. *Multiplication or division of both members of the equation by the same expression provided that this expression is not zero and does not involve the unknown.*

Applications of these operations were given in Sec. 2-12.

If an equation is multiplied or divided by an expression involving the unknown, the derived equation may not be equivalent to the original, as indicated in the following two rules.

3. *If both members of an equation are multiplied by an expression involving the unknown, roots not in the original equation may be introduced. These are called **extraneous roots**.*

4. *If both members of an equation are divided by an expression involving the unknown, the resulting equation may have fewer roots than the original equation.*

The next three examples will illustrate Rule 3.

Example 1. Solve

$$(1) \quad \frac{5E^2 - 7E - 6}{E - 2} = 4.$$

Multiplying both sides by $E - 2$, an expression involving the unknown E , we obtain

$$5E^2 - 7E - 6 = 4E - 8,$$

and hence

$$(2) \quad 5E^2 - 11E + 2 = 0.$$

This when factored gives

$$(5E - 1)(E - 2) = 0.$$

Proceeding as in Sec. 2-13 we obtain

$$5E - 1 = 0 \quad \text{or} \quad E - 2 = 0,$$

$$E = \frac{1}{5} \quad \text{and} \quad E = 2.$$

The solution shows $\frac{1}{5}$ and 2 as roots. But $E = 2$ cannot be a root of the given equation (1), since substitution of this value reduces the left member to $\frac{0}{0}$ which has no meaning (see Sec. 1-7). Substituting $E = \frac{1}{5}$ in the given equation reduces the left member to the right member, since

$$\frac{5 \cdot (\frac{1}{5})^2 - 7(\frac{1}{5}) - 6}{\frac{1}{5} - 2} = 4.$$

Hence we conclude that $E = \frac{1}{5}$ is the only solution of (1). The other answer, $E = 2$, is a root of the derived equation (2) but not of (1), and hence it is an **extraneous root**.

Example 2. Solve

$$\frac{1}{R-2} + \frac{1}{R-3} = \frac{2}{R-4}.$$

To clear fractions we multiply by $(R-2)(R-3)(R-4)$, an expression involving the unknown, and obtain

$$(R-3)(R-4) + (R-2)(R-4) = 2(R-2)(R-3),$$

$$R^2 - 7R + 12 + R^2 - 6R + 8 = 2R^2 - 10R + 12,$$

$$3R = 8,$$

$$R = \frac{8}{3}.$$

Substituting $R = \frac{8}{3}$ into the given equation we obtain

$$\frac{1}{\frac{8}{3}-2} + \frac{1}{\frac{8}{3}-3} = \frac{2}{\frac{8}{3}-4},$$

$$\frac{3}{2} - 3 = -\frac{3}{2}.$$

The multiplier $(R-2)(R-3)(R-4)$, used in clearing fractions of the given equation, is zero for $R = 2, 3, 4$. Since none of these turns up as a root in the solution, no extraneous roots were introduced, even though both members of the given equation are multiplied by an expression involving the unknown R .

Example 3. Solve

$$\frac{2}{x-1} + \frac{5}{x+1} = \frac{4}{x^2-1}.$$

To clear fractions we multiply by $(x-1)(x+1)$, an expression involving the unknown x , and obtain

$$2(x+1) + 5(x-1) = 4,$$

$$2x + 2 + 5x - 5 = 4,$$

$$x = 1.$$

But $x = 1$ is not a solution of the given equation since by substituting $x = 1$ we obtain

$$\frac{2}{0} + \frac{5}{2} = \frac{4}{0},$$

which has no meaning, division by zero being excluded (see Sec. 1-7). Hence $x = 1$ is an extraneous root, and the given equation has no solution.

The next example will illustrate Rule 4.

Example 4. Solve $(x+3)(x-1) = 5(x-1)$.

Dividing both members by $x-1$, an expression involving the unknown x , we have the equation

$$x + 3 = 5,$$

which is satisfied by

$$x = 2.$$

Now $x = 2$ satisfies the given equation but so does $x = 1$. Division of both members by $x-1$ loses the root $x = 1$.

The last example shows that if both members of an equation are divided by an expression involving the unknown, we may lose some roots. A root once lost is not so easily discovered; for that reason, in solving equations, a student should not divide by an expression involving the unknown unless he is certain that by doing so no root is lost. The equation in Example 4 should be worked as follows.

$$(x + 3)(x - 1) = 5(x - 1),$$

$$(x + 3)(x - 1) - 5(x - 1) = 0,$$

$$(x - 1)(x + 3 - 5) = 0,$$

$$(x - 1)(x - 2) = 0,$$

$$x = 1 \quad \text{and} \quad x = 2.$$

EXERCISES

Solve the following equations:

$$1. \frac{a-3}{a+3} = \frac{1}{a-1}.$$

$$2. \frac{E-1}{E+1} = \frac{E-6}{E-3}.$$

$$3. \frac{R+5}{4} - \frac{3}{R+5} = \frac{R-5}{8}.$$

$$4. \frac{1}{K-3} + \frac{3}{K-9} = \frac{4}{K-6}.$$

$$5. \frac{2\mu+3}{5} - \frac{6\mu+22}{15} = \frac{3\mu+17}{5(1-\mu)}.$$

$$6. \frac{6T+7}{12} = \frac{3T-4}{4T-3} + \frac{T}{2}.$$

$$7. \frac{x-1}{x-3} = \frac{(2x-1)^2}{(2x-3)^2}.$$

$$8. \frac{8}{L+3} - \frac{9}{2L+6} = \frac{15}{7L+2}.$$

$$9. \frac{5P+10.5}{P+0.5} + \frac{2P}{2P+1} = 9.$$

$$10. \frac{4}{Z-1} - \frac{3}{Z-2} = \frac{1}{Z-10}.$$

$$11. \frac{x-3}{x-2} - \frac{x+4}{x+2} = \frac{x-14}{x^2-4}.$$

$$12. \frac{x^2}{2x-3} + \frac{x+3}{4x-6} = \frac{x+3}{2}.$$

$$13. \frac{c-2}{c-1} - \frac{c-2}{2c+2} = \frac{c-2}{c+1}.$$

$$14. \frac{T^2}{2T+3} + \frac{2T-3}{4} = \frac{3T-1}{4} - \frac{1}{2}.$$

$$15. \frac{4x^2}{3x-6} - \frac{x+2}{3x} - \frac{4(x^2+2x+4)}{3x} = 0.$$

$$16. \frac{3}{(2x-3)(x+2)} = \frac{4}{(4x-5)(x+2)}.$$

$$17. \frac{M-4}{2M-5} - \frac{M+2}{2M+5} = \frac{3M^2-10}{4M^2-25}.$$

$$18. \frac{x-2}{x+2} + \frac{x+2}{x-2} - \frac{2x^2+x}{x^2-4} = 0.$$

$$19. \frac{2a+3}{3-2a} - \frac{2a-3}{2a+3} = \frac{a^2-22}{4a^2-9}.$$

$$20. \frac{2x-3}{x+4} - \frac{x+4}{2x-3} = \frac{2x^2-19x-1}{2x^2+5x-12}.$$

$$21. \frac{R}{R+3} + \frac{4}{R-3} = \frac{3R-2}{3R-9}.$$

$$22. \frac{2}{x+3} + \frac{3}{x+6} = \frac{4}{x^2+9x+18}.$$

$$23. \frac{3x-4}{2x+3} + \frac{x+3}{2-3x} - 1 = \frac{(x-3)(x-9)}{6x^2+5x-6}.$$

$$24. \frac{2x+3}{x^2-3x} + 3 = \frac{3(x^2-3)}{x^2-9}.$$

$$25. \frac{1}{(R_1+3)(R_1+5)} = \frac{1}{(R_1+9)(R_1-5)}.$$

$$26. \frac{2}{4-9x^2} - \frac{3}{2+3x} = \frac{3x}{4-9x^2}.$$

$$27. \frac{L+2}{L+3} + \frac{L-2}{3-L} = \frac{5}{L^2-9}.$$

$$28. \frac{6}{x+2} - \frac{5}{x} = \frac{5-4x}{x^2+2x}.$$

$$29. \frac{x}{2x-1} - \frac{x-2}{2x+5} + 2 = \frac{8(x^2+x-3)}{4x^2+8x-5}.$$

$$30. \frac{2}{2x^2-x} - \frac{3}{x} + \frac{6}{2x+1} = \frac{6}{4x^2-1}.$$

$$31. \frac{1}{K-1} - \frac{2}{K-2} = \frac{3}{K-3} - \frac{4}{K-4}.$$

$$32. \frac{x+1}{x+2} + \frac{x+3}{x^2+5x+6} = \frac{3}{x+3} + \frac{x^2}{x^2+5x+6}.$$

$$33. \frac{x+3}{x+1} + \frac{x-6}{x-4} = \frac{x+4}{x+2} + \frac{x-5}{x-3}.$$

$$34. \frac{1+3e}{5+7e} - \frac{9-11e}{5-7e} = \frac{14(2e-3)^2}{25-49e^2}.$$

$$35. \frac{2}{2t-1} - \frac{1}{t-3} - \frac{1}{t} + \frac{2}{2t-5} = 0.$$

$$36. \frac{5}{5L+8} - \frac{4}{2L+3} = \frac{1}{5} \left(\frac{3}{L+3} - \frac{8}{L+2} \right).$$

$$37. \frac{2}{3Z+1} = \frac{6Z+1}{9Z^2-3Z+1} + \frac{15+6Z}{27Z^3+1}.$$

$$38. \frac{1-2y}{3(y^2+2y+4)} = \frac{1}{6(2-y)} - \frac{y^2+3}{2y^3-16}.$$

7-6. Application to Solution of Problems. In setting up the equations for many problems, we often obtain equations which are expressed in the form of fractions, of the type discussed in the preceding section. The problems which follow will lead to fractional equations whose solution will depend on the work just completed. Before proceeding, the reader should review Sec. 2-17 in which a discussion of the solution of word problems was given.

Example 1. The denominator of a fraction is two more than the numerator. If both the numerator and the denominator of the fraction are increased by one, the resulting fraction equals $\frac{2}{3}$. Find the fraction.

Let x be the numerator of the fraction. Then $x + 2$ is the denominator. Increasing both of these expressions by 1, we obtain the equation

$$\frac{x + 1}{x + 2 + 1} = \frac{2}{3},$$

$$3x + 3 = 2x + 6,$$

$$x = 3.$$

The fraction is $\frac{3}{5}$.

To check the answer we have

$$\frac{3 + 1}{5 + 1} = \frac{2}{3}.$$

Example 2. After traveling 60 miles at a certain speed, a motorist increases his speed by 5 miles per hour and travels an additional 50 miles. If the total time required for the 110 miles was 5 hours, find the speed for the first 60 miles.

Let x miles per hour be the motorist's speed for the first 60 miles. Then $x + 5$ miles per hour is the speed for the last 50 miles. Remembering from Sec. 2-18 that time is equal to distance divided by speed, we have, as the equation to determine x :

$$\frac{60}{x} + \frac{50}{x + 5} = 5.$$

This equation can be reduced to

$$x^2 - 17x - 60 = 0,$$

$$(x - 20)(x + 3) = 0,$$

$$x = 20 \quad \text{or} \quad x = -3.$$

The two solutions of the equation are $x = 20$ and $x = -3$, but only one of these solutions, $x = 20$, has a meaning in this problem. Since a speed cannot be a negative number, the negative root -3 is meaningless as far as our problem is concerned. Hence the speed during the first 60 miles was 20 miles per hour. This result is correct since

$$\frac{60}{20} + \frac{50}{25} = 5.$$

EXERCISES

1. A certain battery charger can charge a bank of storage batteries in 2 days; a different type of charger can charge them in 3 days. How many days will it take to charge the batteries if both chargers are used simultaneously?
2. The second digit of a number exceeds the first by 2; if the number increased by 6 is divided by the sum of its digits, the quotient is 5. Find the number.
3. The sum of the numerator and the denominator of a proper fraction is 70. If each is increased by 7, the value of the fraction becomes $\frac{3}{4}$. Find the fraction.
4. The difference of the numerator and the denominator of a proper fraction is 24. If each is decreased by 5, the value of the fraction becomes $\frac{1}{3}$. Find the fraction.
5. A 16-qt. mixture of water and alcohol is three-quarters alcohol. How many quarts of water must be added so that the mixture is 20 per cent alcohol?
6. The first digit of a number is 4 less than the second; if the number is divided by the sum of its digits, the quotient is 4. Find the number.
7. The second digit of a number is one-quarter of the first. If the number diminished by 10 is divided by the difference of its digits, the quotient is 12. Find the number.
8. In a two-place number the units digit is 3 more than the ten's digit. The number divided by the sum of its digits gives 3 as a quotient and 4 as the remainder. What is the number?
9. In a given two-place number the ten's digit is 2 more than the units digit. The number divided by the sum of its digits gives 7 as a quotient and 3 as a remainder. What is the number?
10. The denominator of a fraction is 5 more than the numerator. If the denominator is decreased by 20, the resulting fraction, increased by 1, is equal to twice the original fraction. Find the fraction.
11. The denominator of a fraction exceeds the numerator by 4. If both the numerator and denominator of the fraction are increased by 1, the resulting fraction equals $\frac{1}{2}$. Find the fraction.
12. A tank can be filled by one pipe in 9 hours and emptied by another in 21 hours. In what time will it be filled if both pipes are opened?
13. The rate of current of a river is 3 miles per hour. Some men in a boat found that it took them as long to go 8 miles downstream on the river as it did to go 3 miles upstream. At what rate will the boat travel in still water?
14. Two rivers flow at rates of 2 miles per hour and 4 miles per hour respectively. A man finds he can row 16 miles downstream on the second river in the time he takes to row 12 miles downstream on the first river. At what rate can he row in still water?
15. A pontoon bridge across a river 240 ft. wide requires a certain number of pontoons of a standard length. If the pontoons were 3 ft. shorter, 4 more would be required. What is the length of the pontoons?
16. An auto travels 150 miles at a uniform rate. If the rate had been 5 miles per hour more, the journey would have taken one hour less. Find the rate of the automobile.
17. The cost of a certain number of castings was \$36.00. If there had been two more, the cost for each would have been \$3.00 less. Find the number of castings.
18. A motor torpedo boat and a destroyer each ran 180 miles. The torpedo boat ran 6 miles per hour faster than the destroyer and required 1 hour less time for the trip. Find the speed of the torpedo boat.

19. A motorist travels 400 miles from his starting point. Returning the same way, he increased his speed 10 miles per hour and got back in 2 hours' less time. What was his speed each way?

20. The rear wheel of a wagon has a circumference of 2 ft. greater than the front wheel. The smaller wheel makes 20 revolutions more than the other while the wagon travels 1200 ft. Find the circumference of the wheel.

7-7. Applications to Engineering Formulas. Some of the work of the present chapter will now be applied to formulas which are encountered in engineering and in physics. Before proceeding the reader will do well to review Sec. 2-18.

Many formulas are in the form of complex fractions. To work with such formulas the information and experience obtained in Sec. 7-4 are very useful. For example, if the capacitances of three condensers, connected in series, are respectively C_1 , C_2 , and C_3 , then the resultant capacitance C is given by the formula

$$(1) \quad C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}.$$

When C_1 , C_2 , and C_3 are given, we can compute C from (1). When C , C_1 , and C_2 are given, we may regard C_3 as unknown and solve for it in terms of the remaining letters. One thus obtains from (1)

$$C = \frac{1}{\frac{C_2 C_3 + C_1 C_3 + C_1 C_2}{C_1 C_2 C_3}},$$

$$C = \frac{C_1 C_2 C_3}{C_2 C_3 + C_1 C_3 + C_1 C_2},$$

$$C(C_2 C_3 + C_1 C_3 + C_1 C_2) = C_1 C_2 C_3,$$

$$CC_2 C_3 + CC_1 C_3 + CC_1 C_2 = C_1 C_2 C_3,$$

$$CC_2 C_3 + CC_1 C_3 - C_1 C_2 C_3 = -CC_1 C_2,$$

$$C_3(CC_2 + CC_1 - C_1 C_2) = -CC_1 C_2,$$

$$(2) \quad C_3 = \frac{CC_1 C_2}{C_1 C_2 - CC_1 - CC_2}.$$

Formula (2) gives an expression for C_3 in terms of C , C_1 , and C_2 . The above process is called *solving formula (1) for C_3* . One can easily obtain similar expressions for C_1 and for C_2 in terms of the remaining letters, i.e., one can *solve for C_1 and C_2* .

When working with formulas it is important that we do not divide by zero. All the operations with the variables must be carefully examined and division by zero excluded. To exclude division by zero in (1), all the quantities C_1 , C_2 , and C_3 must be different from zero, which is written as $C_1 \neq 0$, $C_2 \neq 0$, and $C_3 \neq 0$, or $C_1 \cdot C_2 \cdot C_3 \neq 0$. These restrictions on the values of C_1 , C_2 , and C_3 have a physical significance, for if one of them, say C_1 , is zero, we then have an open circuit.

By starting with (1) we discover by simple algebraic steps another important fact about electricity represented by (2). It is quite possible that a radio mechanic may know the fact represented by (1), but if he is ignorant of elementary algebra he will not be able to discover (2), even though it is of practical importance to him.

The above example and the exercises which follow are an indication of one of the roles which algebra plays in the engineering sciences. A knowledge of algebra enables the engineer or scientist to derive many facts from a known formula, facts which otherwise would have to be discovered by experiment, often quite laborious.

EXERCISES

In the formulas below, consider all the quantities as given except the one to be solved for. In each case indicate the value or values, if any, of the quantities for which the expressions fail to have a meaning. A hint as to the meaning of some of the formulas is given. If the reader is interested in a more detailed description of the formula, an appropriate textbook should be consulted. Simplify your answer.

GIVEN	SOLVE FOR	DESCRIPTION
1. $E = RI$	I, R	Ohm's law
2. $\mu = \frac{B}{H}$	B, H	Magnetic permeability
3. $R = \frac{K}{d^2}$	d^2	Resistance of wire
4. $R = \frac{K}{A}$	A	Reluctance ("magnetic resistance")
5. $E = \frac{N \cdot \phi}{10^8 \cdot t}$	t	Electromagnetic force generated by cutting magnetic lines
6. $\lambda = \frac{300 \cdot 10^6}{f}$	f	Length of a radio wave
7. $X_c = \frac{1}{2\pi fC}$	f, C	Capacitive reactance
8. $C = \frac{8.84KA}{10^9 d}$	d	Calculation of capacitance

GIVEN	SOLVE FOR	DESCRIPTION
9. $L_{av.} = \frac{1.26N^2A\mu}{10^8l}$	l	Self-inductance of long coils
10. $W_h = \frac{0.796 \cdot A}{10^8}$	A	Hysteresis loss
11. $\frac{R_1}{R_3} = \frac{R_2}{R_x}$	R_x	Wheatstone bridge equation
12. $\frac{E_s}{E_p} = \frac{N_s}{N_p}$	E_p, N_p	Transformer formula
13. $I = \frac{E_x - E_c}{R}$	R, E_x	Current flowing through the armature of a generator
14. $V = \frac{Q}{r_1} - \frac{Q}{r_2}$	r_1, r_2, Q	Potential difference between two point charges
15. $C = 0.0885 \frac{KA}{d}$	K, d	Capacity of a parallel plate condenser
16. $A = \frac{R_L}{r_p + R_L}$	R_L, r_p	Net gain of an amplifier tube
17. $M = \frac{1.26N_1N_2A}{10^8l}$	l, N_1, N_2	Mutual inductance of coils
18. $I_z = M \left(\frac{r^2}{4} + \frac{h^2}{12} \right)$	M	
19. $E = \frac{1}{t} \left(\frac{T^2}{R} - 1 \right)$	t, R	
20. $P = \frac{3.095LWZ}{gD^2}$	L, Z	Determination of viscosity
21. $W = \frac{1}{\frac{1}{n_0} + \alpha Q}$	n_0, Q	
22. $L = \frac{0.8r^2N^2}{6r + 9l + 10t}$	l, t	Inductance of multilayer coil
23. $\frac{R_1}{R_2} = \frac{234.5 + t_1}{234.5 + t_2}$	R_2, t_2	Relation between resistance and temperature
24. $R_s = \frac{I_s R_g}{I' - I_g}$	I', I_g	Ammeter conversion formula
25. $R = \left(\frac{E_b}{E} - 1 \right) R_m$	E_b, E, R_m	Resistance measured with a voltmeter and battery
26. $L = \frac{0.4a^2N^2}{9a + 10b} \mu h$	μ, b	Inductance of a single layer solenoid
27. $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$	C_1, C_2, C_3	Total capacitance in series

GIVEN	SOLVE FOR	DESCRIPTION
28. $R = \frac{L_1}{K_1 A_1} + \frac{L_2}{K_2 A_2}$	L_1, A_2	Resistance to heat flow
29. $\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$	R_1, R_2, R_3	Total resistance in parallel
30. $\frac{R_{t1}}{R_{t2}} = \frac{1 + \alpha t_1}{1 + \alpha t_2}$	α, t_2, R_{t2}	Variation of resistance with temperature
31. $i_{av.} = C \frac{e_2 - e_1}{t_2 - t_1}$	t_1, t_2	Current flowing into a condenser
32. $e_{av.} = L \frac{i_2 - i_1}{t_2 - t_1}$	t_1, t_2	Self-induced voltage of a coil
33. $I = \frac{n_s E}{R + n_s r}$	r, n_s, R	Ohm's law modified for n_s cells in series
34. $I = \frac{E}{R + \frac{r}{n_p}}$	n_p, r, R	Ohm's law modified for n_p cells in parallel
35. $I = \frac{n_s E}{R + \frac{n_s r}{n_p}}$	n_s, n_p, r, R	Ohm's law modified for several cells, n_s in series and n_p in parallel
36. $A = \frac{\frac{W_A}{M_A}}{\frac{W_A}{M_A} + \frac{W_B}{M_B} + \frac{W_C}{M_C}}$	W_A, M_A	Molecular fraction in a mixture

PROGRESS REPORT

In this chapter we extended our domain of knowledge and skill in the operations of

- (a) Expanding products.
- (b) Factoring.
- (c) Simplifying simple and complex fractions.
- (d) Solving fractional equations.

Finally, these skills were applied to mathematical problems and problems in applications.

CHAPTER 8

EXPONENTS AND RADICALS

In Sec. 1-11 we used positive and negative exponents. In practice, however, fractions are used frequently as exponents. Because of the importance of such exponents and radicals in mathematics and their application in engineering, it is necessary to study them in detail. Before proceeding, the reader should carefully review Sec. 1-11.

8-1. Integral Exponents. We shall start by giving a brief summary of the work done in Sec. 1-11.

If a and b are any numbers, and m , n and q are integers, we have the following definitions and laws.

DEFINITIONS

$$1. \underbrace{a \cdot a \cdots a}_{n \text{ factors}} = a^n.$$

$$2. a^{-n} = \frac{1}{a^n}.$$

$$3. a^0 = 1.$$

LAWS OF EXPONENTS

$$1. a^n \cdot a^m = a^{m+n}.$$

$$2. \frac{a^n}{a^m} = a^{n-m}.$$

$$3. \frac{a^n}{a^m} = \frac{1}{a^{m-n}}.$$

$$4. (a^n)^m = a^{mn}.$$

$$5. (a \cdot b)^n = a^n \cdot b^n.$$

$$6. \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

$$7. (a^n b^m)^q = a^{nq} \cdot b^{mq}.$$

Examples.

$$L^3 \cdot L^5 = L^{3+5} \quad \text{Law 1}$$

$$= L^8.$$

$$\left(-\frac{5}{3}R^3\right)^2 = \left(-\frac{5}{3}\right)^2(R^3)^2 \quad \text{Law 7}$$

$$= \frac{25}{9}R^6. \quad \text{Laws 4 and 6}$$

$$(x^2)^3(-2x)^4(-x^2)^2 = x^6(-2)^4x^4x^4 \quad \text{Laws 7 and 4}$$

$$= 16x^{14}. \quad \text{Law 1}$$

$$\frac{(6a^4b^3)^3}{(3a^2b^2)^2(2ab)^5} = \frac{216a^{12}b^9}{9a^4b^4 \cdot 32a^5b^5} \quad \text{Laws 7 and 4}$$

$$= \frac{216a^{12}b^9}{288a^9b^9} \quad \text{Law 1}$$

$$= \frac{3}{4}a^3. \quad \text{Law 2}$$

$$\left(-\frac{R^2}{L^4}\right)^{-3}\left(\frac{R^3}{L^6}\right)^2 = \frac{1}{\left(-\frac{R^2}{L^4}\right)^3} \cdot \left(\frac{R^3}{L^6}\right)^2 \quad \text{Definition 2}$$

$$= \frac{1}{-\frac{R^6}{L^{12}}} \cdot \frac{R^6}{L^{12}} \quad \text{Laws 6 and 4}$$

$$= -\frac{L^{12}}{R^6} \cdot \frac{R^6}{L^{12}} \quad \text{Inverting the fraction}$$

$$= -1. \quad \text{Law 2, Definition 3}$$

EXERCISES

Perform the indicated operations by the laws of exponents. Simplify the results when possible.

- | | | |
|---|--|-----------------------------------|
| 1. $R^5 \cdot R^3$. | 2. $r^4 \cdot r^4$. | 3. $a^{-1} \cdot a^3 \cdot a^6$. |
| 4. $2^2 \cdot 2 \cdot 2^4$. | 5. $(3a)^3$. | 6. $ab^2 \cdot (ab)^2$. |
| 7. $(s_1^2s_2)^3$. | 8. $\frac{Z^5}{Z^2}$. | 9. $\frac{Y^{12}}{Y^8}$. |
| 10. $\frac{R_1^2 \cdot R_2^3}{R_1 \cdot R_2^2}$. | 11. $\frac{X^2 \cdot Y^4}{X^3 \cdot Y^2}$. | 12. $(\frac{1}{3}L)^4$. |
| 13. $(-3R^2)^3$. | 14. $\frac{(Y_1Y_2)^2}{Y_1^2Y_2^3}$. | 15. $(-\frac{1}{4}R^2)^3$. |
| 16. $(4x)^2(0.5x)^3$. | 17. $\left(-\frac{rx^2}{ry^3}\right)^2\left(\frac{rx^3}{ry^2}\right)^{-3}$. | 18. $\frac{R^2r^3}{R^4r^4}$. |

19. $\left(\frac{2s_1}{3s_2}\right)^2 \left(\frac{3s_2}{4s_1}\right)^4.$
20. $(-X_y^4)^4.$
21. $\left[\frac{0.5x^{-2}}{x^3}\right]^2.$
22. $a^4 \cdot b \cdot b^{-2} \cdot a^{-3}.$
23. $\left(\frac{2m^2}{m^2}\right)^2 \left(\frac{m^3}{m^2}\right)^3.$
24. $(R_1^2 R_2^2)^4.$
25. $\frac{X^5}{X^{-2}}.$
26. $\frac{s_1^4 s_2^5}{s_1^2 s_2^4}.$
27. $a^4 \cdot b^2 \cdot a^5 \cdot b^3.$
28. $\frac{s^8}{s^9}.$
29. $r^2 \cdot r^{-2}.$
30. $\left(\frac{R^3}{R_4}\right)^2.$
31. $z^2 \cdot z^{-5} \cdot z^6.$
32. $(3z_1^3 z_2^2 z_3)^2.$
33. $\frac{0.9X^2 Y^4}{0.3Y^3 X}.$
34. $(1.2s^2)^{-2}.$
35. $(-5x)^{-2}(x^2)^3.$
36. $\left(\frac{2r}{3}\right)^4.$
37. $(2r_1^2 r_2^3)^3.$
38. $(s_1^2 \cdot s_2^3)^3.$
39. $(-5R_1^5 R_2)^2.$
40. $(-3R^2 r s)^3.$
41. $\left(\frac{2x}{3y^2}\right)^2 \cdot \left(\frac{3x^2}{y}\right)^3.$
42. $\left(-\frac{R}{4}\right)^2.$
43. $(C^6 \cdot C^{-4})^{-2}.$
44. $X^7 \cdot X^{-5}.$
45. $\frac{r_x^3 r_y^2}{r_x r_y^3}.$
46. $\left(\frac{C^{-1} \cdot b^{-2}}{a^2}\right)^3.$
47. $\frac{R_1^5 R_2^7}{R_1^3 R_2^9}.$
48. $\left(\frac{Z}{2W}\right)^2 \left(\frac{4W}{3Z}\right)^3.$
49. $\left(\frac{XY^2 Z}{X^2 Y^3 Z}\right)^{10}.$
50. $\left(\frac{-r_1^2 r_2^3}{2r_1^2 r_2}\right)^4.$
51. $\left(\frac{Y^{-2} Z^3}{2X^2}\right)^8.$
52. $(sR^2)^2 \cdot (sR^3)^3.$
53. $(sR^3)^2 \cdot (sR^2)^{-2}.$
54. $\frac{4X^3 Y^2}{6X Y^5}.$
55. $\left(-\frac{s^2}{r_x^2 r_y^4}\right)^3.$
56. $(3R + 7)^0.$
57. $(s_1 - s_2)^{-2} \cdot (s_1^2 - s_2^2).$
58. $\left(\frac{\sin A}{\cos A}\right)^{-2} (\sin A)^2.$
59. $\left(\frac{\tan B}{\cot B}\right)^2 (\tan B)^{-2}.$
60. $\left(\frac{\sec \theta}{\csc \theta}\right)^0 \sin \theta.$
61. $(r^2 + r^{-2})(r^2 - r^{-2}).$
62. $(5R_1^2 R_2^3)^2 \cdot (R_3^3 r^4)^3.$
63. $\frac{(sR^2)^3}{(sR^3)^2}.$
64. $\frac{4(r_x^5 r_y^3)^4}{(3r_x r_y^2)^3 (2r_x^2 r_y)^5}.$
65. $\frac{X^6 Y^9}{X^{12} Y}.$
66. $\left(\frac{M_1^2}{M_2}\right)^3 \left(\frac{M_1^3}{M_2}\right)^{-2}.$
67. $\left(\frac{\sin^2 \theta}{\cos \theta}\right)^2 (\cos \theta)^{-2}.$
68. $\left(\frac{\sin^2 A}{\cos A}\right)^3 \left(\frac{\cos^2 A}{\sin^3 A}\right)^2 \cdot \frac{\sin A}{\cos A}.$

8-2. Roots and Radicals. When in the preceding section we wrote $y = x^n$, n any positive integer, we said that the number y is equal to the n th power of the number x . If this equation is written in the order $x^n = y$, then x is said to be an n th root of y . That is, *an n th root of a number is a number whose n th power equals the first number.* Briefly

(1) x is an n th root of y if $x^n = y$.

Thus since $6^2 = 36$, we say that 6 is a second or *square* root of 36. Similarly since $(2E^2)^3 = 8E^6$, the quantity $2E^2$ is a third or *cube* root of $8E^6$.

The symbol $\sqrt[n]{}$ is used to denote the n th root. Thus if $6^2 = 36$, then $6 = \sqrt{36}$, and if $(2E^2)^3 = 8E^6$, then $2E^2 = \sqrt[3]{8E^6}$. The symbol $\sqrt{}$ is called the **radical** sign; the number n placed above the radical sign is called the **index** of the root. In the case of the square root the index is omitted. The term **radical** applies also to expressions of the form $\sqrt[n]{y}$, where the number y is called the **radicand**.

Example 1. Since $3^2 = 9$, 3 is a square root of 9.

Since $(-2)^3 = -8$, -2 is a cube root of -8 .

Since $(-3)^5 = -243$, -3 is a fifth root of -243 .

Since $(-6)^2 = 36$ and $6^2 = 36$, -6 and 6 are square roots of 36.

Since $(-5)^4 = 625$ and $5^4 = 625$, -5 and 5 are fourth roots of 625.

From the preceding example it is obvious that sometimes there exists more than one n th root of a number. When, in a later section, imaginary numbers are introduced, it will be shown that every number has exactly n n th roots. Not all these roots are real numbers. For the present we shall be interested only in those roots which are real numbers. (Real numbers were introduced in Sec. 1-3.)

Since the square of a positive or negative real number is positive, it follows that negative numbers have no real square roots. Thus numbers like -9 , -16 , -25 have no real square roots.

From the last example it follows that a positive number has two square roots, numerically equal but opposite in sign. Thus the square roots of 9 are 3 and -3 . *The positive square root is called the principal square root.* Thus 3 is the principal square root of 9 and is denoted by $\sqrt{9}$.

The term **principal root** is also used for roots of a higher index. Thus, the principal cube root of 8 is 2, denoted by $\sqrt[3]{8}$; similarly, the principal cube root of -8 , denoted by $\sqrt[3]{-8}$, is -2 .

To unify the discussion on principal roots in all cases, we say that *the principal n th root of the number y , denoted $\sqrt[n]{y}$, is positive when y is positive and is negative if y is negative and n odd.* If y is negative and n even, the n th root of y does not exist for the present; this will lead in a later chapter to the consideration of a new kind of numbers, so called **imaginary numbers**.

Example 2. $\sqrt{25} = 5$.

It should be emphasized that $\sqrt{25}$ is *not* equal to -5 , even though -5 is one of the square roots of 25. When the negative square root of 25 is desired, the minus sign is explicitly attached, so that $-\sqrt{25} = -5$, whereas $\sqrt{25} = 5$.

Example 3. By inspection we find

$$\sqrt{49} = 7 \text{ since } 7^2 = 49.$$

$$\sqrt[3]{81} = 3 \text{ since } 3^4 = 81.$$

$$\sqrt[3]{-27} = -3 \text{ since } (-3)^3 = -27.$$

$$\sqrt[5]{-1} = -1 \text{ since } (-1)^5 = -1.$$

These examples illustrate the following:

For y positive and n even or odd, $\sqrt[n]{y}$ is positive.

For y negative and n odd, $\sqrt[n]{y}$ is negative.

For y negative and n even, $\sqrt[n]{y}$ does not exist.

Example 4. By inspection we find

$$\sqrt{\frac{4}{9}} = \frac{2}{3}, \text{ since } \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

$$\sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}, \text{ since } \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}.$$

$$\sqrt{16a^4b^6} = 4a^2b^3, \text{ since } (4a^2b^3)^2 = 16a^4b^6.$$

$$\sqrt{9 \sin^4 \theta} = 3 \sin^2 \theta, \text{ since } (3 \sin^2 \theta)^2 = 9 \sin^4 \theta.$$

EXERCISES

Find the value of each of the following radicals.

- | | | |
|---|---|-------------------------------------|
| 1. $\sqrt{16}$. | 2. $\sqrt{64}$. | 3. $\sqrt{81}$. |
| 4. $\sqrt{625}$. | 5. $\sqrt{400}$. | 6. $\sqrt[3]{1}$. |
| 7. $\sqrt{4}$. | 8. $\sqrt{121}$. | 9. $\sqrt{144}$. |
| 10. $\sqrt{225}$. | 11. $\sqrt{169}$. | 12. $\sqrt{36}$. |
| 13. $\sqrt[3]{8}$. | 14. $\sqrt[3]{-8}$. | 15. $\sqrt[3]{125}$. |
| 16. $\sqrt[3]{-\frac{8}{125}}$. | 17. $\sqrt{-a^3}$. | 18. $\sqrt[3]{-1}$. |
| 19. $\sqrt[3]{64a^3}$. | 20. $\sqrt{0.01E^2}$. | 21. $\sqrt{4a^2b^4}$. |
| 22. $\sqrt{900}$. | 23. $\sqrt{\frac{25}{16} \sin^4 B}$. | 24. $\sqrt[3]{-\frac{1}{8}}$. |
| 25. $\sqrt{\frac{49}{144}}$. | 26. $\sqrt[3]{1000}$. | 27. $\sqrt[3]{16(\cos \theta)^4}$. |
| 28. $\sqrt[3]{81}$. | 29. $\sqrt[3]{10,000}$. | 30. $\sqrt[3]{-0.008E^3}$. |
| 31. $\sqrt[3]{-32}$. | 32. $\sqrt[3]{512 \tan^3 A}$. | 33. $\sqrt[3]{-\frac{27}{8}}$. |
| 34. $\sqrt{\frac{4}{9} X^4 Y^2}$. | 35. $\sqrt[3]{8000}$. | 36. $\sqrt{0.09x^2}$. |
| 37. $\sqrt{625 \cos^4 \theta}$. | 38. $\sqrt{\frac{1}{81} (\sin A)^4}$. | 39. $\sqrt{\frac{9}{16} E^4}$. |
| 40. $\sqrt{0.01E^4}$. | 41. $\sqrt[3]{-343(\tan B)^6}$. | 42. $\sqrt[5]{\frac{32}{243}}$. |
| 43. $\sqrt[3]{-\frac{R^6 \cdot \sin^3 \theta}{S^6 L^{12}}}$. | 44. $\sqrt[5]{-\frac{1}{32} \sin^5 \theta}$. | 45. $\sqrt[5]{-243}$. |

Find the principal square root of:

- | | | |
|---------------------|-----------------------|----------------------|
| 46. 144. | 47. 100. | 48. $\frac{1}{18}$. |
| 49. 81. | 50. 0.0001. | 51. 44. |
| 52. 64. | 53. $\frac{1}{49}$. | 54. 9. |
| 55. 25. | 56. $\frac{1}{121}$. | 57. 225. |
| 58. $\frac{1}{4}$. | 59. 0.01. | 60. $\frac{9}{25}$. |
| 61. 289. | 62. 169. | 63. $\frac{1}{36}$. |
| 64. $\frac{1}{9}$. | 65. 400. | 66. $\frac{9}{16}$. |

Find the principal cube root of:

- | | | |
|----------------------|------------------------|-----------------------|
| 67. 64. | 68. -27. | 69. 8. |
| 70. -125. | 71. $\frac{1}{8}$. | 72. -1. |
| 73. -64. | 74. 216. | 75. $-\frac{8}{27}$. |
| 76. $-\frac{1}{8}$. | 77. $\frac{27}{125}$. | |

Find the real square roots of:

- | | | |
|----------|----------|----------|
| 78. 16. | 79. 81. | 80. 256. |
| 81. 625. | 82. 64. | 83. 4. |
| 84. 49. | 85. 144. | |

8-3. Rational and Irrational Numbers. It was shown in Sec. 1-3 that, by starting with positive integers and the operations of addition, subtraction, multiplication, and division, we arrived at the system of rational numbers. It will be recalled that a rational number was defined to be one which can be expressed as a fraction $\frac{m}{n}$, where the numerator m and the denominator n are integers. Now the operation of extracting the n th root of positive rational numbers and the n th (n odd) root of negative rational numbers leads to new numbers like $\sqrt{2}$, $\sqrt[3]{5}$, etc., called **irrational numbers**. The rational and irrational numbers make up the system of **real numbers**.

It will be seen in Chapter 12 that the n th (n -even) root of negative numbers will lead to a new type of numbers, the so-called imaginaries.

8-4. Fractional Exponents. In the present section we shall introduce fractional exponents. This will unite the subject of radicals and that of exponents under one set of laws.

At present a^2 means a taken twice as a factor, but to say that $a^{\frac{1}{2}}$ means a taken one-half time as a factor is unintelligible. Since $a^{\frac{1}{2}}$ is at present a meaningless symbol, we are free to define it in any manner we please. In mathematics a definition is said to be a good one if it is in harmony with the material previously developed. With this in mind we shall give meanings to fractional exponents which will conform to the laws of integral exponents summarized in Sec. 8-1 of this chapter.

To give a meaning to $a^{\frac{1}{2}}$ by assuming that Law 1 of Sec. 8-1 holds, we write

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a,$$

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a.$$

We see, then, that $a^{\frac{1}{2}}$ is one of the two equal factors of a and hence

$$a^{\frac{1}{2}} = \sqrt{a}.$$

The right-hand side of the last equation is our definition of $a^{\frac{1}{2}}$.

Similarly, to give a meaning to $a^{\frac{1}{3}}$ we write

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = a^1 = a,$$

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a.$$

The expression $a^{\frac{1}{3}}$ is now one of three equal factors of a , and hence

$$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

In general, if n is a positive integer

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots a^{\frac{1}{n}} = a \quad (n \text{ factors}),$$

and hence

$$(1) \quad a^{\frac{1}{n}} = \sqrt[n]{a}.$$

In an analogous way we can define an expression like $a^{\frac{2}{3}}$ by writing

$$a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3}} = a^2,$$

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

In general, if m and n are positive integers,

$$a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdots a^{\frac{m}{n}} = a^m \quad (n \text{ factors})$$

$$(2) \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

The last equation defines a fractional exponent to be a radical. Thus we define $a^{\frac{m}{n}}$ to be the principal n th root of a^m . For $m = 1$ this equation reduces to $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Section 1-11 contained a discussion of negative and zero exponents. This, with the material of the present section, enables us to work with expressions like a^x where a is any **real number** and x any **rational number**.

Summarizing all the definitions concerning exponents so far developed, we have:

DEFINITIONS

a any real number; m and n integers.

$$1. a^n = a \cdot a \cdots a \quad (n \text{ factors}).$$

$$2. a^{-n} = \frac{1}{a^n}.$$

$$3. a^0 = 1.$$

$$4. a^{\frac{1}{n}} = \sqrt[n]{a}.$$

$$5. a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Using these definitions we can now write expressions involving radicals and powers in various forms.

Examples.

$$x \cdot x \cdot x = x^3.$$

Definition 1

$$(-1)^{\frac{1}{3}} = \sqrt[3]{(-1)}$$

Definition 4

$$a^{-4} = \frac{1}{a^4}.$$

Definition 2

$$= -1.$$

$$5E^{-2}L^{-3} = \frac{5}{E^2L^3}.$$

Definition 2

$$\left(\frac{1}{8}\right)^0 = 1.$$

Definition 3

$$8^{\frac{2}{3}} = \sqrt[3]{8^2}$$

Definition 5

$$4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}}$$

Definition 2

$$= \sqrt[3]{64}$$

$$= \frac{1}{\sqrt[3]{4^3}}$$

Definition 5

$$= 4.$$

$$(5x + 9)^0 = 1.$$

Definition 3

$$= \frac{1}{\sqrt[3]{64}}$$

$$\frac{a^0}{b^{-5}} = a^0 \cdot b^5$$

Definition 2

$$= \frac{1}{8}.$$

$$= b^5.$$

Definition 3

$$\frac{3x^2y^{-3}}{z^{-1}} = \frac{3x^2z}{y^3}.$$

Definition 2

EXERCISES

Find the value of each expression by changing from a fractional exponent to a radical or from a negative to a positive exponent.

1. $27^{\frac{1}{3}}$.

2. $49^{\frac{1}{2}}$.

3. 4^{-2} .

4. 8^{-1} .

5. $\left(\frac{1}{4}\right)^{\frac{1}{2}}$.

6. $8^{\frac{1}{3}}$.

7. $(-1)^{-\frac{3}{2}}$.

8. $32^{\frac{1}{5}}$.

9. $32^{-\frac{1}{5}}$.

10. $125^{-\frac{1}{3}}$.

11. $7a^0$.

12. $\frac{3}{2^{-3}}$.

13. $(3x^2 + 7x + 1)^0$.

14. 2.4^{-2} .

15. $(0.125)^{-\frac{1}{3}}$.

16. $(\frac{1}{85})^{-\frac{1}{2}}$.

19. $(0.0049)^{\frac{1}{2}}$.

22. 10^{-4} .

25. $(\frac{1}{84})^{-\frac{1}{2}}$.

17. $(-\frac{1}{7})^{-1}$.

20. $(0.0081)^{-\frac{1}{2}}$.

23. $1^{-1} + 2^{-2}$.

26. $(\frac{1}{84})^{-\frac{1}{2}}$.

18. 1021^0 .

21. $(0.008)^{\frac{1}{2}}$.

24. $(-\frac{1}{5})^{-1}$.

27. $(0.0001)^{\frac{1}{2}}$.

Change the following into identical expressions without zero or negative exponents and simplify.

28. $\frac{S^0}{r^{-4}}$.

29. $\frac{4^{-2}}{4^{-1}}$.

30. L^{-4} .

31. r^2S^{-5} .

32. $\frac{\cos \theta}{5x^{-3}y^{-5}}$.

33. $\frac{m^2n^{-4}}{p^{-3}}$.

34. $\frac{2r^{-3}S^5}{4r^4S}$.

35. $5(\sin \theta)^{-2}$.

36. $6^{-2}cy^{-4}$.

37. $\frac{5x^2y^{-3}}{x^{-4}y^2}$.

38. $\frac{2^{-3}L^{-1}}{5^{-1}S^{-4}}$.

39. $\frac{3x^{-2}y^{-3}}{6^{-1}z^{-2}}$.

Write the following in forms without denominators, using negative exponents if necessary.

40. $\frac{S}{r}$.

41. $\frac{x^{-3}}{y^{-4}}$.

42. $\frac{r}{S^2}$.

43. $\frac{1}{N^3}$.

44. $\frac{7}{y^3}$.

45. $\frac{3}{L^2}$.

46. $\frac{y^2}{6^2}$.

47. $\frac{a^2}{x^4}$.

48. $\frac{a^3}{xyz}$.

49. $\frac{1}{x^2y^2z^2}$.

50. $\frac{1}{Sr}$.

51. $\frac{P}{S^2r}$.

52. $\frac{5k}{x^2y^4}$.

53. $\frac{1}{r_1^2r_2^2r_3^2}$.

54. $\frac{S}{r_1^2r_2^4}$.

Write the following expressions with radicals instead of fractional exponents.

55. $r^{\frac{1}{2}}$.

56. $Z^{\frac{1}{2}}$.

57. $L^{\frac{5}{7}}$.

58. $A^{\frac{4}{3}}$.

59. $5x^{\frac{1}{2}}$.

60. $7W^{\frac{1}{2}}$.

61. $7S^{\frac{1}{2}}$.

62. $Ky^{\frac{3}{2}}$.

63. $(2a)^{\frac{1}{2}}$.

64. $18W^{\frac{1}{2}}$.

65. $(3Sr)^{\frac{1}{2}}$.

66. $(2Lx^3)^{\frac{1}{2}}$.

67. $(6 \cdot \sin \theta)^{\frac{1}{2}}$.

68. $(7abc)^{\frac{1}{2}}$.

69. $(2S_1 + 3S_2)^{\frac{1}{2}}$.

70. $(L_x + L_y)^{\frac{1}{2}}$.

71. $2(\cos \theta)^{\frac{1}{2}}$.

72. $(3r^2 + S)^{\frac{1}{2}}$.

Using fractional exponents, write the following expressions without radicals.

73. $\sqrt{S^3}$.

74. $\sqrt{r^5}$.

75. $\sqrt[3]{N^4}$.

76. $\sqrt[3]{Z^2}$.

77. $\sqrt[5]{x^8}$.

78. $\sqrt[5]{M}$.

79. $\sqrt[3]{P}$.

80. $\sqrt[3]{a^5}$.

81. $\sqrt[3]{x^{12}}$.

82. $\sqrt{S_1 + S_2}$.

83. $\sqrt[3]{x + y}$.

84. $\sqrt[3]{(L_x + K)^2}$.

85. $\sqrt[3]{M^3 + N^3}$.

86. $\sqrt[3]{(x - 3y)^3}$.

87. $\sqrt[3]{(a - b)^2}$.

88. $\sqrt[3]{\sin \theta}$.

89. $\sqrt[3]{\tan^2 \theta}$.

90. $\sqrt{\sin A + \cos A}$.

8-5. The Laws of Exponents. In Sec. 8-1 of this chapter a summary of the laws of exponents was given. These laws were true only for positive and negative integral values of the exponents. Later, in Sec. 8-4, fractional exponents were defined in such a way that they were in harmony with the first law of Sec. 8-1. Although we leave the proof to more advanced work in mathematics, it can be shown that our definition of fractional exponents of Sec. 8-4 is consistent with all the laws of exponents. To state this fact explicitly we write:

Laws 1 to 7 of Sec. 8-1 are true when the exponents m , n , and q are rational numbers, i.e., integers, zero, fractional, or negative.

The way the negative and fractional exponents have been introduced exemplifies an important characteristic of mathematics. This is known as the **principle of permanence of formal laws of algebra**. This same principle helped us when we enlarged our number system from positive integers to that of the real number system. Its object is to make a few laws suffice where otherwise we would need many. With appropriate definitions for the negative and fractional exponents, this principle achieves concision by applying a formula already in existence to a much greater variety of cases, thus making the introduction of new formulas unnecessary.

With this generalization in mind, the definitions given in Sec. 8-4 and the laws given in Sec. 8-1 form the basis for all the work in the remainder of this chapter and for any work on exponents in this book. Subsequent allusions to a definition or law of exponents will always refer to Secs. 8-4 and 8-1, and to Sec. 8-6, which follows.

Examples. Perform the indicated operations by the laws of exponents.

$$\begin{aligned} S^{\frac{1}{2}}S^{\frac{1}{2}} &= S^{\frac{1}{2}+\frac{1}{2}} && \text{Law 1} \\ &= S^1. \end{aligned}$$

$$\begin{aligned} \frac{R^{\frac{1}{2}}}{R^{\frac{1}{2}}} &= R^{\frac{1}{2}-\frac{1}{2}} && \text{Law 2} \\ &= R^0. \end{aligned}$$

$$\begin{aligned} (4^{\frac{3}{4}})^{-2} &= 4^{\frac{3}{4}(-2)} && \text{Law 4} \\ &= 4^{-\frac{3}{2}} \\ &= \frac{1}{8^{\frac{1}{2}}}. && \text{Definition 2} \end{aligned}$$

$$\begin{aligned} (9xy)^{\frac{1}{2}} &= 9^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}} && \text{Law 5} \\ &= 3x^{\frac{1}{2}}y^{\frac{1}{2}}. \end{aligned}$$

$$\left(\frac{M^2}{N^{-3}}\right)^{\frac{1}{2}} = \frac{(M^2)^{\frac{1}{2}}}{(N^{-3})^{\frac{1}{2}}} \quad \text{Law 6}$$

$$= \frac{M^{\frac{2}{2}}}{N^{-1}} \quad \text{Law 4}$$

$$= M^{\frac{1}{2}}N. \quad \text{Definition 2}$$

$$(5^{\frac{1}{2}}x^{\frac{3}{2}}y)^4 = (5^{\frac{1}{2}})^4 \cdot (x^{\frac{3}{2}})^4 \cdot (y)^4 \quad \text{Law 7}$$

$$= 25x^3y^4. \quad \text{Law 4}$$

$$(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}}) = x + x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y \quad \text{Law 1}$$

$$= x - y.$$

$$\left(\frac{a^{-1} \cdot b}{a^3 \cdot b^{-3}}\right)^{\frac{1}{2}} = (a^{-4} \cdot b^4)^{\frac{1}{2}} \quad \text{Law 2}$$

$$= a^{-2}b^2 \quad \text{Law 7}$$

$$= \frac{b^2}{a^2}. \quad \text{Definition 2}$$

Using negative or fractional exponents we can factor certain expressions. This will be illustrated by the following examples.

Example 1.

$$\begin{aligned} R_1 - R_2 &= (R_1^{\frac{1}{2}})^2 - (R_2^{\frac{1}{2}})^2 \\ &= (R_1^{\frac{1}{2}} - R_2^{\frac{1}{2}})(R_1^{\frac{1}{2}} + R_2^{\frac{1}{2}}). \end{aligned}$$

Example 2.

$$\begin{aligned} 25x^{-4} - 9y^2 &= (5x^{-2})^{-2} - (3y)^2 \\ &= (5x^{-2} - 3y)(5x^{-2} + 3y). \end{aligned}$$

Example 3.

$$\begin{aligned} 4L^2 + 4LR^{-\frac{1}{2}} + R^{-1} &= (2L)^2 + 2 \cdot (2L)(R^{-\frac{1}{2}}) + (R^{-\frac{1}{2}})^2 \\ &= (2L + R^{-\frac{1}{2}})^2. \end{aligned}$$

The laws of exponents are useful in evaluating certain numerical expressions.

Example 4.

$$(64)^{\frac{1}{2}} = (2^6)^{\frac{1}{2}} = 2^{6 \cdot \frac{1}{2}} = 2^3 = 8.$$

Example 5.

$$\left(\frac{1}{125}\right)^{\frac{2}{3}} = \left(\frac{1}{5^3}\right)^{\frac{2}{3}} = \frac{1}{5^{3 \cdot \frac{2}{3}}} = \frac{1}{5^2} = \frac{1}{25}.$$

Example 6.

$$(32)^{\frac{1}{5}} \cdot (16)^{-\frac{1}{4}} = (2^5)^{\frac{1}{5}} \cdot (2^4)^{-\frac{1}{4}} = 2^1 \cdot 2^{-1} = 1.$$

EXERCISES

Perform the indicated operations by the laws of exponents and express the results without negative or zero exponents.

1. $(ab^{\frac{1}{2}})^0$.
2. $x^0(y^{\frac{1}{2}})^4$.
3. $y^4 \cdot y^{-\frac{1}{2}}$.
4. $(2a^{\frac{1}{2}})^6$.
5. $2^3 \cdot 2^{-5}$.
6. $\left(\frac{b^0 c^{-2}}{b^{-2} \cdot c^0}\right)^{\frac{1}{2}}$.
7. $2L^{\frac{1}{2}} \cdot L^{-\frac{5}{2}}$.
8. $\left(\frac{r^3}{s^6}\right)^{\frac{1}{2}}$.
9. $\frac{m^x}{m^{-x}}$.
10. $(-125x^{15})^{\frac{1}{3}}(-64x^{12})^{-\frac{1}{3}}$.
11. $K^{-1} \cdot \frac{6}{K^{-\frac{2}{3}}}$.
12. $(25R^{-2})^{-\frac{3}{2}}$.
13. $\frac{E^{\frac{3}{2}}}{E^{\frac{1}{2}}}$.
14. $(64x^{-3}y^6)^{\frac{2}{3}}$.
15. $(16L^4L^2)^{\frac{1}{4}}$.
16. $(27a^6b^3)^{\frac{2}{3}}$.
17. $(r^{\frac{2}{3}} \cdot r^{-\frac{1}{6}})^6$.
18. $(x^{\frac{1}{2}})^{\frac{2}{3}}(x^{-\frac{1}{3}})^{\frac{1}{2}}$.
19. $(32L^{-5}b^{10})^{\frac{1}{5}}$.
20. $(81m^4n^2)^{\frac{1}{3}}$.
21. $\left(\frac{4K}{9y}\right)^{\frac{1}{2}} \cdot \left(\frac{4K^{\frac{1}{2}}}{12y^{\frac{1}{2}}}\right)^{-1}$.
22. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
23. $(K_1^{-1} + K_2^{-1})(K_1 + K_2)$.
24. $(r^4 - L^4)^{\frac{2}{3}}(r^2 \cdot L^2)^{-\frac{2}{3}}$.
25. $(c^x + c^{-x})(c^x - c^{-x})$.
26. $(L^{\frac{1}{2}} + L^{\frac{1}{4}}r^{\frac{1}{4}} + r^{\frac{1}{4}})(L^{\frac{1}{4}} - r^{\frac{1}{4}})$.
27. $(-125K^{15})^{\frac{1}{3}}(-216K^{12})^{-\frac{1}{3}}$.
28. $(x - y^{\frac{1}{2}})(x^2 + xy^{\frac{1}{2}} + y)$.
29. $[(c^x + c^{-x})^2 - 4]^{\frac{1}{2}}$.
30. $(K_1^{\frac{1}{2}} + K_2^{-\frac{1}{2}})(K_1^{\frac{1}{2}} - K_2^{-\frac{1}{2}})$.

Find one value of:

31. $16^{\frac{3}{4}}$.
32. $8^{\frac{1}{2}}$.
33. $81^{\frac{1}{4}}$.
34. $27^{\frac{2}{3}}$.
35. $32^{\frac{3}{5}}$.
36. $216^{\frac{2}{3}}$.
37. $9^{-\frac{2}{3}}(\frac{1}{3})^{-3}$.
38. $8^{\frac{2}{3}}(4)^{-\frac{1}{2}}$.
39. $16^{-\frac{3}{4}}(\frac{1}{8})^{-\frac{5}{8}}$.

Factor each of the following expressions into two factors, using negative or fractional exponents.

40. $x^{\frac{2}{3}} - 9y^{\frac{2}{3}}$.
41. $9L^{\frac{4}{5}} - 4r^{\frac{2}{5}}$.
42. $a^2 - b^{-2}$.
43. $r^8 - 2r^4L^{\frac{1}{2}} + L^{\frac{1}{2}}$.
44. $25K^{-6} - 4r^2$.
45. $1 + 8a^{-\frac{5}{2}} + 16a^{-5}$.
46. $x^{\frac{4}{3}} + 5x^{\frac{2}{3}}y + 6y^2$.
47. $K^{12} + 6K^6r^{-\frac{7}{2}} + 9r^{-7}$.
48. $L_1 - 3L_1^{\frac{1}{2}}L_2^{\frac{1}{2}} - 4L_2$.
49. $3 - 4y^{\frac{1}{3}} + y^{\frac{2}{3}}$.
50. $4L^{\frac{2}{3}} - 20L^{\frac{1}{3}}r^{\frac{1}{3}} + 25r^{\frac{2}{3}}$.
51. $125 - x^{\frac{2}{3}}$.

8-6. Changes in Radical Form. By means of Definition 5, fractional exponents were identified with radicals. Although it is possible with the aid of this definition to express a radical as a power, it is convenient to retain the radical form in many operations. Definition 5, together with the laws of exponents, makes it possible to write any radical in a

variety of equivalent forms. These can be condensed in the following laws of radicals which follow directly from the laws of exponents.

LAWS OF RADICALS

$$8. (\sqrt[n]{a})^n = \sqrt[n]{a^n} = a.$$

$$9. \sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}.$$

$$10. \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

$$11. \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

$$12. \sqrt[km]{a^{kn}} = \sqrt[m]{a^n}.$$

These laws enable us to write radicals in a number of equivalent forms. The four ways in which radicals are ordinarily changed are given below with accompanying illustrative examples.

(a) *Removing factors from the radicand.* This is usually accomplished by making use of Law 9, for any factor which is a perfect n th power can be thus removed from the radicand.

Examples.

$$\begin{aligned}\sqrt{28} &= \sqrt{2^2 \cdot 7} \\ &= \sqrt{2^2} \cdot \sqrt{7} && \text{Law 9} \\ &= 2\sqrt{7}. && \text{Law 8}\end{aligned}$$

$$\begin{aligned}\sqrt[3]{40x^5y^3} &= \sqrt[3]{2^3 \cdot x^3 \cdot y^3 \cdot 5 \cdot x^2} && \text{Law 1} \\ &= \sqrt[3]{2^3 \cdot x^3 \cdot y^3} \cdot \sqrt[3]{5x^2} && \text{Law 9} \\ &= 2xy\sqrt[3]{5x^2}. && \text{Laws 8 and 9}\end{aligned}$$

$$\begin{aligned}\sqrt[4]{162E^5R^6L^7} &= \sqrt[4]{3^4 \cdot 2E^4R^4L^4ER^2L^3} && \text{Law 1} \\ &= \sqrt[4]{3^4E^4R^4L^4} \cdot \sqrt[4]{2ER^2L^3} && \text{Law 9} \\ &= 3ERL\sqrt[4]{2ER^2L^3}. && \text{Laws 8 and 9}\end{aligned}$$

$$\begin{aligned}\sqrt{a^2b^4 + 9b^4} &= \sqrt{b^4(a^2 + 9)} \\ &= \sqrt{b^4} \sqrt{a^2 + 9} && \text{Law 9} \\ &= b^2\sqrt{a^2 + 9} && \text{Law 8}\end{aligned}$$

$$\begin{aligned}\sqrt{\sin^3 \theta - \sin^2 \theta \cos \theta} &= \sqrt{\sin^2 \theta (\sin \theta - \cos \theta)} \\ &= \sqrt{\sin^2 \theta} \sqrt{\sin \theta - \cos \theta} && \text{Law 9} \\ &= \sin \theta \sqrt{\sin \theta - \cos \theta}. && \text{Law 8}\end{aligned}$$

(b) *Introducing quantities under the radical.* Any quantity multiplying a radical may be introduced under the radical sign. To accomplish this, the quantity has to be raised to a power corresponding to the index of the radical.

Examples.

$$7\sqrt{2} = \sqrt{7^2} \cdot \sqrt{2} \quad \text{Law 8}$$

$$= \sqrt{49 \cdot 2} \quad \text{Law 9}$$

$$= \sqrt{98}.$$

$$4ab\sqrt[3]{5ab} = \sqrt[3]{(4ab)^3} \cdot \sqrt[3]{5ab} \quad \text{Law 8}$$

$$= \sqrt[3]{(4ab)^3 \cdot 5ab} \quad \text{Law 9}$$

$$= \sqrt[3]{64a^3b^3 \cdot 5ab} \quad \text{Laws 7 and 4}$$

$$= \sqrt[3]{320a^4b^4}. \quad \text{Law 1}$$

(c) *Eliminating fractions from the radicand.* In order to simplify computations, a radical should usually be written without fractions under the radical sign. A fraction may be eliminated from the radicand by multiplying numerator and denominator of this fraction by a quantity that will make the denominator a perfect n th power where n is the index of the radical.

Examples.

$$\sqrt{\frac{2}{5}} = \sqrt{\frac{2}{5} \cdot \frac{5}{5}} \quad \text{Multiplication by } \frac{5}{5} = 1$$

$$= \sqrt{\frac{10}{5^2}} \quad \text{Definition 1}$$

$$= \frac{\sqrt{10}}{\sqrt{5^2}} \quad \text{Law 11}$$

$$= \frac{1}{5}\sqrt{10}. \quad \text{Law 8}$$

$$\sqrt{\frac{E}{R}} = \sqrt{\frac{E R}{R R}} \quad \text{Multiplication by } \frac{R}{R} = 1$$

$$= \sqrt{\frac{ER}{R^2}} \quad \text{Definition 1}$$

$$= \frac{\sqrt{ER}}{\sqrt{R^2}} \quad \text{Law 11}$$

$$= \frac{1}{R}\sqrt{ER}. \quad \text{Law 8}$$

$$\sqrt[3]{\frac{4ab}{9xy^7}} = \sqrt[3]{\frac{4ab}{9xy^7} \cdot \frac{3x^2y^2}{3x^2y^2}} \quad \text{Multiplication by } \frac{3x^2y^2}{3x^2y^2} = 1$$

$$= \sqrt[3]{\frac{12abx^2y^2}{3^3x^3y^9}} \quad \text{Law 1}$$

$$= \frac{\sqrt[3]{12abx^2y^2}}{\sqrt[3]{3^3x^3y^9}} \quad \text{Law 11}$$

$$= \frac{1}{3xy^2} \sqrt[3]{12abx^2y^2}. \quad \text{Law 8}$$

$$\sqrt{\frac{\sin \theta}{\cos \theta}} = \sqrt{\frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\cos \theta}} \quad \text{Multiplication by } \frac{\cos \theta}{\cos \theta} = 1$$

$$= \sqrt{\frac{\sin \theta \cdot \cos \theta}{\cos^2 \theta}} \quad \text{Law 1}$$

$$= \frac{\sqrt{\sin \theta \cdot \cos \theta}}{\sqrt{\cos^2 \theta}} \quad \text{Law 11}$$

$$= \frac{1}{\cos \theta} \sqrt{\sin \theta \cdot \cos \theta}. \quad \text{Law 8}$$

This procedure is known as **rationalizing the denominator**, for in the final form of the expression no radical appears in the denominator.

(d) *Reducing the radical to one with a lower index.* Using Law 12, the index of many radicals may be reduced.

Examples.

$$\sqrt[9]{27} = \sqrt[9]{3^3} \quad \text{Definition 1}$$

$$= \sqrt[3]{3^1} \quad \text{Law 12}$$

$$= \sqrt{3}.$$

$$\sqrt[4]{49x^2y^2} = \sqrt[4]{(7xy)^2} \quad \text{Law 7}$$

$$= \sqrt[2]{(7xy)^1} \quad \text{Law 12}$$

$$= \sqrt{7xy}.$$

$$\sqrt[4]{\sin^2 A \cdot \cos^2 A} = \sqrt[4]{(\sin A \cdot \cos A)^2} \quad \text{Law 7}$$

$$= \sqrt{\sin A \cos A}. \quad \text{Law 12}$$

It is not possible to make a general definition of the **simplest** form of a radical which will be applicable in all problems. In most problems, however, the changes in cases (a), (c), and (d) above are said to *simplify radicals*.

EXERCISES

Simplify by removing factors from the radicands.

- | | | |
|------------------------------------|----------------------------------|------------------------------------|
| 1. $\sqrt{200}$. | 2. $\sqrt{96}$. | 3. $\sqrt[3]{16}$. |
| 4. $\sqrt[3]{54}$. | 5. $\sqrt[4]{32}$. | 6. $\sqrt[5]{96}$. |
| 7. $\sqrt[3]{-16}$. | 8. $\sqrt{24R^3}$. | 9. $\sqrt[3]{x^7}$. |
| 10. $\sqrt{8E^5}$. | 11. $\sqrt[3]{-27L^7}$. | 12. $\sqrt[3]{135 \sin \theta}$. |
| 13. $\sqrt{5(\cos A)^3}$. | 14. $\sqrt{20(\tan \theta)^3}$. | 15. $\sqrt{9a^2x^2 \cos \theta}$. |
| 16. $\sqrt{18a^6 \sin^3 \theta}$. | 17. $\sqrt[5]{216x^3y^6z^9}$. | 18. $\sqrt[4]{81x^5y^9}$. |
| 19. $\sqrt[3]{0.008M^5}$. | 20. $\sqrt{x^2 + 4x^2y}$. | 21. $\sqrt{9 - 9x}$. |
| 22. $\sqrt{M^6 + M^3N}$. | 23. $\sqrt{(x^2 - y^2)^3}$. | 24. $\sqrt[3]{54a^3 - 27a^3b^3}$. |

Change each expression to a radical whose coefficient is 1.

- | | | |
|---|--|--|
| 25. $2\sqrt{3}$. | 26. $3\sqrt[3]{2}$. | 27. $5\sqrt{2}$. |
| 28. $3\sqrt{2y}$. | 29. $a\sqrt{bc}$. | 30. $2\sqrt[4]{3x}$. |
| 31. $5\sqrt[3]{a}$. | 32. $2x\sqrt[4]{3x^3}$. | 33. $2E^2R\sqrt[3]{ERL}$. |
| 34. $5\sqrt{3x}$. | 35. $4x^2\sqrt{\frac{3a}{2x}}$. | 36. $\frac{3}{2}m\sqrt{12m}$. |
| 37. $2E\sqrt[3]{7}$. | 38. $6E\sqrt{\frac{5R}{2E}}$. | 39. $(1 + y)\sqrt{1 + y}$. |
| 40. $\frac{m^2}{3n}\sqrt{\frac{3n}{m}}$. | 41. $\sin \theta \sqrt{\frac{\cos \theta}{\sin \theta}}$. | 42. $\frac{a}{\tan A} \sqrt{\frac{(\tan A)^3}{a^2}}$. |

Rationalize the denominators, i.e., eliminate fractions from the radicand, and remove factors from the radicands.

- | | | |
|--|--|--|
| 43. $\sqrt{\frac{2}{3}}$. | 44. $\sqrt{\frac{1}{2}}$. | 45. $\sqrt{\frac{1}{3}}$. |
| 46. $\sqrt{\frac{A}{3}}$. | 47. $\sqrt{\frac{3}{8}}$. | 48. $\sqrt{\frac{3}{7}}$. |
| 49. $\sqrt{\frac{3x}{5}}$. | 50. $\sqrt{\frac{1}{5R^3}}$. | 51. $\sqrt{\frac{a^3b^3}{2}}$. |
| 52. $\sqrt{\frac{s}{r}}$. | 53. $\frac{5}{\sqrt{3}}$. | 54. $\sqrt{\frac{3}{3^2}}$. |
| 55. $\sqrt[3]{\frac{7}{9}}$. | 56. $\sqrt[3]{\frac{27}{16E^4}}$. | 57. $\sqrt[3]{\frac{1}{4}}$. |
| 58. $\sqrt[4]{\frac{\sin A}{3}}$. | 59. $\sqrt[3]{\frac{5x}{16y^2}}$. | 60. $\sqrt[3]{\frac{-16L^5}{81E^2}}$. |
| 61. $\sqrt[4]{\frac{m}{27}}$. | 62. $\sqrt{\frac{\sin \theta}{\cos \theta}}$. | 63. $\sqrt{\frac{a}{\sin A}}$. |
| 64. $\sqrt{\frac{\tan \theta}{\cot \theta}}$. | 65. $\sqrt{\frac{5a + 1}{8b^5}}$. | 66. $\sqrt{\frac{mn}{m + n}}$. |

Reduce each radical to lowest order and remove factors from the radicands.

87. $\sqrt[3]{64}$.

68. $\sqrt[3]{25}$.

69. $\sqrt[3]{125}$.

70. $\sqrt[3]{9}$.

71. $\sqrt[3]{36}$.

72. $\sqrt[3]{81}$.

73. $\sqrt[3]{9x^8}$.

74. $\sqrt[3]{8E^3}$.

75. $\sqrt[3]{36L^4}$.

76. $\sqrt[3]{25x^2y^4}$.

77. $\sqrt[3]{27x^3y^9}$.

78. $\sqrt[4]{\frac{49m^2n^2}{16}}$.

79. $\sqrt[3]{\sin^2 \theta}$.

80. $\sqrt[3]{\sin^6 \theta \cos^2 \theta}$.

81. $\sqrt[6]{\frac{27x^3y^3}{64}}$.

82. $\sqrt[3]{36E^2R^2}$.

83. $\sqrt[3]{9 \tan^2 A}$.

84. $\sqrt[6]{\frac{\sin^3 \theta}{\cos^3 \theta}}$.

85. $\sqrt[3]{27 \cot^3 A}$.

86. $\sqrt[3]{121 \sin^2 \theta \cdot \cos^2 \theta}$.

87. $\sqrt[4]{\frac{36x^2y^2}{a^4b^4}}$.

Using the changes discussed in cases (a), (c), and (d), simplify the following radicals.

88. $\frac{1}{\sqrt{2}}$.

89. $\sqrt{\frac{5a^2}{3}}$.

90. $\sqrt{\frac{x^3y^3}{16}}$.

91. $\sqrt{\frac{3x}{2y^2}}$.

92. $\sqrt{\frac{1}{xy^2}}$.

93. $\sqrt{\frac{1}{5E^3}}$.

94. $\sqrt[3]{\frac{8m^3}{3x^4}}$.

95. $\sqrt[5]{128L^{-6}}$.

96. $\sqrt[3]{\frac{1}{3a^5}}$.

97. $\sqrt[3]{\frac{7a}{125x}}$.

98. $\sqrt{\frac{2}{5}x^2y^3}$.

99. $\sqrt[4]{\frac{25}{18}E^6}$.

100. $\sqrt[3]{\frac{8a}{3}}$.

101. $\sqrt[3]{\frac{125x^3}{216y^3}}$.

102. $\sqrt{\frac{4m^3n}{9}}$.

103. $2\sqrt[5]{\frac{E^6R}{81}}$.

104. $\sqrt[3]{\frac{(x-y)^3z^3}{16}}$.

105. $\sqrt[5]{\frac{32a^3y^6}{81a^2}}$.

106. $\sqrt{\frac{a^2b}{a+b}}$.

107. $\sqrt{1 + \frac{E^2}{1+2E}}$.

108. $\sqrt[3]{\frac{1}{R} + 1}$.

109. $\sqrt{(x^3 + y^3)(x + y)^3}$.

110. $\sqrt{(m+n)(m^2 - mn - 2n^2)}$.

111. $2\sqrt[3]{\frac{2a^3b^2c}{3x^3y^2z}}$.

8-7. Addition and Subtraction of Radicals. Terms containing the same radicals can be added or subtracted. Radicals are the **same** if they can be simplified so as to have the same radicand and the same index. In adding or subtracting expressions containing radicals, *simplify each radical by the methods of Sec. 8-6 and collect all multiples of the same radical.*

Examples.

$$2\sqrt{6} + 9\sqrt{\frac{2}{3}} - \sqrt[3]{36}$$

$$= 2\sqrt{6} + 9\sqrt{\frac{2}{3} \cdot \frac{3}{3}} - \sqrt[3]{6^2}$$

$$= 2\sqrt{6} + 3\sqrt{6} - \sqrt{6}$$

$$= 4\sqrt{6}.$$

Case (c) of Sec. 8-6 for the second term and case (d) for the third term

Collecting multiples of the same radical

$$3\sqrt{125} + \sqrt{75} - 3\sqrt{20} - \sqrt{675}$$

$$= 3\sqrt{5 \cdot 5^2} + \sqrt{3 \cdot 5^2} - 3\sqrt{5 \cdot 2^2} - \sqrt{3 \cdot 15^2}$$

$$= 15\sqrt{5} + 5\sqrt{3} - 6\sqrt{5} - 15\sqrt{3}$$

$$= 9\sqrt{5} - 10\sqrt{3}.$$

Case (a) of Sec. 8-6 for all terms

Collecting multiples of the same radical

$$\sqrt[3]{16E^2} - E\sqrt[3]{4E} + \sqrt[3]{64E^3}$$

$$= \sqrt[3]{(4E)^2} - E\sqrt[3]{4E} + \sqrt[3]{(4E)^3}$$

$$= \sqrt[3]{4E} - E\sqrt[3]{4E} + \sqrt[3]{4E}$$

$$= (2 - E)\sqrt[3]{4E}.$$

Case (d) for the first and last term

Collecting multiples of the same radical and factoring

EXERCISES

Perform the indicated additions and subtractions where possible.

1. $\sqrt{50} + 7\sqrt{2} - \sqrt{8}.$

2. $5\sqrt[3]{81} - \sqrt[3]{24} + 3\sqrt[3]{3}.$

3. $\sqrt[3]{5a} + 2x\sqrt[3]{5a}.$

4. $\sqrt{9 \sin \theta} - \sqrt{4 \sin \theta}.$

5. $5\sqrt[3]{4} + 2\sqrt[3]{32} - \sqrt[3]{108}.$

6. $4\sqrt{147} + 3\sqrt{75} + \sqrt{192}.$

7. $\sqrt{\frac{3}{2}} + 5\sqrt{24}.$

8. $5\sqrt{\frac{3}{2}} + \sqrt{60} - 3\sqrt{\frac{5}{2}}.$

9. $\sqrt{25a^2 \cos \theta} + \sqrt{9b^2 \cos \theta}.$

10. $E\sqrt{E} - \sqrt{4E^3} + \sqrt{9E^3}.$

11. $2\sqrt{\frac{5}{3}} + \sqrt{60} + \sqrt{\frac{3}{5}} + \sqrt{\frac{4}{15}}.$

12. $\sqrt{a^2b^2c} - a\sqrt{4c} + b\sqrt{a^2c}.$

13. $\sqrt{\sin^3 A} + \sqrt{\sin A \cdot \cos^2 A}.$

14. $\sqrt{\frac{x-y}{x+y}} - \sqrt{\frac{x+y}{x-y}}.$

15. $\sqrt{25 \sin x} + 2\sqrt{9 \sin x} - 3\sqrt{4 \sin x}.$

16. $\sqrt[3]{81a^5} - \sqrt[3]{16a} + \sqrt[3]{256a^5}.$

17. $3\sqrt{125m^3n^2} + n\sqrt{20m^3} - \sqrt{500m^3n^2}.$

18. $\sqrt{32x^4y^5} + 6\sqrt{72y} + 3\sqrt{128x^2y^3}.$

19. $2\sqrt[3]{E^6R} - 3E^2\sqrt[3]{64R} + 5E\sqrt[3]{E^3R}.$

20. $y^2\sqrt{8x^5y} + xy\sqrt{50x^3y^3} - x^2\sqrt{128xy^5}.$

21. $a^2\sqrt[3]{32a^2} + a\sqrt[3]{108a^5} + \sqrt[3]{500a^8}.$

22. $5\sqrt{\frac{7}{3}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{5}{14}}.$

23. $\sqrt[3]{\frac{2}{3}} + \sqrt[3]{\frac{4}{3}} + \sqrt[3]{\frac{8}{3}}.$

$$24. \sin^2 \theta \sqrt{150 \sin \theta} - \sqrt{54 \sin^5 \theta} - \sin \theta \sqrt{24 \sin^3 \theta}.$$

$$25. \sqrt{50b^4 - 75b^2y} - \sqrt{32b^2y^4 - 48y^5}.$$

8-8. Multiplication of Radicals. When the indices of the radicals are alike, they are multiplied according to Law 9 (Sec. 8-6). Thus to find the product of two or more radicals of the same index, multiply the coefficients to obtain the coefficient of the product and multiply the radicals by means of Law 9 to obtain the radical of the product. Simplify the result when necessary.

Examples.

$$\begin{aligned} 5\sqrt{2} \cdot 3\sqrt{5} &= 15\sqrt{2} \cdot \sqrt{5} && \text{Multiplying the coefficients} \\ &= 15\sqrt{10}. && \text{Law 9} \end{aligned}$$

$$\begin{aligned} 2\sqrt[3]{3} \cdot 5\sqrt[3]{4} \cdot 3\sqrt[3]{18} &= 2 \cdot 5 \cdot 3\sqrt[3]{3} \sqrt[3]{4} \sqrt[3]{18} && \text{Multiplying the coefficients} \\ &= 30\sqrt[3]{216} && \text{Law 9} \\ &= 30 \cdot 6 && \text{Law 8} \\ &= 180. \end{aligned}$$

$$\begin{aligned} \sqrt[4]{8R^3} \sqrt[4]{4R^2E} &= \sqrt[4]{32R^5E} && \text{Law 9} \\ &= 2R\sqrt[4]{2RE}. && \text{Simplifying} \end{aligned}$$

To multiply radicals of different indices, we must first express them (by Law 12) as radicals of a common index and proceed as above. This common index is the lowest common multiple (L.C.M.) of the indices of the original radicals. In performing operations with radicals it is sometimes easier to replace the radicals by fractional exponents and to use the laws of exponents.

Examples.

$$\begin{aligned} \sqrt{2} \cdot \sqrt[3]{4} &= \sqrt[6]{2} \cdot \sqrt[6]{4} && \text{The L.C.M. of the indices 2 and 3 is 6} \\ &= \sqrt[6]{2^3} \cdot \sqrt[6]{4^2} && \text{Law 12} \\ &= \sqrt[6]{2^3 \cdot 4^2} && \text{Law 9} \\ &= 2\sqrt[6]{2}. && \text{Simplifying} \end{aligned}$$

$$\begin{aligned} \sqrt[3]{4n} \sqrt[4]{8n^3} &= \sqrt[12]{(4n)^4} \cdot \sqrt[12]{(8n^3)^3} && \text{The L.C.M. of the indices 3 and 4 is 12} \\ &= \sqrt[12]{(2^2n)^4 \cdot (2^3n^3)^3} && \text{Law 9} \\ &= 2n\sqrt[12]{32n}. && \text{Simplifying} \end{aligned}$$

$$\begin{aligned} \sqrt{\sin \theta} \sqrt[3]{\sin^3 \theta} &= \sqrt[6]{\sin^2 \theta} \cdot \sqrt[6]{\sin^3 \theta} && \text{Law 12} \\ &= \sin \theta \sqrt[6]{\sin \theta}. && \text{Law 9 and simplifying} \end{aligned}$$

EXERCISES

Find the following products and simplify.

1. $2\sqrt{2} \cdot 5\sqrt{6}$.
2. $2\sqrt{3} \cdot 4\sqrt{2}$.
3. $4\sqrt{5} \cdot 3\sqrt{2}$.
4. $2\sqrt{5} \cdot 5\sqrt{5}$.
5. $\sqrt[3]{a^2} \cdot a\sqrt[3]{a}$.
6. $3\sqrt{7} \cdot 4\sqrt{7}$.
7. $3\sqrt[3]{9} \cdot 2\sqrt[3]{3}$.
8. $2\sqrt[3]{2} \cdot 5\sqrt[3]{4}$.
9. $4\sqrt[3]{x^2y^4} \cdot 2\sqrt[3]{xy^2}$.
10. $4\sqrt{\frac{8}{5}} \cdot 5\sqrt{\frac{2}{5}}$.
11. $\sqrt{4x^2} \cdot x\sqrt{3x^3}$.
12. $\sqrt[5]{16} \cdot 3\sqrt[5]{2}$.
13. $2\sqrt{2} \cdot 3\sqrt{3} \cdot 5\sqrt{6}$.
14. $\sqrt[3]{2E^2R} \cdot \sqrt[3]{4E^4R^4}$.
15. $\sqrt[5]{4a^2b^2} \cdot \sqrt[5]{2a^3b^2} \cdot \sqrt[5]{4ab^2}$.
16. $2\sqrt[4]{ab^2c^3d^4} \cdot 4\sqrt[4]{a^2bcd}$.
17. $3\sqrt[4]{27R^3x} \cdot \sqrt[4]{3Rx^5}$.
18. $2\sqrt[5]{8R^4E^2} \cdot \sqrt[5]{-4R^2E^3}$.
19. $2\sqrt[3]{2ab^2c} \cdot \sqrt[3]{4ab^3c^2} \cdot 3\sqrt[3]{8ab^2c^2}$.
20. $(\sqrt{2} + \sqrt{3})(2\sqrt{2} - 5\sqrt{3})$.
21. $\sqrt{3E} \sqrt[3]{9E^2}$.
22. $\sqrt[4]{4} \cdot \sqrt[6]{5}$.
23. $\sqrt[4]{x^3} \sqrt[5]{3x^5}$.
24. $\sqrt[3]{2\sin^2\theta} \sqrt[4]{4\sin\theta}$.
25. $\sqrt[4]{\frac{2}{3}} \cdot \sqrt[3]{\frac{1}{2}}$.
26. $(\sqrt{18} + 2\sqrt{72} - 3\sqrt{8}) \cdot \sqrt{2}$.
27. $\sqrt[5]{a^4b^3} \sqrt[3]{a^3b^5}$.
28. $\sqrt[3]{3\tan^2B} \cdot \sqrt[4]{9\tan^3B}$.
29. $\sqrt{6m} \sqrt[4]{24m^3n}$.
30. $\sqrt[3]{(E_1 + E_2)^2(E_1 - E_2)} \sqrt[3]{(E_1 + E_2)(E_1 - E_2)^2}$.
31. $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2$.
32. $(\sqrt{5x} - \sqrt{6})(\sqrt{5x} + \sqrt{6})$.
33. $\sqrt[4]{8\cos^3A} \sqrt{2\cos A}$.
34. $(\sqrt{2L} + \sqrt{5L})^2$.
35. $(\sqrt{a} + \sqrt{b})^3$.
36. $(2\sqrt[3]{\sin\theta} - 3)(5\sqrt[3]{\sin^2\theta} + 2)$.
37. $\sqrt[5]{a^2b^3c^4} \sqrt[4]{a^3b^2c}$.
38. $\sqrt[3]{\sin^2\theta} \sqrt{\sin\theta \cdot \cos\theta}$.
39. $m\sqrt{n}(m\sqrt{mn} + n\sqrt{mn} + \sqrt{mn})$.
40. $(\sqrt{a} - \sqrt{b} + \sqrt{ab})(2\sqrt{b})$.
41. $\sqrt{L} \cdot \sqrt[3]{L^2R} \cdot \sqrt[4]{L^3R^2}$.
42. $\sqrt{5 + \sqrt{4}} \sqrt{5 - \sqrt{4}}$.
43. $\sqrt{9 - \sqrt{17}} \cdot \sqrt{9 + \sqrt{17}}$.
44. $\sqrt[3]{8\sqrt{\sin^9A}}$.
45. $\sqrt[5]{32\sqrt{\cot^5\theta}}$.
46. $\sqrt[4]{\sqrt{(x-y)^8}}$.
47. $\sqrt[3]{\sqrt[5]{32a^{45}}}$.
48. $\sqrt[7]{\sqrt[3]{(3a-28)^{14}}}$.
49. $\sqrt[3]{R^3L^6\sqrt[4]{256a^8}}$.
50. $\sqrt[8]{R^2\sqrt[3]{R^2}}$.

8-9. Division of Radicals. When the dividend and the divisor are monomials containing radicals of the same index, the division may be performed by the use of Law 11. When the radicals are of different indices, express them as radicals of the same index by the method of the preceding section or change to fractional exponents.

Examples.

$$\frac{\sqrt{28}}{\sqrt{12}} = \sqrt{\frac{28}{12}} \quad \text{Law 11}$$

$$= \sqrt{\frac{7}{3}}$$

$$= \frac{1}{3}\sqrt{21}. \quad \text{Rationalizing}$$

$$\frac{\sqrt{4ER} \sqrt[3]{2ER}}{\sqrt[6]{4E^5R^3}}$$

The L.C.M. of the indices 2, 3, and 6 is 6

$$= \frac{\sqrt[6]{(4ER)^3} \sqrt[6]{(2ER)^2}}{\sqrt[6]{4E^5R^3}} \quad \text{Law 12}$$

$$= \sqrt[6]{\frac{64E^3R^3 \cdot 4E^2R^2}{4E^5R^3}}$$

Laws 9 and 11

$$= \sqrt[6]{64R^2}$$

Laws 1 and 3

$$= 2\sqrt[3]{R}.$$

Simplifying

$$\frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{8}{5}}} = \sqrt{\frac{\frac{3}{4}}{\frac{8}{5}}}$$

Law 11

$$= \sqrt{\frac{3}{4} \cdot \frac{5}{8}}$$

Inverting the divisor

$$= \frac{1}{4}\sqrt{\frac{15}{2}}$$

Simplifying

$$= \frac{1}{8}\sqrt{30}.$$

Rationalizing

One important case in division of radicals occurs when the divisor is a binomial in which one or both of the terms contain a square root. In such a case the division is performed by rationalizing the divisor. This is done as follows. *Multiply numerator and denominator of the resulting fraction by the denominator with the sign between its terms changed. Simplify the numerator and the denominator.* Similar methods can be applied to other special forms.

Examples.

$$\frac{2}{3 + \sqrt{5}}$$

$$= \frac{2}{3 + \sqrt{5}} \cdot \frac{3 - \sqrt{5}}{3 - \sqrt{5}}$$

Multiplying divisor and dividend by $3 - \sqrt{5}$

$$= \frac{2(3 - \sqrt{5})}{9 - 5}$$

Multiplying

$$= \frac{1}{2}(3 - \sqrt{5}).$$

Reducing

$$\frac{\sqrt{5} + 6\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$$

$$= \frac{\sqrt{5} + 6\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}} \cdot \frac{2\sqrt{5} + 3\sqrt{2}}{2\sqrt{5} + 3\sqrt{2}}$$

Multiplying divisor and
dividend by $2\sqrt{5} + 3\sqrt{2}$

$$= \frac{2 \cdot 5 + 3\sqrt{10} + 12\sqrt{10} + 3 \cdot 6 \cdot 2}{20 - 18}$$

Multiplying

$$= \frac{1}{2}(46 + 15\sqrt{10}).$$

Reducing

$$\frac{\sqrt{1+a} - \sqrt{1-a}}{\sqrt{1+a} + \sqrt{1-a}}$$

$$= \frac{\sqrt{1+a} - \sqrt{1-a}}{\sqrt{1+a} + \sqrt{1-a}} \cdot \frac{\sqrt{1+a} - \sqrt{1-a}}{\sqrt{1+a} - \sqrt{1-a}}$$

Multiplying divisor and
dividend by $\sqrt{1+a} - \sqrt{1-a}$

$$= \frac{(1+a) - 2\sqrt{1-a^2} + (1-a)}{(1+a) - (1-a)}$$

Multiplying

$$= \frac{2 - 2\sqrt{1-a^2}}{2a}$$

Reducing

$$= \frac{1 - \sqrt{1-a^2}}{a}.$$

EXERCISES

Divide and reduce to simplest form.

1. $\frac{\sqrt{12}}{\sqrt{3}}.$

2. $\frac{\sqrt{98}}{\sqrt{2}}.$

3. $\frac{\sqrt[3]{135a}}{\sqrt[3]{5a}}.$

4. $\frac{\sqrt{27xy}}{\sqrt{3xy}}.$

5. $\frac{\sqrt[3]{16 \sin^2 \theta}}{\sqrt[3]{2 \sin \theta}}.$

6. $\frac{\sqrt[4]{a^5 \sin^3 A}}{\sqrt[4]{a \sin A}}.$

7. $\frac{\sqrt[5]{64E^6}}{\sqrt[5]{2E}}.$

8. $\frac{5\sqrt{32}}{4\sqrt{45}}.$

9. $\frac{15\sqrt{6} - 9\sqrt{15}}{3\sqrt{3}}.$

10. $\frac{5\sqrt{8 \cos \theta} - \sqrt{18 \cos \theta}}{\sqrt{2 \cos \theta}}.$

11. $\frac{\sqrt{x^2 y^2}}{\sqrt{xy}}.$

12. $\frac{6\sqrt{5} + 18\sqrt{7} + 36}{2\sqrt{3}}.$

13. $\frac{3 \tan \theta}{\sqrt{\tan \theta}}.$

14. $\frac{6mn^2}{\sqrt{12mn^3}}.$

15. $\frac{2}{\sqrt{6} - 2}.$

16. $\frac{11}{5 + \sqrt{3}}.$

17. $\frac{\sqrt{30x} \sqrt[3]{24x^2} \sqrt[3]{75x}}{\sqrt[5]{5x}}.$

$$18. \frac{3\sqrt{y} \cdot \sqrt[4]{xy^2}}{\sqrt[3]{x^2y}}.$$

$$20. \frac{4 + \sqrt{2}}{4 - \sqrt{2}}.$$

$$22. \frac{\sqrt{2} + \sqrt{5}}{3\sqrt{5} - \sqrt{2}}.$$

$$24. \frac{\sqrt[6]{4ax^3y^2} \cdot \sqrt[3]{a^2y^2}}{\sqrt[4]{2a^3x}}.$$

$$26. \frac{\sqrt{R}}{\sqrt{R} + 1 + 1}.$$

$$28. \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}.$$

$$30. \frac{1}{\sqrt{3} - \sqrt{2}} \div \frac{1 + \sqrt{6}}{4 - \sqrt{6}}.$$

$$32. \frac{\sqrt{\sin \theta} - \sqrt{\cos \theta}}{\sqrt{\sin \theta} + \sqrt{\cos \theta}}.$$

$$34. \frac{1}{a - \sqrt{a^2 - x^2}}.$$

$$36. \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}.$$

$$38. \frac{\sqrt[3]{E^2} - \sqrt[3]{E}}{1 - \sqrt{E}}.$$

$$19. \frac{\sqrt[6]{4ax^3y}}{\sqrt[4]{2a^3x}}.$$

$$21. \frac{\sqrt{8} + \sqrt{3}}{\sqrt{8} - \sqrt{3}}.$$

$$23. \frac{\sqrt{2R - 3L}}{\sqrt[3]{4R^2 - 9L^2}}.$$

$$25. \frac{\sqrt{2} + \sqrt{E}}{\sqrt{E} + 2\sqrt{2}}.$$

$$27. \frac{1}{\sqrt{E} + \sqrt{R}}.$$

$$29. \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}}.$$

$$31. \frac{4\sqrt{3} - 3\sqrt{2}}{\sqrt{6}} \div \frac{\sqrt{10}}{4\sqrt{3} + 3\sqrt{2}}.$$

$$33. \frac{\sqrt{x^2 + y^2} + y}{\sqrt{x^2 + y^2} - y}.$$

$$35. \frac{5}{3\sqrt{2} - \sqrt{13}}.$$

$$37. \frac{\sqrt{\sin A - \cos A} + \sqrt{\sin A}}{\sqrt{\sin A - \cos A} - \sqrt{\sin A}}.$$

$$39. \frac{2x + \sqrt{1 - 4x^2}}{2x - \sqrt{1 - 4x^2}}.$$

8-10. Equations Involving Radicals. An equation in which the unknown quantity appears under a radical sign is called a **radical equation**. The radical equations needed in elementary mathematics and its applications contain no radicals other than square roots. We shall therefore limit our discussion to such equations only.

If the equation contains only *one* square radical, *transpose the terms of the equation so that the radical term stands alone in one member; then square both members and solve the resulting equation.*

The process of squaring is equivalent to a multiplication. In Sec. 7-5 we saw that if both members of an equation are multiplied by an expression involving the unknown, roots not in the original equation may be introduced. These are called extraneous roots and must be rejected. Therefore, it is essential to test each root obtained by substituting in the *original* equation.

The following examples should make the above procedure clear.

Example 1. Solve the equation $\sqrt{3x - 5} - 2 = 0$.

Arrange the equation so that the radical $\sqrt{3x - 5}$ will stand alone on one side of the equation:

$$\sqrt{3x - 5} = 2;$$

squaring,

$$3x - 5 = 4$$

and

$$x = 3.$$

Substituting $x = 3$ in the original equation:

$$\sqrt{9 - 5} - 2 = 0.$$

Therefore $x = 3$ is a root of the given equation.

Example 2. Solve the equation $\sqrt{R - 4} + 1 = 0$.

Transposing

$$\sqrt{R - 4} = -1.$$

Squaring

$$R - 4 = 1.$$

Thus

$$R = 5.$$

Substituting $R = 5$ in the original equation we get $\sqrt{5 - 4} + 1 = 0$ or $1 + 1 = 0$, and therefore $R = 5$ is an extraneous root and must be rejected. We conclude that the original equation has no solution. That this must be the case is obvious from a careful inspection of the equation $\sqrt{R - 4} + 1 = 0$. In order that this equation be satisfied, $\sqrt{R - 4}$ must equal -1 , which is not possible according to Sec. 8-2, since $\sqrt{R - 4}$ is the positive root of $R - 4$.

Example 3. Solve the equation $\sqrt{4x + 1} + 5 = x$.

Transposing

$$\sqrt{4x + 1} = x - 5.$$

Squaring

$$4x + 1 = x^2 - 10x + 25.$$

Collecting terms

$$x^2 - 14x + 24 = 0.$$

Factoring

$$(x - 2)(x - 12) = 0.$$

Therefore

$$x = 2 \quad \text{or} \quad x = 12.$$

Substituting $x = 2$ in the given equation, we find $\sqrt{8 + 1} + 5 = 2$, and therefore $x = 2$ is extraneous and must be rejected. Substituting $x = 12$ in the given equation,

$$\sqrt{48 + 1} + 5 = 12.$$

Therefore $x = 12$ is a root of the given equation.

To solve an equation containing two or more square radicals, the general procedure is to isolate one radical at a time on one side of the equation and continue as in the case of an equation containing only one radical. This process is repeated until all radicals are eliminated.

Example 4. Solve the equation $\sqrt{E-12} - \sqrt{E} = -2$.

Arrange the equation so that the more complicated radical term $\sqrt{E-12}$ will stand alone on one side of the equation.

$$\sqrt{E-12} = \sqrt{E} - 2.$$

Squaring

$$E - 12 = E - 4\sqrt{E} + 4.$$

Transposing and dividing by 4

$$\sqrt{E} = 4.$$

Squaring

$$E = 16.$$

Substituting $E = 16$ in the original equation

$$\sqrt{16-12} - \sqrt{16} = -2.$$

Therefore $E = 16$ is a root of the given equation.

Example 5. Solve the equation $\sqrt{x-1} + \sqrt{3x+3} = 4$.

Transposing

$$\sqrt{3x+3} = 4 - \sqrt{x-1}.$$

Squaring

$$3x + 3 = 16 - 8\sqrt{x-1} + x - 1.$$

Collecting terms

$$4\sqrt{x-1} = 6 - x.$$

Squaring

$$16(x-1) = 36 - 12x + x^2.$$

Collecting terms

$$x^2 - 28x + 52 = 0.$$

Factoring

$$(x-26)(x-2) = 0.$$

Therefore

$$x = 26 \quad \text{or} \quad x = 2.$$

Substituting $x = 26$ in the given equation, we find:

$$\sqrt{25} + \sqrt{81} = 4,$$

and therefore $x = 26$ is extraneous and must be rejected as a root.

Substituting $x = 2$,

$$\sqrt{1} + \sqrt{9} = 4.$$

Therefore $x = 2$ is a root of the given equation.

EXERCISES

Solve the following equations:

1. $\sqrt{x-2} = 7$.
2. $\sqrt{h+5} = \sqrt{28}$.
3. $2\sqrt{w-3} = 4$.
4. $\sqrt{3m+4} - m = 0$.
5. $5 - \sqrt{3v} = 4$.
6. $4 = \sqrt[3]{3x+7}$.
7. $\sqrt{L_x^2-5} - L_x + 1 = 0$.
8. $\sqrt{r+9} = 5\sqrt{r-3}$.
9. $\sqrt{L+4} + \sqrt{L} = 3$.
10. $\sqrt{4p-11} = 2\sqrt{p-1}$.
11. $7 = \sqrt{2x-5}$.
12. $\sqrt{9+x} = 9$.
13. $\sqrt{E+20} - \sqrt{E-1} = 3$.
14. $\sqrt{5E+10} - \sqrt{5E} = 2$.
15. $\sqrt{32+z} + \sqrt{z} = 16$.
16. $\sqrt{N-2} + \sqrt{2N-3} = 1$.
17. $\sqrt{2y+3} + \sqrt{y-2} + 2 = 0$.
18. $\sqrt{K+1} + \sqrt{K+2} = 3$.
19. $2\sqrt{R+6} + 3\sqrt{R+1} = 0$.
20. $\sqrt{x^2-7} + x = 7$.
21. $\sqrt{2y+14} + \sqrt{2y+35} = 7$.
22. $2x-3 = \sqrt{x^2+6x-6}$.
23. $\sqrt{x+5} - \sqrt{x-2} = 1$.
24. $\sqrt{4E-3} = 6 - \sqrt{4E-8}$.
25. $\sqrt{10x-1} - 2 = \sqrt{x}$.
26. $3\sqrt{R^2-4} - \sqrt{9R^2+4R-24} = 0$.
27. $E + 3\sqrt{E} - 10 = 0$.
28. $\sqrt{x-1} - \sqrt{x+4} + 1 = 0$.
29. $\sqrt{x} + \sqrt{x+4} = \frac{2}{\sqrt{x}}$.
30. $\sqrt{R} - \sqrt{R-5} = \sqrt{5}$.
31. $\sqrt{2E+20} - 9 = \sqrt{2E+13}$.
32. $\sqrt{H} - \sqrt{H+2} = 2$.
33. $\sqrt{\sqrt{3E+7}} = 4$.
34. $\frac{\sqrt{x}}{\sqrt{x+6}} = \frac{5}{6}$.
35. $\frac{\sqrt{p}-3}{\sqrt{p}+3} = \frac{\sqrt{p}-1}{\sqrt{p}-2}$.
36. $\frac{\sqrt{5L^2+16}}{\sqrt{3L+25}} = \frac{4}{5}$.
37. $2\sqrt{R^2-R+6} - 3R = 7 - 4R$.
38. $\sqrt{x-1} + \sqrt{x-4} = \sqrt{2x-1}$.

8-11. Numerical Computations. It was indicated in Sec. 8-3 that the operation of extracting roots lead to numbers like $\sqrt{2}$, $\sqrt[3]{5}$, etc., called irrational numbers. It is shown in books on more advanced mathematics that such numbers cannot be represented in forms of fractions like $\frac{m}{n}$ where m and n are integers. However, these irrational numbers can be approximated as closely as we please by rational numbers, i.e., fractions. Several methods are available for such computations. For our purpose it will be sufficient to use the slide rule for finding square and cube roots as explained in Chapter 1. If in computing square roots greater accuracy is required, the student should use Table 1 in the Appendix.

In evaluating square roots, it is often necessary to modify or simplify the radicand. The following examples will indicate some of the devices employed. The student should compare the answers of the following examples with those obtained by using the slide rule.

Example 1. Evaluate $\sqrt{120}$ using tables.

Although the number 120 is not in the table, we can find $\sqrt{120}$ by writing

$$\begin{aligned}\sqrt{120} &= \sqrt{4 \cdot 30} = 2\sqrt{30} \\ &= 2 \cdot 5.477 = 10.954.\end{aligned}$$

Example 2. Evaluate $\sqrt{3\frac{1}{2}}$, using tables.

We have $\sqrt{3\frac{1}{2}} = \sqrt{\frac{7}{2}} = \frac{\sqrt{7}}{\sqrt{2}}$, and although both $\sqrt{7}$ and $\sqrt{2}$ can now be found,

it is advisable to rationalize our expression so as to avoid division by $\sqrt{2} = 1.414$.

Thus:

$$\sqrt{3\frac{1}{2}} = \frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14}}{2} = \frac{3.742}{2} = 1.871.$$

Example 3. Evaluate $\frac{3}{5 - \sqrt{7}}$, using tables.

To avoid division by $5 - \sqrt{7} = 5 - 2.646 = 2.354$, rationalize the given expression. Thus:

$$\begin{aligned}\frac{3}{5 - \sqrt{7}} &= \frac{3(5 + \sqrt{7})}{(5 - \sqrt{7})(5 + \sqrt{7})} = \frac{15 + \sqrt{63}}{18} \\ &= \frac{15 + 7.937}{18} = \frac{22.937}{18} = 1.274.\end{aligned}$$

EXERCISES

Using the table or the slide rule, evaluate the following expressions.

- | | | |
|---|--|--|
| 1. $\sqrt{170} - \sqrt{110}$. | 2. $\frac{1}{\sqrt{2}}$. | 3. $\frac{1}{\sqrt{3}}$. |
| 4. $\frac{7}{\sqrt{5}}$. | 5. $\frac{1}{\sqrt{7}}$. | 6. $\frac{3\sqrt{5}}{\sqrt{2}}$. |
| 7. $\frac{\sqrt[3]{7}}{\sqrt{5}}$. | 8. $\frac{1}{\sqrt[3]{100}}$. | 9. $\frac{1}{\sqrt[3]{5}}$. |
| 10. $\frac{\sqrt[3]{2}}{\sqrt[3]{-25}}$. | 11. $\frac{7}{1 + \sqrt{2}}$. | 12. $\frac{\sqrt[3]{3}}{\sqrt[3]{49}}$. |
| 13. $\frac{1}{\sqrt{3} - \sqrt{11}}$. | 14. $\frac{\sqrt{5}}{\sqrt{3} - \sqrt{2}}$. | 15. $\frac{\sqrt[3]{-3}}{\sqrt[3]{-12}}$. |
| 16. $\frac{2 + \sqrt{5}}{2 - \sqrt{5}}$. | 17. $\frac{1}{1 + \sqrt{10}}$. | 18. $\frac{\sqrt{76} + \sqrt{31}}{\sqrt{3}}$. |

19. $\frac{1}{2 + 3\sqrt{3}}.$

20. $\frac{5(1 + \sqrt{6})}{\sqrt{2} + \sqrt{3}}.$

21. $\frac{\sqrt{11} - \sqrt{5}}{\sqrt{2}}.$

22. $\frac{2\sqrt{5} - \sqrt{3}}{2\sqrt{5} + \sqrt{3}}.$

23. $\frac{1 + \sqrt{\frac{1}{2}}}{1 + \sqrt{\frac{1}{3}}}.$

24. $\frac{1}{\sqrt{3} - \sqrt{2} + \sqrt{5}}.$

8-12. Applications to Formulas and Problems. Some of the results obtained in this chapter will now be applied to the transformation of formulas which are encountered in engineering and in physics. It is of great importance to be able to transpose a formula from one form to another.

For example, in radio engineering the inductance of a single-layer-wound air-core coil is given by the equation

$$L = \frac{(rN)^2}{9r + 10l}$$

where N is the number of turns, r the radius of coil in inches, l the length of winding in inches, and L the inductance in microhenries. When r , N , and l are given, we can compute L from the above formula. When L , r , and l are given, we may regard N as the unknown and solve for it in terms of the known quantities. We thus obtain

$$r^2 N^2 = L(9r + 10l),$$

$$N^2 = \frac{L(9r + 10l)}{r^2},$$

$$N = \frac{1}{r} \sqrt{L(9r + 10l)}.$$

In the last step only the positive square root was taken, since N is the number of turns and hence should be positive. Thus by transformation, a desired quantity in a formula can be isolated.

EXERCISES

In the formulas below, express the required quantity in terms of the others.

GIVEN	SOLVE FOR	DESCRIPTION
1. $A = \pi r^2$	r	Area of a circle
2. $P = \frac{E^2}{R}$	E	Electrical power
3. $S = 4\pi r^2$	r	Surface of a sphere
4. $W = \frac{E^2}{R}$	E	Ohm's law for d.-c. circuits

GIVEN	SOLVE FOR	DESCRIPTION
5. $V = \pi r^2 h$	r	Volume of a circular cylinder
6. $D = 1.063\sqrt{h}$	h	A formula used in navigation
7. $I = \sqrt{\frac{W}{R}}$	W, R	Ohm's law for d.-c. circuits
8. $P = I^2 R$	I	Electrical power
9. $R = \frac{K \cdot l}{d^2}$	d	Resistance of a wire
10. $c^2 = a^2 + b^2$	a, b	Pythagorean theorem
11. $F = \frac{M_1 M_2}{D^2}$	D	Force between two magnets
12. $V = a^3$	a	Volume of a cube
13. $V = \frac{4}{3}\pi r^3$	r	Volume of a sphere
14. $E = \frac{1}{2}mv^2$	v	Kinetic energy
15. $F = \frac{mv^2}{r}$	v	Centripetal force
16. $s = \frac{1}{2}gt^2$	t	Freely falling body
17. $Z = \sqrt{R^2 + X^2}$	R, X	Impedance of a resistance and reactance in series
18. $F_r = \frac{159}{\sqrt{LC}}$	L, C	Frequency
19. $d = \frac{Wl^2}{8d}$	l	
20. $H = \frac{d^2 n}{2.5}$	d	Horsepower for an automobile engine
21. $\lambda = 1884\sqrt{LC}$	L, C	Wavelength in terms of inductance and capacity
22. $K = \frac{M}{\sqrt{L_1 \cdot L_2}}$	L_1, L_2	The coefficient of coupling between two coils
23. $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$	a, b	
24. $t = 2\pi\sqrt{\frac{L}{g}}$	L, g	Time for one complete swing of a pendulum
25. $P_{\max.} = \frac{(\mu E_s)^2}{4r_p}$	E_s	Vacuum tube formula for maximum power output
26. $F = \frac{1}{2\pi\sqrt{LC}}$	L, C	Resonant frequency of a capacitor, inductor and resistor in series in an a.-c. circuit

GIVEN	SOLVE FOR	DESCRIPTION
27. $L = \frac{(rN)^2}{9r + 10l}$	r	The inductance of a single-layer-wound air-core coil
28. $F = \frac{8.94B^2A}{10^8}$	B	Tractive force of electromagnets
29. $L_{av.} = \frac{1.26N^2A\mu}{10^8 \cdot l}$	N	Inductance of long coils
30. $P = \frac{3.095L\omega z}{g \cdot D^2}$	D	Determination of viscosity
31. $L = \frac{4\pi^2n^2r^2\mu}{10^9l}$	n, r	Inductance of long coils
32. $L = 0.0251d^2n^2lk$	d, n	Inductance of a single-layer solenoid
33. $H = 0.24I^2RT$	I	Heating effect of a current
34. $V^2 = v_0^2 + 2gh$	V, v_0	
35. $Z = \frac{RX}{\sqrt{R^2 + X^2}}$	R, X	Impedance of a resistance and reactance in parallel
36. $Z = \sqrt{R^2 + (X_L - X_C)^2}$	R	Impedance of an inductance, resistance and capacity in series
37. $r = \sqrt{F} \left(\frac{1}{d_1} + \frac{1}{d_2} \right) \cdot 10^{-3}$	F	Radio-frequency resistance of a copper concentric transmission line
38. $S = \frac{n \cdot W \cdot 33 \cdot 10^8}{r^4}$	r	Steel springs
39. $E = \frac{1}{t} \left(\frac{T^2}{R} - 1 \right)$	T	
40. $L = \frac{0.315a^2N^2}{6a + 9b + 10c} \mu h$	N	The inductance of a coil
41. $I_L = \frac{E_0}{\sqrt{2(R_0^2 + w^2L_0^2)}}$	R_0, L_0	Alternating current of a class C amplifier
42. $P = R_l \left(\frac{\mu E_s}{r_p + R_l} \right)^2$	E_s	Vacuum tube formula for power output
43. $I = \frac{E}{\sqrt{R^2 + \left(2\pi FL - \frac{1}{2\pi fC} \right)^2}}$	R	Ohm's law for alternating-current circuits
44. $I_p + I_s = K \left(E_s + \frac{E_p}{\mu} \right)^{\frac{2}{3}}$	E_s, E_p	Triode formula
45. $r_p = \frac{\sqrt{a + b[a + b(\mu + 1)]^{\frac{2}{3}} \cdot 10^8}}{A_1 \sqrt{E_p + E_s}}$	$E_p + E_s$	Plate resistance of a tube

8-13. Related Graphs: The Power Function. Graphs of various algebraic functions were discussed in Chapter 3. In the present section we shall draw the graphs of the function

$$y = x^n$$

for different values of n . This function is sometimes called a **power function**.

Graph of $y = x^2$

x	$y = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

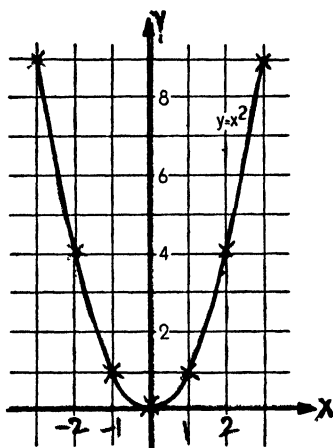
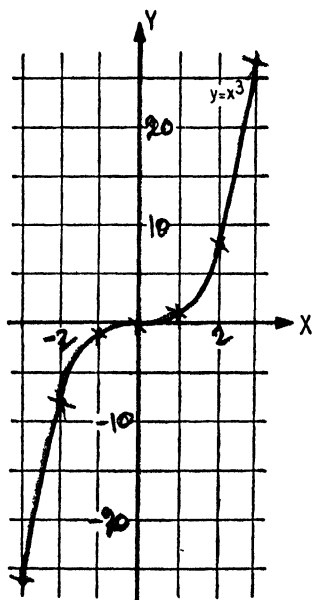


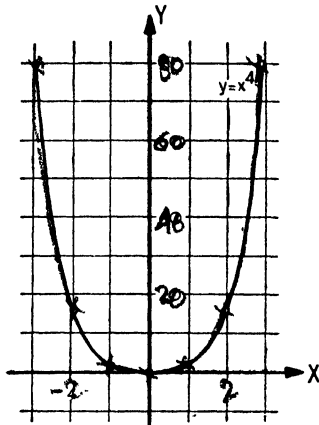
FIG. 8-1.



Graph of $y = x^3$

x	$y = x^3$
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

FIG. 8-2.



Graph of $y = x^4$

x	$y = x^4$
-3	81
-2	16
-1	1
0	0
1	1
2	16
3	81

FIG. 8-3.

Graph of $y = x$. This is a straight line passing through the origin.

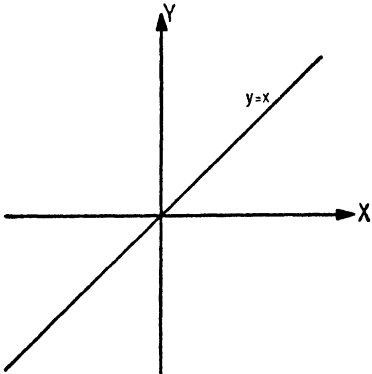


FIG. 8-4.

Graph of $y = \sqrt{x}$. In this case the function $y = \sqrt{x}$ is defined for positive values of x only.

x	$y = \sqrt{x}$
5	2.23
4	2
3	1.73
2	1.41
1	1
0	0

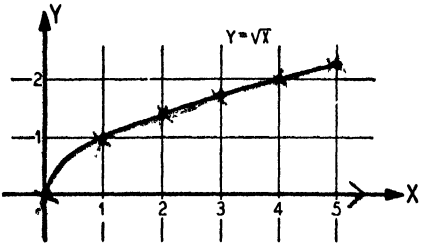


FIG. 8-5.

Graph of $y = \sqrt[3]{x}$

x	$y = \sqrt[3]{x}$
-5	-1.71
-4	-1.59
-3	-1.44
-2	-1.26
-1	-1
0	0
1	1
2	1.26
3	1.44
4	1.59
5	1.71

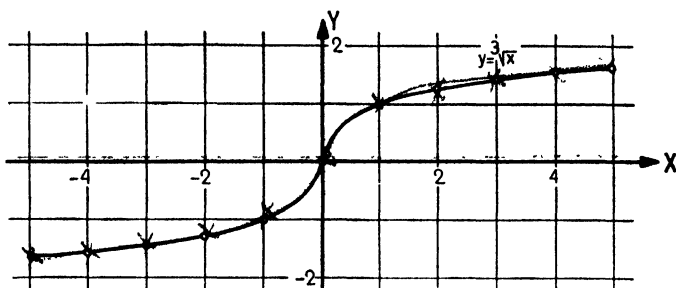


FIG. 8-6.

We now combine the six graphs given in this section in one figure. Thus in Fig. 8-7 are shown the graphs of $y = x^n$ for $n = \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4$. The student will be able to deduce for himself the relative positions and forms of the graphs of $y = x^n$ as n takes on various possible values.

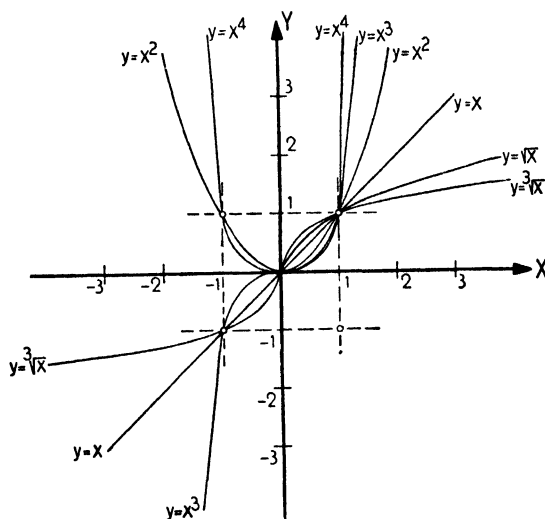


FIG. 8-7.

These graphs occur frequently, and it is therefore worth while to remember the graphs given in this section and to be able to sketch quickly the graphs of $y = x^n$ for various positive values of n .

EXERCISES

Draw graphs of each of the following functions.

- $y = \sqrt[4]{x}$.
- $y = x^5$.

- $y = x^{\frac{3}{4}}$.
- $y = x^6$.

8-14. Related Graphs: The Exponential Function. Up to the present we have considered the exponent n in the equation

$$y = x^n$$

as a constant and the base x as a variable. These roles can now be interchanged, i.e., consider the base as a constant and the exponent as a variable. When this is done, we obtain the equation

$$y = a^x,$$

called an **exponential function**. We shall consider only the exponential functions whose bases are positive numbers different from unity.

In this connection it should be pointed out that the exponent x in $y = a^x$, can assume also irrational values. The quantity $y = a^x$ can, in this case, be computed with any desired accuracy by replacing the irrational number x by a rational approximation: For example $3^{\sqrt{2}}$ is approximately $3^{1.41}$ because 1.41 is an approximation of $\sqrt{2}$.

Exponential functions occur frequently in engineering and it is important to be able to sketch them quickly. Consider, for example, the two functions

$$y = 2^x \quad \text{and} \quad y = 3^x$$

for which we obtain the following tables of values.

x	$y = 2^x$	x	$y = 3^x$
-3	$\frac{1}{8}$	-3	$\frac{1}{27}$
-2	$\frac{1}{4}$	-2	$\frac{1}{9}$
-1	$\frac{1}{2}$	-1	$\frac{1}{3}$
0	1	0	1
1	2	1	3
2	4	2	9
3	8	3	27

The corresponding graphs are drawn in Fig. 8-8.

The functions of the form $y = 2^x$ and $y = 3^x$ are exponential functions $y = a^x$. All such functions have the property that they intersect the y -axis in one point P whose coordinates are $(0, 1)$. At the point P (in Fig. 8-8) draw tangent lines to the two curves $y = 2^x$ and $y = 3^x$. These tangent lines form the angles α and β with the x -axis, called **angles of inclination**.

Measuring, the student will find that $\alpha < 45^\circ$ and $\beta > 45^\circ$. For different values of a , graphs of a^x may be plotted. The graphs will resemble those given in Fig. 8-8 and will intersect at point P . The tangent lines to these curves at the point will have various angles of inclination. We shall call the particular curve for which this angle of

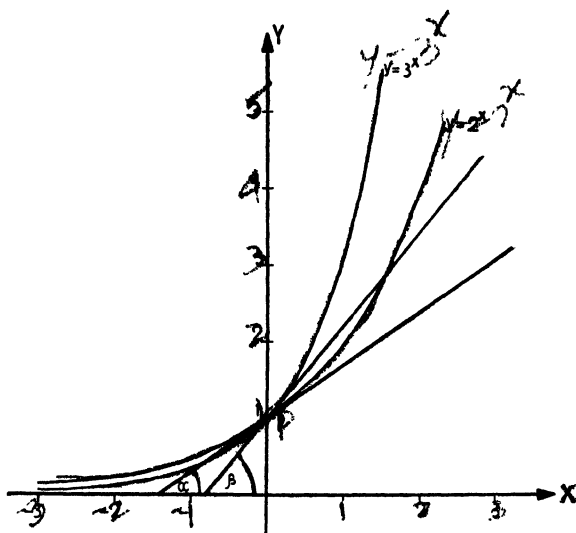


FIG. 8-8.

inclination is 45° the standard curve. From Fig. 8-8 it is obvious that this standard curve is of the form $y = a^x$ where $2 < a < 3$. In books on more advanced mathematics it is shown that for this standard curve

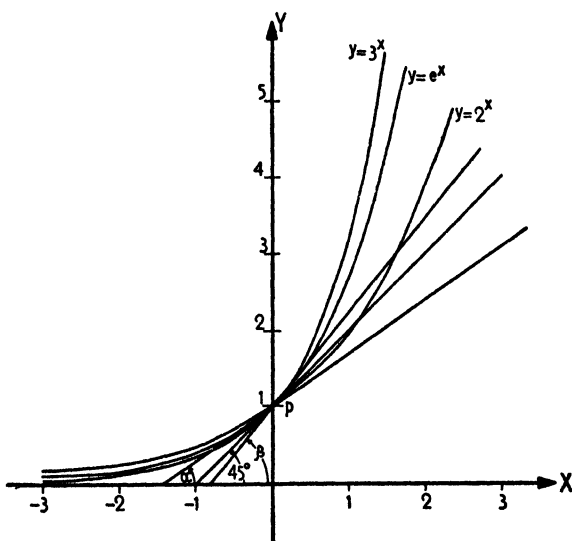


FIG. 8-9.

$a = 2.718$, approximately. Because of the importance of this number, a special notation e is introduced. Thus $e = 2.718$ and $y = e^x$ is the

standard curve described above. In Fig. 8-9 $y = 2^x$, $y = 3^x$, and $y = e^x$ are all plotted on the same set of axes.

The student should also plot the functions $y = 2^{-x}$, $y = e^{-x}$, $y = 3^{-x}$ and compare them with Fig. 8-9.

EXERCISES

Construct the graphs of the following functions.

1. $y = 2^{-x}$.

2. $y = 4^x$.

3. $y = 5^x$.

4. $y = 3^{-x}$.

5. $y = 4^{-x}$.

6. $y = e^{-x}$.

7. $y = (\frac{1}{2})^x$.

8. $y = (\frac{1}{3})^x$.

9. $y = 2^{-x^2}$.

PROGRESS REPORT

Exponents were introduced in Chapters 1 and ~~2~~ in order to have a convenient notation for certain complicated computations, and a few rules involving exponents were derived. It is a fundamental idea in mathematics, whenever a symbol or an operation is defined, to extend its use as far as possible without essentially changing the rules concerning the symbol or the operation. Thus the use of exponents was extended in this chapter to all real numbers so that the same laws for operating with exponents hold in all cases.

In addition to exponents, radicals were introduced, which are often very useful, although they are only another way of writing exponents.

The greater part of the chapter was devoted to a detailed study of the formal rules which are involved in the operations with exponents and radicals. Equations containing radicals were also discussed.

An important step was taken in the last few sections of this chapter where in the equation $y = u^x$ the quantity u or the quantity x was regarded as a variable. Thus the power function and the exponential function were introduced and studied.

A great many engineering formulas showed the usefulness of being familiar with exponents and radicals.

CHAPTER 9

LOGARITHMS

In the applications of mathematics to the other sciences, many problems involve long and tedious numerical computations. Logarithms, which will be studied in this chapter, can be used to simplify many of these computations, and they can be used to establish results to any desired degree of accuracy.

Logarithms are also used in many theoretical discussions in engineering and the other sciences. The distance covered during acceleration or deceleration of an automobile, train, or ship varies not directly but as an exponential function of time. The response of the human senses to stimuli is not directly proportional to the stimulus, but proportional to the logarithm of the stimulus. Many engineering formulas are most simply and conveniently expressed in logarithmic form.

9-1. Introduction. The discussions in this chapter will be based upon those of the preceding chapter on exponents.

Consider the two numbers $a = B^m$ and $c = B^n$. In writing $a = B^m$ we are associating an exponent m with the number a , and in writing $c = B^n$ we are associating an exponent n with the number c . By the laws of exponents in Chapter 8,

$$(1) \qquad ac = B^m \cdot B^n = B^{m+n},$$

$$(2) \qquad \frac{a}{c} = \frac{B^m}{B^n} = B^{m-n},$$

$$(3) \qquad a^k = (B^m)^k = B^{mk}.$$

From (1) we see that the multiplication of a and c corresponds to the addition of the associated exponents m and n . From (2) we see that the division of a by c corresponds to the subtraction of n from m . From (3) we see that the operation of raising a to the k th power corresponds to multiplying the exponent m (which is associated with a) by k .

It is easier to add than multiply, easier to subtract than divide, and easier to multiply than raise to a power. Therefore, provided that we had some way of associating with any number a an exponent m such that $a = B^m$, formula (1) would supply a convenient method for replac-

ing multiplication by addition; formula (2) would supply a method for replacing division by subtraction; and formula (3) would supply a method for replacing raising to a power by multiplication. For example, let us choose $B = 2$. Then for $a = 2^m$, the following table can be easily compiled.

TABLE 1

$a = 2^m$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512
m	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9

The use of this table is suggested in the following examples.

Example 1.

$$\begin{aligned}\frac{1}{32} \times 128 &= 2^{-5} \times 2^7 && \text{(By Table 1)} \\ &= 2^{7-5} && \text{(By the laws of exponents)}\end{aligned}$$

$$= 2^2 = 4. \quad \text{(By Table 1)}$$

Example 2.

$$8^3 = (2^3)^3 \quad \text{(By Table 1)}$$

$$= 2^9 \quad \text{(By the laws of exponents)}$$

$$= 512. \quad \text{(By Table 1)}$$

Of course, if the process suggested in this way is to have any practical value, the table must give exponents m for any a that occurs in computations. It will now be shown how a simple table of this kind for positive, two-digit numbers can be constructed on the basis of what the reader already knows.

In Chapter 8 it was shown that when a positive number B is given, $a = B^m$ has a definite meaning for any number m . Regarding m as the independent variable and a as the dependent variable, this function can be plotted as shown in Sec. 8-14. Figure 9-1 shows the graph of the function $a = 2^m$.

From Fig. 9-1, we see that (a) for every value of m there is a corresponding value of a , (b) the values of a are always positive, (c) as m increases, a increases, and (d) as m decreases, a decreases, approaching zero as m becomes very small, i.e., negative with a very large absolute value.

Suppose we now attempt to find an exponent m such that $5 = 2^m$. In Fig. 9-1 the straight line perpendicular to the a -axis at $a = 5$ intersects the curve at P , whose abscissa is about $m = 2.3$. It can be concluded, therefore, that $5 = 2^{2.3}$ approximately. Actually the exact

value of m is irrational, and in calculus methods are developed to compute m to as many digits as desired, but this rough approximation is sufficient here.

By drawing a careful graph like Fig. 9-1 to a large scale, the exponents in the following table can be easily determined.

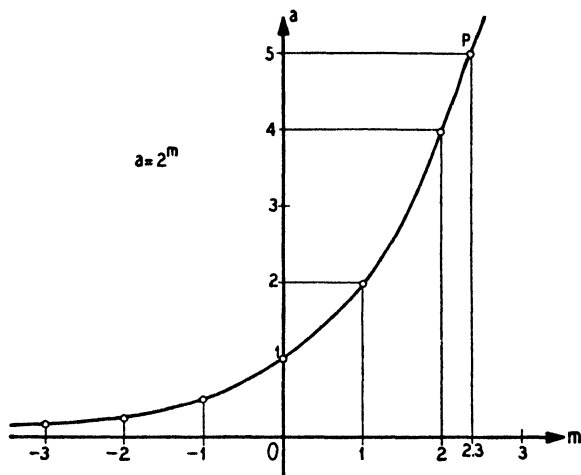


FIG. 9-1.

TABLE 2

$$a = 2^m$$

a	m	a	m	a	m	a	m	a	m	a	m
1.0	0.00	2.5	1.32	4.0	2.00	5.5	2.46	7.0	2.81	8.5	3.09
1.1	0.14	2.6	1.38	4.1	2.04	5.6	2.49	7.1	2.83	8.6	3.10
1.2	0.26	2.7	1.43	4.2	2.07	5.7	2.51	7.2	2.85	8.7	3.12
1.3	0.38	2.8	1.49	4.3	2.10	5.8	2.54	7.3	2.87	8.8	3.14
1.4	0.49	2.9	1.54	4.4	2.14	5.9	2.56	7.4	2.89	8.9	3.15
1.5	0.58	3.0	1.59	4.5	2.17	6.0	2.59	7.5	2.91	9.0	3.17
1.6	0.68	3.1	1.63	4.6	2.20	6.1	2.61	7.6	2.93	9.1	3.18
1.7	0.77	3.2	1.68	4.7	2.23	6.2	2.63	7.7	2.94	9.2	3.20
1.8	0.85	3.3	1.72	4.8	2.26	6.3	2.66	7.8	2.96	9.3	3.22
1.9	0.93	3.4	1.77	4.9	2.29	6.4	2.68	7.9	2.98	9.4	3.23
2.0	1.00	3.5	1.81	5.0	2.32	6.5	2.70	8.0	3.00	9.5	3.25
2.1	1.07	3.6	1.85	5.1	2.35	6.6	2.72	8.1	3.02	9.6	3.26
2.2	1.14	3.7	1.89	5.2	2.38	6.7	2.74	8.2	3.04	9.7	3.28
2.3	1.20	3.8	1.93	5.3	2.40	6.8	2.77	8.3	3.05	9.8	3.29
2.4	1.26	3.9	1.96	5.4	2.43	6.9	2.79	8.4	3.07	9.9	3.31
2.5	1.32	4.0	2.00	5.5	2.46	7.0	2.81	8.5	3.09	10.0	3.32

Computations may be performed by the use of Table 2 as illustrated in the following examples.

Example 3.

$$\begin{aligned}
 67 \times 84 &= 6.7 \times 10 \times 8.4 \times 10 \\
 &= 6.7 \times 8.4 \times 10^2 \\
 &= 2^{2.74} \times 2^{3.07} \times 10^2 && \text{(By Table 2)} \\
 &= 2^{2.74+3.07} \times 10^2 && \text{(By the laws of exponents)} \\
 &= 2^{5.81} \times 10^2.
 \end{aligned}$$

Since the exponent 5.81 is not in the tables and since $10 = 2^{3.32}$, we can write $5.81 = 2.49 + 3.32$ and we have

$$\begin{aligned}
 67 \times 84 &= 2^{2.49+3.32} \times 10^2 \\
 &= 2^{2.49} \times 2^{3.32} \times 10^2 && \text{(By the laws of exponents)} \\
 &= 2^{2.49} \times 10 \times 10^2 && \text{(By Table 2)} \\
 &= 5.6 \times 10^3. && \text{(By Table 2)}
 \end{aligned}$$

A slide rule computation easily verifies this to be the correct result.

Example 4.

$$\begin{aligned}
 67^3 &= (6.7 \times 10)^3 \\
 &= 6.7^3 \times 10^3 && \text{(By the laws of exponents)} \\
 &= (2^{2.74})^3 \times 10^3 && \text{(By Table 2)} \\
 &= 2^{8.22} \times 10^3 && \text{(By the laws of exponents)} \\
 &= 2^{1.58+2(3.32)} \times 10^3 \\
 &= 2^{1.58} \times (2^{3.32})^2 \times 10^3 && \text{(By the laws of exponents)} \\
 &= 2^{1.58} \times 10^2 \times 10^3 && \text{(By Table 2)} \\
 &= 3.0 \times 10^5. && \text{(By Table 2)}
 \end{aligned}$$

Example 5.

$$\begin{aligned}
 (5.6)^{3.2} &= (2^{2.49})^{3.2} && \text{(By Table 2)} \\
 &= 2^{2.49 \times 3.2} && \text{(By the laws of exponents)} \\
 &= 2^{7.97} \\
 &= 2^{1.33+2(3.32)} \\
 &= 2^{1.33} \times (2^{3.32})^2 && \text{(By the laws of exponents)} \\
 &= 2.5 \times 10^2. && \text{(By Table 2)}
 \end{aligned}$$

Example 6.

$$\begin{aligned}
 \frac{510}{67} &= \frac{5.1 \times 10^2}{6.7 \times 10} \\
 &= \frac{5.1}{6.7} \times 10 \\
 &= \frac{2^{2.35}}{2^{2.74}} \times 10 && \text{(By Table 2)} \\
 &= 2^{2.35-2.74} \times 10 && \text{(By the laws of exponents)} \\
 &= 2^{-0.39} \times 2^{3.32} && \text{(By Table 2)} \\
 &= 2^{3.32-0.39} && \text{(By the laws of exponents)} \\
 &= 2^{2.93} \\
 &= 7.6 && \text{(By Table 2)}
 \end{aligned}$$

Examples 4 and 6 can be readily checked on a Mannheim slide rule. However, Example 5 cannot be checked on a Mannheim rule.

From these considerations we see clearly how the use of exponents simplifies long computations by reducing the complexity of the operations involved. The purpose of this chapter is to develop a system with the same properties as the one above, but which can be used for practical computations when more accuracy is desired than can be obtained by slide rule computations. Thus we shall reduce, for example, the multiplication of 895.2 by 95.34 to the addition of two numbers. In this way tedious computations can be greatly simplified, with a corresponding saving in time and energy. Although we shall concentrate on computations in which four-digit accuracy is required, the methods developed will apply no matter what degree of accuracy is required.

EXERCISES

Perform the following computations by the use of Table 2. Express your results in scientific notation correct to two significant digits.

- | | | |
|-----------------------------------|----------------------------------|-------------------------------------|
| 1. 6.1×2.3 . | 2. 7.5×26 . | 3. 8.3×460 . |
| 4. $47 \times 1.8 \times 32$. | 5. $(6.5)^3$. | 6. $(1.8)^{4.3}$. |
| 7. $(6.7)^2 \times (1.4)^{1.5}$. | 8. $(42)^{1.3} \times (1.7)^3$. | 9. $57 \div 16$. |
| 10. $690 \div 48$. | 11. $57 \div 490$. | 12. $96 \div 510$. |
| 13. $(36)^3 \div 480$. | 14. $(46)^{2.5} \div (5.7)^3$. | 15. $(5.6)^3(3.7)^2 \div (7.3)^2$. |

9-2. The Definition of a Logarithm. By referring to Fig. 9-1, the reader will see that the function $a = B^m$, for any positive B greater than 1, has the properties noted in the preceding section for $a = 2^m$:

- For every value of m there is a corresponding value of a .
- The values of a are positive.
- As m increases, a increases.

(d) As m decreases, a decreases, approaching zero as m is negative and its absolute value becomes very large. Thus a system of computation like that devised in Sec. 9-1 can be devised for any positive number B greater than 1. The number B is called the **base** of the system. For any given B , the fundamental question in constructing the system is the same: Given any a , can an m be found such that $a = B^m$?

From (b) above we see that a must be positive. As in Sec. 9-1, we see that if $a > 0$ and $B > 1$ are given, then an exponent m can be found such that $a = B^m$.

In Fig. 9-1, $a = B^m$ was considered as a function with m as the independent variable and a as the dependent variable. If a is given and the corresponding value of m is desired, we are considering a as the independent variable and m as the dependent variable. We thus have the **inverse function**, $m = f(a)$. For simplicity in theoretical discussions and in later computational work we shall introduce a notation for this inverse function.

If $a = B^m$, we say that a is equal to B raised to the exponent m . Inversely, m is the exponent to which B must be raised to obtain a . If the term **exponent** is replaced by **logarithm**, we may say that m is the *logarithm to which B must be raised to obtain a* . This latter statement is conveniently abbreviated by

$$(1) \qquad m = \log_B a.$$

For the sake of brevity, (1) is often read " m equals the logarithm to the base B of a ." To summarize:

IN SYMBOLS

$$B^m = a$$

$$m = \log_B a$$

IN WORDS

B raised to the exponent m equals a

$\left\{ \begin{array}{l} m \text{ equals the exponent to which } B \text{ must be raised to get } a \\ m \text{ equals the logarithm to the base } B \text{ of } a. \end{array} \right.$

By the definition above, only positive numbers have logarithms. In advanced treatises logarithms are defined for negative numbers, but such an extension of the above definition is beyond the scope of this book. The logarithms of positive numbers are sufficient for all computational work.

Example. The exponential expressions given below are restated to the right in logarithmic notation.

$$2^3 = 8 \quad \text{is equivalent to} \quad \log_2 8 = 3.$$

$$2^2 = 4 \quad \text{is equivalent to} \quad \log_2 4 = 2.$$

$$2^1 = 2 \quad \text{is equivalent to} \quad \log_2 2 = 1.$$

$$2^0 = 1 \quad \text{is equivalent to} \quad \log_2 1 = 0.$$

$$2^{-1} = \frac{1}{2} \quad \text{is equivalent to} \quad \log_2 \frac{1}{2} = -1.$$

$$2^{-2} = \frac{1}{4} \quad \text{is equivalent to} \quad \log_2 \frac{1}{4} = -2.$$

Table 1 of Sec. 9-1 gives values of a and m for the relation $a = 2^m$. Since $m = \log_2 a$, the columns headed with an m could also be labeled $\log_2 a$. The same can be said of Table 2. The information stated above can be read from either of these tables.

Information gained from Table 2 can be stated in two ways as follows.

$$2.5 = 2^{1.32} \quad \text{is equivalent to} \quad \log_2 2.5 = 1.32.$$

$$5.7 = 2^{2.51} \quad \text{is equivalent to} \quad \log_2 5.7 = 2.51.$$

Likewise we see that:

$$1000 = 10^3 \quad \text{is equivalent to} \quad \log_{10} 1000 = 3.$$

$$100 = 10^2 \quad \text{is equivalent to} \quad \log_{10} 100 = 2.$$

$$0.001 = 10^{-3} \quad \text{is equivalent to} \quad \log_{10} 0.001 = -3.$$

$$9 = 3^2 \quad \text{is equivalent to} \quad \log_3 9 = 2.$$

$$125 = 5^3 \quad \text{is equivalent to} \quad \log_5 125 = 3.$$

EXERCISES

Write each of the following exponential expressions in logarithmic notation.

- | | | |
|-------------------------------|------------------------------|--------------------------------|
| 1. $2^2 = 4$. | 2. $2^5 = 32$. | 3. $3^3 = 27$. |
| 4. $10^2 = 100$. | 5. $10^{-2} = 0.01$. | 6. $2^{-2} = \frac{1}{4}$. |
| 7. $8^2 = 64$. | 8. $6^3 = 216$. | 9. $10^{-3} = 0.001$. |
| 10. $1.2^2 = 1.44$. | 11. $11^2 = 121$. | 12. $5^{-2} = \frac{1}{25}$. |
| 13. $15^2 = 225$. | 14. $10^4 = 10,000$. | 15. $10^{-1} = \frac{1}{10}$. |
| 16. $2^7 = 128$. | 17. $2^{-3} = \frac{1}{8}$. | 18. $3^4 = 81$. |
| 19. $3^{-3} = \frac{1}{27}$. | 20. $6^2 = 36$. | |

Write each of the following logarithmic expressions in exponential form.

- | | | |
|--|-------------------------------------|---|
| 21. $\log_2 32 = 5$. | 22. $\log_3 9 = 2$. | 23. $\log_5 125 = 3$. |
| 24. $\log_{10} 100 = 2$. | 25. $\log_{10} 1000 = 3$. | 26. $\log_8 64 = 2$. |
| 27. $\log_2 \frac{1}{4} = -2$. | 28. $\log_3 \frac{1}{9} = -2$. | 29. $\log_7 49 = 2$. |
| 30. $\log_{11} \frac{1}{121} = -2$. | 31. $\log_{15} \frac{1}{15} = -1$. | 32. $\log_5 25 = 2$. |
| 33. $\log_{10} 0.001 = -3$. | 34. $\log_{1.2} 1.2 = 1$. | 35. $\log_{1.2} \frac{1}{1.2} = -1$. |
| 36. $\log_2 \frac{1}{32} = -5$. | 37. $\log_{10} 0.001 = -3$. | 38. $\log_{10} \sqrt{10} = \frac{1}{2}$. |
| 39. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$. | 40. $\log_{10} 0.01 = -2$. | |

Find the values of the following logarithms.

- | | | |
|------------------------------|----------------------------|-----------------------|
| 41. $\log_5 \frac{1}{125}$. | 42. $\log_5 25$. | 43. $\log_9 81$. |
| 44. $\log_9 \sqrt{9}$. | 45. $\log_2 \frac{1}{8}$. | 46. $\log_{10} 100$. |
| 47. $\log_3 27$. | 48. $\log_3 \frac{1}{9}$. | 49. $\log_4 16$. |
| 50. $\log_4 2$. | 51. $\log_{14} 1$. | 52. $\log_7 49$. |
| 53. $\log_{10} 1000$. | 54. $\log_{10} 0.001$. | 55. $\log_8 64$. |
| 56. $\log_3 2$. | 57. $\log_8 4$. | 58. $\log_5 216$. |
| 59. $\log_3 81$. | 60. $\log_5 125$. | |

Find the number x such that:

- | | | |
|--------------------------------|-------------------------|--------------------------------|
| 61. $\log_2 x = 3$. | 62. $\log_3 x = 2$. | 63. $\log_5 x = -1$. |
| 64. $\log_{10} x = -2$. | 65. $\log_3 x = 2$. | 66. $\log_3 x = 4$. |
| 67. $\log_5 x = 3$. | 68. $\log_5 x = -2$. | 69. $\log_{1.5} x = 2$. |
| 70. $\log_3 x = 0$. | 71. $\log_{10} x = 0$. | 72. $\log_6 x = -2$. |
| 73. $\log_2 x = 5$. | 74. $\log_2 x = -3$. | 75. $\log_2 x = \frac{1}{2}$. |
| 76. $\log_3 x = \frac{1}{3}$. | 77. $\log_{10} x = 2$. | 78. $\log_{15} x = 1$. |
| 79. $\log_{20} x = 0$. | | |

9-3. Simple Properties of Logarithms. In all the discussions which follow in this chapter, the reader should constantly remind himself that *a logarithm is an exponent*.

If $a = B^m$, then $m = \log_B a$. Obviously, then

$$(1) \quad B^{\log_B a} = a.$$

Similarly, replacing a by B^m in $\log_B a$, we have

$$(2) \quad \log_B B^m = m.$$

Example.

$$10^{\log_{10} 8} = 8.$$

$$3^{\log_3 9^2} = 9^2 = 81.$$

$$\log_2 2^3 = 3.$$

$$\log_3 9^{-3} = \log_3 (3^2)^{-3} = \log_3 3^{-6} = -6.$$

$$\log_5 \sqrt[3]{5} = \log_5 5^{\frac{1}{3}} = \frac{1}{3}.$$

$$\log_6 \frac{1}{\sqrt{6}} = \log_6 6^{-\frac{1}{2}} = -\frac{1}{2}.$$

EXERCISES

Find the values of the following expressions.

- | | | |
|-----------------------------------|--|---|
| 1. $\log_3 27^{-3}$. | 2. $\log_9 9^{\frac{3}{5}}$. | 3. $\log_4 64^{-1}$. |
| 4. $\log_9 1$. | 5. $\log_3 9^{\frac{3}{2}}$. | 6. $\log_9 \sqrt[3]{9}$. |
| 7. $\log_3 3^{-7}$. | 8. $\log_B B^3$. | 9. $\log_5 (5^3 \times 5^7)$. |
| 10. $\log_e e^x$. | 11. $\log_e e^{\sin \theta}$. | 12. $\log_{10} (10^{-27} \times 10^{15})$. |
| 13. $\log_B (B^2)^{10}$. | 14. $\log_7 \frac{1}{\sqrt[4]{7}}$. | 15. $\log_B B$. |
| 16. $\log_A A^{x^2 + y^2}$. | 17. $\log_e (e^2)^{2x + 1}$. | 18. $\log_e e^{-x}$. |
| 19. $\log_e (e^{-x} \cdot e^x)$. | 20. $\log_{10} \left(\frac{10^2 \times 10^{-3}}{10^{-2}} \right)$. | 21. $2^{\log_2 10}$. |
| 22. $10^{\log_{10} 67}$. | 23. $15^{\log_{15} 46}$. | 24. $8^{\log_3 16}$. |
| 25. $20^{\log_{20} 49}$. | 26. $e^{\log_e 2}$. | 27. $2^{\log_2 x}$. |
| 28. $5^{\log_5 (x^2 + 1)}$. | 29. $B^{\log_B 56}$. | 30. $B^{\log_B (2x + 1)}$. |
| 31. $5^{\log_5 (a^2 + b^2)}$. | 32. $e^{\log_e \sqrt{x^2 + 1}}$. | 33. $A^{\log_A A^2}$. |
| 34. $B^{\log_B 1}$. | 35. $49^{\log_{49} 10}$. | 36. $8^{\log_3 64}$. |

9-4. The Graph of the Function $m = \log_B a$. In Fig. 9-1 the function $a = 2^m$ was plotted, where m is the independent variable and a the dependent variable. The inverse function is $m = \log_2 a$. In this case a is the independent variable and m the dependent variable. Table 1 of Sec. 9-1 gives the information necessary to plot the graph of $m = \log_2 a$, shown in Fig. 9-2.

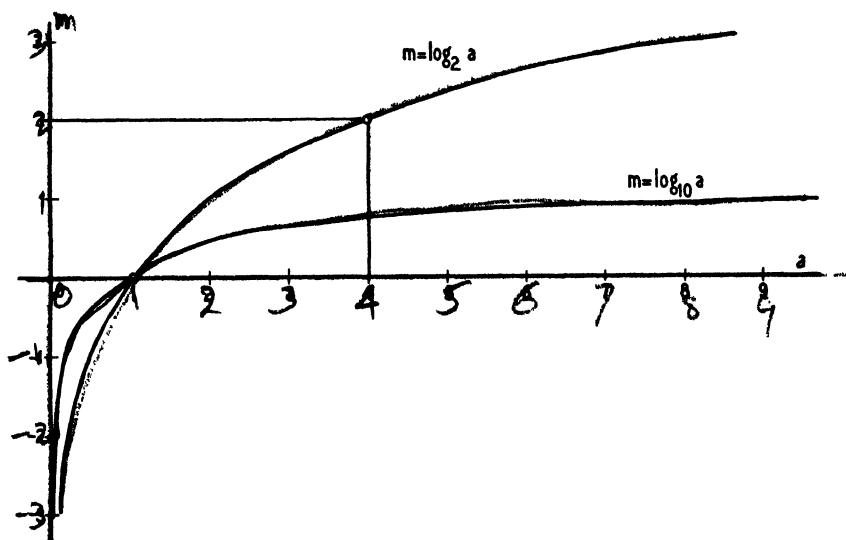


FIG. 9-2.

In like manner, the graph of $m = \log_{10} a$ can be plotted from the following table:

a	0.001	0.01	0.1	1	10	100	1000
$m = \log_{10} a$	-3	-2	-1	0	1	2	3

The graph is shown in Fig. 9-2.

Similar tables and graphs can be obtained for any function of the form $m = \log_B a$, where $B > 1$.

A study of the graphs in Fig. 9-2 yields the following information about the functions $m = \log_B a$, where $B > 1$.

(a) For every positive value of the independent variable a there is a corresponding value of the dependent variable m .

(b) When $a > 1$, $m > 0$; when $a = 1$, $m = 0$; and when $a < 1$, $m < 0$.

(c) When a increases, m increases. When a approaches zero, m becomes negatively infinite.

(d) Since the graph extends only to the right of the y -axis, there are no logarithms of negative numbers by this definition.

EXERCISES

Plot on the same coordinate system the graphs of the following functions; x is the independent variable, and y the dependent variable.

1. $y = 3^x$ and $y = \log_3 x$.

2. $y = 10^x$ and $y = \log_{10} x$.

3. $y = 1.5^x$ and $y = \log_{1.5} x$.

4. $y = 5^x$ and $y = \log_5 x$.

5. $y = 4^x$ and $y = \log_4 x$.

6. Use the graph of $y = \log_3 x$ to find approximate values of $\log_3 4$, $\log_3 2$, $\log_3 \frac{1}{2}$, and $\log_3 6$.

7. Use the graph of $y = \log_4 x$ to find approximate values of $\log_4 5$, $\log_4 1.5$, $\log_4 6$, and $\log_4 3$.

8. What can be inferred from Fig. 9-2 about the relation between the values of $\log_2 x$ and $\log_{10} x$?

9-5. Properties of Logarithms. The equations

$$(1) \quad \log_B a = m, \quad \log_B c = n$$

are equivalent to the equations

$$(2) \quad a = B^m, \quad c = B^n.$$

Since

$$ac = B^m B^n = B^{m+n},$$

it follows that

$$(3) \quad \log_B ac = \log_B B^{m+n} = m + n = \log_B a + \log_B c.$$

This proves:

Property 1. The logarithm of a product is equal to the sum of the logarithms of the factors, all logarithms being taken to the same base. Symbolically,

$$\log_B (ac) = \log_B a + \log_B c.$$

Since

$$\frac{a}{c} = \frac{B^m}{B^n} = B^{m-n},$$

it follows that

$$\log \left(\frac{a}{c} \right) = \log_B B^{m-n} = m - n = \log_B a - \log_B c,$$

giving:

Property 2. The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator, all logarithms being taken to the same base. Symbolically,

$$\log_B \left(\frac{a}{c} \right) = \log_B a - \log_B c.$$

Since

$$a^k = (B^m)^k = B^{mk},$$

it follows that

$$\log_B a^k = \log_B B^{mk} = mk = k \log_B a,$$

giving:

Property 3. The logarithm of a power of a number is equal to the exponent times the logarithm of the number, all logarithms being taken to the same base. Symbolically,

$$\log_B a^k = k \log_B a.$$

An important special case of the last property is the following.

$$\log_B \sqrt[q]{a} = \frac{1}{q} \log_B a.$$

There is no simple formula for the logarithm of a sum in terms of the logarithms of the members of the sum. The value of $\log_B (a + b)$ cannot be computed if only $\log_B a$ and $\log_B b$ are known.

Example 1. Express $\log_B 2\pi\sqrt{\frac{L}{g}}$ as the sum of logarithms of first powers of single letters or numbers.

$$\log_B 2\pi\sqrt{\frac{L}{g}} = \log_B 2 + \log_B \pi\sqrt{\frac{L}{g}} \quad (\text{By property 1})$$

$$= \log_B 2 + \log_B \pi + \log_B \sqrt{\frac{L}{g}} \quad (\text{By property 1})$$

$$= \log_B 2 + \log_B \pi + \log_B \sqrt{L} - \log_B \sqrt{g} \quad (\text{By property 2})$$

$$= \log_B 2 + \log_B \pi + \frac{1}{2} \log_B L - \frac{1}{2} \log_B g. \quad (\text{By property 3})$$

Example 2. Express $\frac{1}{3} \log_{10} 2 - \log_{10} 3 + 3 \log_{10} 5$ as a single logarithm.

$$\begin{aligned} \frac{1}{3} \log_{10} 2 - \log_{10} 3 + 3 \log_{10} 5 \\ = \log_{10} 2^{\frac{1}{3}} - \log_{10} 3 + \log_{10} 5^3 \quad (\text{By property 3}) \end{aligned}$$

$$= \log_{10} \frac{2^{\frac{1}{3}}}{3} + \log_{10} 5^3 \quad (\text{By property 2})$$

$$= \log_{10} \frac{2^{\frac{1}{3}} \cdot 5^3}{3}. \quad (\text{By property 1})$$

Example 3. Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, and $\log_{10} 7 = 0.8451$, find the value of $\log_{10} \frac{12}{7}$.

$$\begin{aligned} \log_{10} \frac{12}{7} &= \log_{10} \frac{3 \times 2^2}{7} = \log_{10} 3 + 2 \log_{10} 2 - \log_{10} 7 \\ &= 0.4771 + 2(0.3010) - 0.8451 = 0.4771 + 0.6020 - 0.8451 \\ &= 0.2340. \end{aligned}$$

Example 4. Find the value of $\log_2 8^3$.

$$\log_2 8^3 = 3 \log_2 8 = 3 \log_2 2^3 = 3 \times 3 = 9.$$

Also

$$\log_2 8^3 = \log_2 (2^3)^3 = \log_2 2^9 = 9.$$

Example 5. Find the value of $\log_2 (16 \times 8)$.

$$\begin{aligned}\log_2 (16 \times 8) &= \log_2 16 + \log_2 8 \\ &= \log_2 2^4 + \log_2 2^3 = 4 + 3 = 7.\end{aligned}$$

EXERCISES

Express each of the following logarithms as the sum of logarithms of first powers of single letters or numbers.

- | | | |
|---------------------------------------|--|--|
| 1. $\log_B PQR$. | 2. $\log_B \frac{CD}{E}$. | 3. $\log_B \frac{C}{DE}$. |
| 4. $\log_B C^4$. | 5. $\log_B (C^3 D^5)$. | 6. $\log_{10} \frac{1}{C^n}$. |
| 7. $\log_{10} C^{-n}$. | 8. $\log_{10} \sqrt[q]{C^5}$. | 9. $\log_2 \frac{1}{\sqrt[q]{C}}$. |
| 10. $\log_3 \sqrt[3]{C^2 D^4}$. | 11. $\log_4 \frac{Q^n \cdot \sqrt[m]{R}}{S^5}$. | 12. $\log_B \sqrt{\frac{QR^n}{S^m}}$. |
| 13. $\log_B \sqrt[5]{\frac{CD}{E}}$. | 14. $\log_B (\sqrt[3]{x} \cdot \sqrt[5]{y})$. | 15. $\log_2 \sqrt[3]{\frac{A^2 S^6}{T}}$. |

Express each of the following as a single logarithm

- | | |
|---|--|
| 16. $\log_{10} 2 + \log_{10} 3 - \log_{10} 5$. | 17. $\log_2 5 + \log_2 6 - \log_2 7$. |
| 18. $\log_e x + \log_e y$. | 19. $3 \log_e x + 2 \log_e y$. |
| 20. $5 \log_2 a + 3 \log_2 b - 2 \log_2 C$. | 21. $\log_e x + \log_e y + \log_e z$. |
| 22. $\frac{1}{2} \log_a x + \frac{1}{3} \log_a y - \frac{1}{4} \log_a z$. | 23. $\frac{1}{3} \log_a 3 + \log_a \pi - \log_a x$. |
| 24. $\frac{1}{2} [\log_{10} (s - a) + \log_{10} (s - b) + \log_{10} (s - c) - \log_{10} s]$. | |
| 25. $\frac{1}{2} \log_{10} 3 + \frac{1}{2} \log_{10} 5 - \frac{1}{8} \log_{10} 15$. | |
| 26. $\frac{1}{2} \log_2 5 - \log_2 7 + 2 \log_2 14 - \frac{1}{3} \log_2 35$. | |
| 27. $\frac{1}{2} \log_B x + \frac{1}{3} \log_B y - \frac{1}{4} \log_B z$. | 28. $\log_3 a + \log_3 b - \frac{1}{2} \log_3 C$. |
| 29. $2 \log_B a + \frac{1}{3} \log_B b - 5 \log_B C$. | 30. $\log_B \sin x - \log_B \cos x$. |

Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 5 = 0.6990$, and $\log_{10} 7 = 0.8451$. Compute the values of the following expressions.

- | | | |
|---|--|--|
| 31. $\log_{10} 14$. | 32. $\log_{10} 18$. | 33. $\log_{10} \sqrt{3}$. |
| 34. $\log_{10} \frac{1}{3}$. | 35. $\log_{10} \frac{1}{4}$. | 36. $\log_{10} 125$. |
| 37. $\log_{10} \frac{\sqrt[3]{7}}{\sqrt[3]{3}}$. | 38. $\log_{10} \frac{7^{4.5} \sqrt{2}}{5^3 \cdot \sqrt[3]{3}}$. | 39. $\log_{10} \sqrt{210}$. |
| 40. $\log_{10} (7 \sqrt[3]{15})$. | 41. $\log_{10} \frac{48}{\sqrt{5}}$. | 42. $\log_{10} \frac{35^3}{\sqrt{15}}$. |
| 43. $\log_{10} \sqrt[3]{168}$. | 44. $\log_{10} 126^3$. | 45. $\log_{10} \frac{\sqrt{7}}{120}$. |

Evaluate each of the following expressions.

46. $\log_8 4^{\frac{1}{2}}$.

47. $\log_4 \sqrt{8}$.

48. $\log_{16} \sqrt[3]{\frac{1}{8}}$.

49. $\log_9 81^{3.2}$.

50. $\log_{12} \sqrt[2]{144^3}$.

51. $\log_{81} 27$.

52. $\log_{16} 8^5$.

53. $\log_3 \sqrt{3} 27$.

54. $\log_{2\sqrt{2}} 8^{\frac{3}{2}}$.

55. $\log_{\frac{2}{3}} (\frac{4}{9})^{-3}$.

56. $\log_2 (16 \times 2^2)^3$.

57. $\log_{10} (100)^{\frac{3}{2}} - \log_{3\sqrt{2}} (18)^{\frac{3}{2}}$.

58. $\log_8 (\frac{1}{2})^{-4} - \log_{16} (32)^{-5}$.

59. $\log_5 125^{\frac{3}{2}} - \log_5 \sqrt{25^3}$.

60. $\log_{49} 7 + \log_{64} 8$.

61. Given $i = i_0 e^{-\frac{Rt}{L}}$, show that $\log_e i = \log_e i_0 - \frac{Rt}{L}$.

62. Given $i = i_0 (1 - e^{-\frac{Rt}{L}})$, show that $\log_e (i_0 - i) = \log_e i_0 - \frac{Rt}{L}$.

63. Given $q = q_0 e^{-\frac{t}{CR}}$, show that $\log_e q = \log_e q_0 - \frac{t}{CR}$.

64. Given $\log_e (q_0 - q) = \log_e q_0 - \frac{t}{CR}$, show that $q = q_0 (1 - e^{-\frac{t}{CR}})$.

65. Given $\log_e i = \log_e q_0 - \log_e C - \log_e R - \frac{t}{CR}$, show that $i = \frac{q_0}{CR} e^{-\frac{t}{CR}}$.

9-6. Simple Logarithmic Computations. Since logarithms are exponents, computations by their aid can be performed as suggested by the exponential computations of Sec. 9-1. In that section we used the relation $a = 2^m$, whence $m = \log_2 a$. Thus, if we replace m by $\log_2 a$ in Tables 1 and 2 of that section, these tables become tables of logarithms to the base 2. The following examples indicate the parallel between logarithmic and exponential computations, using these tables.

Example 1. Multiply 64×8 , using Table 1 of Sec. 9-1.

EXPONENTIAL COMPUTATION	LOGARITHMIC COMPUTATION
$64 \times 8 = 2^6 \times 2^3$ (Table 1)	$\log_2 (64 \times 8) = \log_2 64 + \log_2 8$ (Property 1)
$= 2^{6+3}$ (Laws of exponents)	$\log_2 64 = 6$ (Table 1)
$= 2^9$	$\log_2 8 = 3$ (Table 1)
$= 512$ (Table 1)	$\log_2 (64 \times 8) = 9$
	$64 \times 8 = 512$ (Table 1)

Example 2. Multiply $2.5 \times 1.9 \times 5.2$ using Table 2 of Sec. 9-1.

EXPONENTIAL COMPUTATION	
$2.5 \times 1.9 \times 5.2$	
$= 2^{1.32} \times 2^{0.93} \times 2^{2.38}$ (Table 2)	
$= 2^{1.32+0.93+2.38}$ (Laws of exponents)	
$= 2^{4.63}$	
$= 2^{1.31+3.32}$	
$= 2^{1.31} \times 2^{3.32}$	
$= 2.5 \times 10$ (Table 2)	

LOGARITHMIC COMPUTATION

$$\begin{aligned}
 \log_2 (2.5 \times 1.9 \times 5.2) &= \log_2 2.5 + \log_2 1.9 + \log_2 5.2. && \text{(Property 1)} \\
 \log_2 2.5 &= 1.32 && \text{(Table 2)} \\
 \log_2 1.9 &= 0.93 && \text{(Table 2)} \\
 \log_2 5.2 &= 2.38 && \text{(Table 2)} \\
 &= 4.63 \\
 &= 1.31 + 3.32. \\
 \log 2.5 &= 1.31 && \text{(Table 2)} \\
 \log 10 &= 3.32 && \text{(Table 2)} \\
 \log (2.5 \times 10) &= 4.63. && \text{(Property 1)} \\
 2.5 \times 1.9 \times 5.2 &= 2.5 \times 10.
 \end{aligned}$$

We will find in the following sections that logarithmic notation in extended computations simplifies the mechanical details of the work very considerably. The examples given above are too simple, however, to illustrate this fact.

EXERCISES

Perform the following computations by means of logarithms to the base 2 as given in Table 2 of Sec. 9-1. Compare your computations with the exponential computations of these same exercises which you performed previously.

Exercises 1-15 of Sec. 9-1.

9-7. Systems of Logarithms. The logarithms to a given base of all positive numbers constitute a **system of logarithms**. For illustrative purposes up to this point we have used the base 2. In practical applications, however, two other bases are most frequently used.

The **common system of logarithms** employs the base 10. Since 10 is also the base of the ordinary number system, many simplifications in the use of logarithms occur when this base is used. Therefore this base is the most convenient for computational purposes. It is sometimes called the **Briggsian system** after its inventor Henry Briggs (1560-1631).

The **natural system of logarithms** employs as its base the number e discussed in Chapter 8. This number is irrational, and its value to 10 decimal places is 2.71828 18285. This system is the most convenient for theoretical purposes and will be met in the study of the calculus. It is sometimes called the **Napierian system**, after John Napier (1550-1617).

When the base is not expressed, the base 10 is understood. Thus $\log 2 = 0.3010$ means $\log_{10} 2 = 0.3010$. The word logarithm will also mean common logarithm unless otherwise stated. For the natural system we write $\log_e a$ as $\ln a$. Since these conventions are not entirely general, it is well to check the convention in any given text.

As we have seen, a table of logarithms must be at hand if they are to be used for numerical work. Very extended tables have been com-

puted for common logarithms and on a smaller scale for natural logarithms. The methods used in computing these tables depend upon calculus and will not be discussed here.

9-8. Common Logarithms. Characteristic and Mantissa. We saw in Chapter 1 that a given positive number N can be written in scientific notation as

$$(1) \quad N = M \times 10^n,$$

where M is between 1 and 10, and n is an integer which may be positive or negative. Then

$$\log N = \log (M \times 10^n) = \log M + \log 10^n,$$

and thus

$$(2) \quad \log N = n + \log M.$$

The graph of the function $m = \log a$ in Fig. 9-2 shows that $\log 1 = 0$, $\log 10 = 1$, and that the logarithm of a number between 1 and 10 is between 0 and 1. Then, since M is between 1 and 10, $\log M$ is between 0 and 1. Actually, $\log M$ may be an irrational number, but practically we shall consider only the first few places of its decimal equivalent. Thus (2) states that *the logarithm of a positive number N can be written as the sum of an integer n (positive or negative) and a decimal fraction which is positive and between 0 and 1.*

The integral part n of the logarithm is called the **characteristic**. The portion of the logarithm which is the decimal fraction between 0 and 1 is called the **mantissa**.

Example. Given that $\log 2.56 = 0.4082$, we find that:

$$\begin{aligned} \log 25.6 &= \log (10 \times 2.56) = \log 10 + \log 2.56 = 1 + 0.4082. \\ \log 256 &= \log (100 \times 2.56) = \log 100 + \log 2.56 = 2 + 0.4082. \\ \log 2560 &= \log (1000 \times 2.56) = \log 1000 + \log 2.56 = 3 + 0.4082. \\ \log 0.256 &= \log (0.1 \times 2.56) = \log 0.1 + \log 2.56 = -1 + 0.4082. \\ \log 0.0256 &= \log (0.01 \times 2.56) = \log 0.01 + \log 2.56 = -2 + 0.4082. \\ \log 0.00256 &= \log (0.001 \times 2.56) = \log 0.001 + \log 2.56 = -3 + 0.4082. \end{aligned}$$

Summarizing this information in a table, we have:

NUMBER	CHARACTERISTIC	MANTISSA
2560	3	0.4082
256	2	0.4082
25.6	1	0.4082
2.56	0	0.4082
0.256	-1	0.4082
0.0256	-2	0.4082
0.00256	-3	0.4082

9-9. Rules for Finding the Characteristic. The definition of scientific notation supplies the rule for finding the characteristic of such numbers.

If N is given in scientific notation as

$$(1) \quad N = M \times 10^n,$$

then the characteristic of $\log N$ is the integer n .

Since the characteristic of $\log N$ is the integer n of (1), the rule of Sec. 1-12 can be used to find its value for numbers given in positional notation. For the sake of convenience it will be restated here.

Let N be a positive number given in positional notation.

(a) *If $N > 1$, the characteristic of $\log N$ is 1 less than the number of digits to the left of the decimal point of N .*

(b) *If $N < 1$, the characteristic of $\log N$ is a negative number whose absolute value is one more than the number of zeros between the decimal point and the left-most significant digit.*

We see, then, that the characteristic of a logarithm depends only upon the position of the decimal point, and not upon the digits in a given number.

When the characteristic is negative it will be often more convenient to use an equivalent form, writing the characteristic as the difference of two positive numbers, the number to be subtracted being a multiple of 10. Thus we replace -1 by $9 - 10$, -2 by $8 - 10$, -4 by $6 - 10$, etc.

Example.

N	CHARACTERISTIC OF $\log N$
2.56	0
256	2
0.256	-1 or 9 - 10
0.00256	-3 or 7 - 10
35670	4
0.00008	-5 or 5 - 10
2.3×10^4	4
2.3×10^{-3}	-3 or 7 - 10
2.3×10^{-11}	-11 or 9 - 20
2.5×10^{-31}	-31 or 9 - 40

EXERCISES

Determine the characteristics of the logarithms of the following numbers.

- | | | |
|-----------------------|--------------------------|--------------------------|
| 1. 56.8. | 2. 679. | 3. 59.2. |
| 4. 0.438. | 5. 0.095. | 6. 0.00067. |
| 7. 466.72. | 8. 0.000056. | 9. 56,785. |
| 10. 0.423. | 11. 0.0597. | 12. 2.367. |
| 13. 31.56. | 14. π . | 15. 5π . |
| 16. 2.3×10 . | 17. 3.76×10^2 . | 18. 5.82×10^5 . |

19. 6.72×10^{-1} .	20. 8.97×10^{-3} .	21. 9×10^{-3} .
22. 8×10^{-6} .	23. 9.2×10^5 .	24. 8.567×10^3 .
25. 9.23×10^{-3} .	26. 8.76×10^{-4} .	27. 9.47×10^{-18} .
28. 87.63×10^{-32} .	29. 9.76×10^{15} .	30. 2.314×10^{-18} .

Place the decimal point in the sequence of digits 5689 as determined by the following characteristics:

31. 3.	32. 1.	33. 2.
34. 0.	35. $9 - 10$.	36. $8 - 10$.
37. 5.	38. $6 - 10$.	39. $7 - 10$.
40. 4.	41. 10.	42. 14.
43. 18.	44. $2 - 20$.	

9-10. Properties of the Mantissa. We have seen that the characteristic depends only on the position of the decimal point. Now we shall show that the mantissa is independent of the position of the decimal point.

If two positive numbers N_1 and N_2 have the same significant digits arranged in the same order, then regardless of the position of the decimal point in each number, we say that N_1 and N_2 have the *same sequence of digits*. For example, the numbers 85.6, 856, 0.00856, 8560, and 85,600,000 have the same sequence of digits.

The two numbers N_1 and N_2 can be written in scientific notation as:

$$(1) \quad N_1 = M_1 \times 10^{n_1},$$

$$(2) \quad N_2 = M_2 \times 10^{n_2},$$

where M_1 and M_2 are between 1 and 10, and n_1 and n_2 are integers. Now, if N_1 and N_2 have the same sequence of integers, then $M_1 = M_2$. The following example illustrates this fact.

Example 1.

N (in positional notation)	$N = M \times 10^n$ (in scientific notation)	M	n
85.6	8.56×10	8.56	1
856	8.56×10^2	8.56	2
0.00856	8.56×10^{-3}	8.56	-3
8560	8.56×10^3	8.56	3
85,600,000	8.56×10^7	8.56	7

All the numbers given above have the same sequence of digits, and hence all the numbers M are the same.

From (1) and (2),

$$\log N_1 = n_1 + \log M_1,$$

$$\log N_2 = n_2 + \log M_2,$$

and thus the characteristics of $\log N_1$ and $\log N_2$ are n_1 and n_2 respectively. The mantissa of $\log N_1$ is $\log M_1$, and the mantissa of $\log N_2$ is $\log M_2$. If N_1 and N_2 have the same sequence of digits, $M_1 = M_2$, and therefore $\log M_1 = \log M_2$.

Since the mantissa is positive by its definition, we now have these two important properties of the mantissa:

Property 1. The mantissa is always a positive number.

Property 2. The common logarithms of numbers which have the same sequence of digits have the same mantissas.

The example in Sec. 9-8 illustrates Property 2. This property of the mantissa, which holds only for common logarithms, makes common logarithms superior for computational purposes to natural logarithms or to logarithms with any other base.

Example 2. Given that $\log 8.56 = 0.9325$, we may obtain the information given below:

N	CHARACTERISTIC OF $\log N$	MANTISSA OF $\log N$	$\log N$
85.6	1	0.9325	1.9325
856	2	0.9325	2.9325
0.00856	7 - 10	0.9325	7.9325 - 10
8560	3	0.9325	3.9325
85,600,000	7	0.9325	7.9325
8.56×10^{-18}	2 - 20	0.9325	2.9325 - 20

$\log 0.00856 = 7.9325 - 10$ has actually the value -2.0675 , but written in this way the logarithm does not exhibit its characteristic and mantissa. Hence we always write $\log 0.00856$ as $7.9325 - 10$, so that the characteristic is clearly 7 - 10, and the mantissa is 0.9325.

Example 3. If the logarithm of a number is -2.3765 , what is the characteristic and mantissa?

The entire quantity -2.3765 is negative, and the mantissa is always positive. If we rewrite -2.3765 as $7.6235 - 10$, then it can be seen that the characteristic is 7 - 10, and the mantissa is 0.6235.

EXERCISES

State the characteristic and mantissa of each of the following logarithms.

- | | | |
|-------------------|-------------------|-------------------|
| 1. 2.9325. | 2. 3.60912. | 3. 0.315796. |
| 4. 0.00323. | 5. 5.6194. | 6. 8.7325 - 10. |
| 7. 1.5692 - 10. | 8. 4.6357 - 10. | 9. 5.6392. |
| 10. -0.4631. | 11. -3.2294. | 12. -4.6732 + 10. |
| 13. -5.6093 + 10. | 14. -9.2317. | 15. -4.6735 + 10. |
| 16. -5.6932 + 20. | 17. 3.2159. | 18. -1.23452. |
| 19. -2.3157. | 20. -8.6732 + 20. | |

Given $\log 892 = 2.9504$, $\log 5.76 = 0.7604$, $\log 13.4 = 1.1271$, and $\log 45.9 = 1.6618$, find the logarithms of the following numbers:

- | | | |
|-----------------------------|-----------------------------|--------------------------|
| 21. 8.92. | 22. 576. | 23. 8920. |
| 24. 0.000892. | 25. 134. | 26. 4590. |
| 27. 0.459. | 28. 0.00459. | 29. 0.892. |
| 30. 0.134. | 31. 57,600. | 32. 0.00134. |
| 33. 0.00576. | 34. 4590. | 35. 8.92×10^8 . |
| 36. 1.34×10^{-5} . | 37. 5760. | 38. 4,590,000. |
| 39. 5.76×10^5 . | 40. 1.34×10^{10} . | |

9-11. Using a Table of Logarithms. The mantissas of the logarithms of most numbers are unending decimal fractions. Methods are developed in advanced treatises for computing the values of these mantissas to as many decimal places as desired. Such computed values are usually given in a table of mantissas, called also a **table of logarithms**. These are called four-place, five-place, etc., according to the number of decimal places in the tabulated mantissas. While the mantissas are decimal fractions, the decimal point is usually omitted in the table for convenience in printing.

A four-place table is sufficient for most engineering purposes, and such a table is attached to this book (Table 2). We shall illustrate the use of this table, and the student can adapt these methods to the use of other tables. In the four-place table, the mantissas are given for the logarithms of all numbers which are written with a sequence of three significant digits. The table is arranged so that the first two significant digits of the given number are located in the left-most column of the table, and the third digit is located at the top. For example, if the given number has the sequence of digits 738, then 73 is located in the column to the left and 8 is located at the top of the table. Going down from this 8 to the line corresponding to 73, the mantissa 8681 is found.

To find the logarithm of a number.

(a) *Determine the characteristic by the rules of Sec. 9-9.*

(b) *Find the mantissa corresponding to the given sequence of digits by using a table of logarithms.*

Example 1. To find $\log 73.8$:

(a) By the rule of Sec. 9-9 the characteristic is 1.

(b) As we saw above the mantissa is 8681.

Hence

$$\log 73.8 = 1.8681.$$

In the same way it is found that

$$\log 7.38 = 0.8681,$$

$$\log 73800 = 4.8681,$$

$$\log 0.738 = 9.8681 - 10,$$

$$\log 0.000738 = 6.8681 - 10.$$

The process of finding the number corresponding to a given logarithm is the inverse of the process given above. The number obtained is sometimes called the **antilogarithm** (abbreviated **antilog**) of the given logarithm.

To find the number corresponding to a given logarithm:

(a) *Determine the sequence of digits corresponding to the given mantissa by using the table.*

(b) *Place the decimal point in the position given by the characteristic and the rules of Sec. 9-9.*

Example 2. Find x such that $\log x = 1.5453$.

From the table it is found that 5453 is the common mantissa of all numbers with the sequence of digits 351. Because a positive characteristic of a logarithm is 1 less than the number of places to the left of the decimal point of the corresponding number, x has two places to the left of the decimal, for the characteristic of $\log x$ is 1. Hence

$$x = 35.1.$$

In the same way it is found that if

$$\log x = 2.5453, x = 351;$$

$$\log x = 4.5453, x = 35100;$$

$$\log x = 8.5453 - 10, x = 0.0351;$$

$$\log x = 9.5453 - 10, x = 0.351;$$

$$\log x = 7.5453 - 10, x = 0.00351.$$

EXERCISES

Using a four-place table, find the logarithm of each of the following numbers.

1. 3.24.

2. 5.76.

3. 78.2.

4. 95.3.

5. 495.

6. 6780.

7. 0.00672.

8. 95,900.

9. 0.864.

10. 0.000675.

11. 69.5.

12. 0.0892.

13. 4.23×10^5 .

14. 5.96×10^{-3} .

15. 6.43×10^8 .

Using a four-place table, find the number of three significant digits corresponding to each of the following logarithms:

16. 2.5198.

17. 1.5922.

18. $9.2810 - 10$.

19. 4.6532.

20. 2.9053.

21. $8.8261 - 10$.

22. $7.8943 - 10$.

23. $6.9430 - 10$.

24. 5.5315.

25. 0.6021.

26. 3.4378.

27. 0.5378.

28. $9.6474 - 10$.

29. 2.8976.

30. 1.9350.

9-12. Interpolation. If a number has four significant figures, the mantissa of its logarithm cannot be found directly from a four-place table. The mantissa of the logarithm of a four-digit number lies between the

mantissas of two consecutive three digit numbers which are given in the table. Thus, to find the mantissa of the logarithm of a four-digit number, the process of **interpolation** can be employed. This process was used in Chapter 4 in connection with the trigonometric tables.

The following examples illustrate the process of interpolation:

Example 1. Find $\log 35.64$.

The characteristic of this logarithm is 1. The logarithm of the number with the digits 356 has the mantissa 5514; the logarithm of the number with the digits 357 has the mantissa 5527. We can then set up a scheme similar to that used in Chapter 4:

$$0.10 \left[\begin{array}{c} 0.04 \left[\begin{array}{c} \log 35.60 = 1.5514 \\ \log 35.64 = \dots \\ \log 35.70 = 1.5527 \end{array} \right] x \\ \log 35.70 = 1.5527 \end{array} \right] 0.0013$$

Since the logarithm increases by 0.0013 as the number increases by 0.10, then as the number increases by 0.04 the logarithm will increase by

$$x = \frac{0.04}{0.10} \times 0.0013 = 0.00052.$$

Since this last statement is only approximately true (see below), and since the four-place mantissas given in the table are rounded off from longer decimals, we round off the result of this interpolation to four places and call $x = 0.0005$. Adding this amount to $\log 35.60$ we obtain

$$\log 35.64 = 1.5519.$$

A shortened scheme can be set up as follows.

NUMBER	MANTISSA
10	4
$\left[\begin{array}{c} 3560 \\ 3564 \\ 3570 \end{array} \right]$	$\left[\begin{array}{c} 5514 \\ \dots \\ 5527 \end{array} \right] x$
	13

$$x = \frac{4}{10} \times 13 = 5.2 \text{ or } 5.$$

Then

$$\log 35.64 = 1.5519.$$

Example 2. Find $\log 0.01947$.

Setting up the simplified scheme,

NUMBER	MANTISSA
10	7
$\left[\begin{array}{c} 1940 \\ 1947 \\ 1950 \end{array} \right]$	$\left[\begin{array}{c} 2878 \\ \dots \\ 2900 \end{array} \right] x$
	22

$$x = \frac{7}{10} \times 22 = 15.4 \text{ or } 15.$$

Adding 15 to 2878 we obtain 2893. Since the characteristic of $\log 0.01947$ is -2 or $8 - 10$, then

$$\log 0.01947 = 8.2893 - 10.$$

Example 3. Find x such that $\log x = 4.8079$.

The mantissa of this logarithm is 8079. From the table we find that 8079 lies between 8075 and 8082, which correspond to the numbers 642 and 643 respectively. Setting up the scheme, we have:

NUMBER	MANTISSA
$10 \left[x \left[\begin{array}{c} 6420 \\ \dots \\ 6430 \end{array} \right] \right]$	$\left[\begin{array}{c} 8075 \\ 8079 \\ 8082 \end{array} \right] \begin{array}{c} 4 \\ 7 \end{array}$

Proceeding as before,

$$x = \frac{4}{7} \times 10 = 5\frac{5}{7} \text{ or } 6,$$

approximately. Adding 6420 and 6 we obtain 6426, the sequence of digits in x .

Since the characteristic of $\log x$ is 4, x has five digits in front of the decimal point consequently

$$x = 64,260.$$

Example 4. Find x such that $\log x = 8.4864 - 10$.

Setting up the scheme, we have:

NUMBER	MANTISSA
$10 \left[x \left[\begin{array}{c} 3060 \\ \dots \\ 3070 \end{array} \right] \right]$	$\left[\begin{array}{c} 4857 \\ 4864 \\ 4871 \end{array} \right] \begin{array}{c} 7 \\ 14 \end{array}$

$$x = \frac{7}{14} \times 10 = 5.$$

The sequence of digits is 3065, and since the characteristic is $8 - 10$,

$$x = 0.03065.$$

Example 5. Find $\log 557.5$.

NUMBER	MANTISSA
$10 \left[5 \left[\begin{array}{c} 5570 \\ 5575 \\ 5580 \end{array} \right] x \right]$	$\left[\begin{array}{c} 7459 \\ \dots \\ 7466 \end{array} \right] 7$

$$x = \frac{5}{10} \times 7 = 3.5.$$

Since the result of the interpolation is 3.5, we round off to 4, in accordance with our convention of rounding off to the even digit. *This is done before the correction is added to the smaller mantissa.* Hence the result is

$$\log 557.5 = 2.7463.$$

By the process of interpolation, a four-place mantissa can be found for a number of four significant figures when a four-place table is used. Similarly, five-place mantissas can be found by interpolation from five-place tables when numbers of five significant figures are given, etc. That accurate results by interpolation are limited to one digit more than the number of digits for which the table is tabulated is a consequence of two facts:

(a) The mantissas given in the tables are approximations, rounded off from longer decimals.

(b) The process of interpolation is based on the assumption that for small differences in numbers, the changes in the mantissas are proportional to the changes in the numbers. This assumption is not strictly true, but the results based on it are sufficiently accurate for practical purposes if we consider numbers of four significant digits with four-place tables, numbers of five significant figures with five-place tables, etc. For this assumption to be precisely true, the curve of $y = \log x$ would have to be a straight line. Figure 9-2, of course, shows that this curve is not a straight line. However, small sections of it are very nearly straight, and this assumption merely states that they are close enough to being straight that we may consider them so for practical purposes.

After some practice, the student will be able to perform the process of interpolation mentally. Some tables contain aids to interpolation in the form of tables of *proportional parts*. When the process of interpolation is well understood, the use of such aids is immediately clear. Therefore they will not be discussed here.

EXERCISES

Find the value of the following.

- | | | |
|----------------------------------|-------------------------------------|--------------------------------------|
| 1. $\log 7.564$. | 2. $\log 85.96$. | 3. $\log 107.9$. |
| 4. $\log 423,500$. | 5. $\log 1,487,000$. | 6. $\log 86.43$. |
| 7. $\log 0.2372$. | 8. $\log 0.8142$. | 9. $\log 0.09355$. |
| 10. $\log 0.0005415$. | 11. $\log 0.1056$. | 12. $\log 0.00007365$. |
| 13. $\log 149.6$. | 14. $\log 2.358$. | 15. $\log 389.4$. |
| 16. $\log 5012$. | 17. $\log 64,010$. | 18. $\log 700,500$. |
| 19. $\log 3.142$. | 20. $\log 2.718$. | 21. $\log (3.572 \times 10^{12})$. |
| 22. $\log 0.1956$. | 23. $\log 0.007948$. | 24. $\log (7.314 \times 10^{-20})$. |
| 25. $\log (5.321 \times 10^2)$. | 26. $\log (5.295 \times 10^2)$. | 27. $\log (6.829 \times 10^{-5})$. |
| 28. $\log (6.493 \times 10^5)$. | 29. $\log (8.476 \times 10^{-8})$. | 30. $\log (8.561 \times 10^{-2})$. |
| 31. $\log (9.111 \times 10^3)$. | 32. $\log (9.401 \times 10^{-3})$. | 33. $\log (2.147 \times 10)$. |
| 34. $\log (1.143 \times 10^2)$. | | |

Find the numbers whose logarithms are given by:

- | | | |
|------------------|------------------|------------------|
| 35. 0.8165. | 36. 3.4900. | 37. 5.8551. |
| 38. 1.0186. | 39. 0.4774. | 40. 21.7785. |
| 41. 8.6086 - 10. | 42. 9.9158 - 10. | 43. 6.3040 - 10. |
| 44. 9.5296 - 10. | 45. 6.6994 - 10. | 46. 0.8457 - 26. |
| 47. 0.5983. | 48. 3.4138. | 49. 1.0150. |
| 50. 5.2930. | 51. 4.3002. | 52. 0.0770. |
| 53. 8.8424 - 10. | 54. 9.4105 - 10. | 55. 6.6004 - 10. |
| 56. 8.9830 - 10. | 57. 9.7997 - 10. | 58. 3.1450 - 10. |

9-13. Computations by Means of Logarithms. The examples of this section are designed to illustrate a number of important details of procedure in logarithmic computations. It is very important that work of this kind be carried out neatly and systematically. Several convenient ways of arranging the work are indicated in the examples. Before using the tables, the student should first analyze the problem to see what operations are involved. Then he should prepare a skeleton outline of the work to be performed, in which each operation is indicated and in which a place is provided for each number and logarithm entering into the computation. Finally, this outline can be filled in from the tables and the work completed. This procedure is the most conducive to speed and accuracy.

Example 1. Compute $89.46 \times 0.04137 \times 0.3124$ by logarithms.

If N denotes the result of this computation, then by Property 1 of Sec. 9-5,

$$\log N = \log 89.46 + \log 0.04137 + \log 0.3124.$$

Before using the tables, we prepare the following outline.

$$\begin{array}{rcl} \log 89.46 & = & \\ \log 0.04137 & = & \\ \log 0.3124 & = & \\ \hline \log N & = & \\ N & = & \end{array}$$

The characteristics can be filled in easily, and finally the table is used to complete the work. The completed computation is given below.

$$\begin{array}{rcl} \log 89.46 & = & 1.9516 \\ \log 0.04137 & = & 8.6167 - 10 \\ \log 0.3124 & = & 9.4947 - 10 \\ \hline \log N & = & 20.0630 - 20 = 0.0630 \\ N & = & 1.156. \end{array}$$

An abbreviated form is very useful. In it the numbers are always to the left of the vertical line, their corresponding logarithms on the same horizontal line to the right of the vertical line. This form for this example is given below:

NUMBERS	LOGARITHMS
89.46	1.9516
0.04137	8.6167 - 10
0.3124	9.4947 - 10
N	$20.0630 - 20 = 0.0630$
	$N = 1.156.$

Example 2. Compute $\frac{0.08942 \times 3.592}{105.2 \times 0.5127}$ by logarithms.

If N denotes the result of this computation, then by Property 2 of Sec. 9-5,

$$\log N = \log \text{numerator} - \log \text{denominator},$$

where

$$\log \text{numerator} = \log 0.08942 + \log 3.592,$$

$$\log \text{denominator} = \log 105.2 + \log 0.5127.$$

The computation is carried out below:

$$\begin{array}{rcl} \log 0.08942 & = & 8.9514 - 10 \\ \log 3.592 & = & 0.5553 \\ \hline \log \text{numerator} & = & 9.5067 - 10 = 9.5067 - 10 \\ \log 105.2 & = & 2.0220 \\ \log 0.5127 & = & 9.7099 - 10 \\ \hline \log \text{denominator} & = & 11.7319 - 10 = 1.7319 \\ \log N & & = 7.7748 - 10 \\ N & = & 0.005954. \end{array}$$

The abbreviated form for this computation can be set up as follows.

NUMBERS	LOGARITHMS	
0.08942	8.9514 - 10	
3.592	0.5553	
	<hr/>	
	9.5067 - 10	9.5067 - 10
105.2	2.0220	
0.5127	9.7099	
	<hr/>	
	11.7319	1.7319
	<hr/>	
N		7.7748 - 10
	<hr/>	
	$N = 0.005954.$	

Example 3. Compute $\frac{47.82}{815.1}$ by logarithms.

If N denotes the result of this computation,

$$\log N = \log 47.82 - \log 815.1.$$

Then from

NUMBERS	LOGARITHMS
47.82	1.6796
815.1	2.9112

we see that the logarithm 2.9112 which is to be subtracted is larger than 1.6796.

If we replace 1.6796 by $11.6796 - 10$, the computation can be carried out as follows.

NUMBERS	LOGARITHMS
47.82	11.6796 - 10
815.1	2.9112
	<hr/>
N	8.7684 - 10

$$N = 0.05867.$$

Example 4. Compute $\sqrt[3]{0.1084} \times (0.4231)^5$ by logarithms.
If N denotes the result of this computation,

$$\log N = \log \sqrt[3]{0.1084} + \log (0.4231)^5,$$

and by Property 3 of Sec. 9-5

$$\log N = \frac{1}{3} \log 0.1084 + 5 \log 0.4231.$$

Now $\log 0.1084 = 9.0350 - 10$. To find $\frac{1}{3} \log 0.1084$, it is more convenient to write $\log 0.1084 = 29.0350 - 30$, and then $\frac{1}{3} \log 0.1084 = \frac{1}{3} (29.0350 - 30) = 9.6783 - 10$.

The complete computation is carried out below.

NUMBERS	LOGARITHMS
0.1084	29.0350 - 30
$\sqrt[3]{0.1084}$	9.6783 - 10
0.4231	9.6264 - 10
$(0.4231)^5$	48.1320 - 50
	<hr/>
N	17.8103 - 20
N	7.8103 - 10
	$N = 0.006461.$

Example 5. Compute $\frac{(0.04163)^3 \times \sqrt[5]{0.001574}}{(0.3157)^2 \times 5.123}$ by logarithms.

If N denotes the result of this computation, then

$$N = \log \text{numerator} - \log \text{denominator},$$

$$N = [3 \log 0.04163 + \frac{1}{5} \log 0.001574] - [2 \log 0.3157 + \log 5.123].$$

The computation is tabulated below.

NUMBERS	LOGARITHMS
0.04163	8.6194 - 10
$(0.04163)^3$	25.8582 - 30
0.001574	7.1970 - 10
$\sqrt[5]{0.001574}$	1.4394 - 2
	<hr/>
<i>Numerator</i>	15.2976 - 20
	15.2976 - 20

NUMBERS	LOGARITHMS	
0.3157	9.4993 - 10	
(0.3157) ²	18.9986 - 20	8.9986 - 10
5.123		0.7095
<i>Denominator</i>	9.7081 - 10	9.7081 - 10
<i>N</i>		5.5895 - 10
$N = 0.00003886.$		

EXERCISES

Compute each of the following expressions by a four-place table of logarithms. Each number is assumed to be accurate to four significant figures, and hence each result should have four significant figures.

1. $3.756 \times 4.821 \times 378.0.$
2. $0.0004376 \times 86.41.$
3. $0.4241 \times 96.47 \times 0.03765.$
4. $0.0007156 \times 47.56 \times 0.6510.$
5. $476.3 \times 0.005681 \times 43.79.$
6. $1.262 \times 80.72 \times 0.06254.$
7. $65.37 \times 41.52 \times 0.03102.$
8. $2.736 \times 3.829 \times 4.562.$
9. $4.832 \times 8.976 \times 504.2.$
10. $6.004 \times 0.00005682 \times 814.2.$
11. $\frac{75.32}{9.426}.$
12. $\frac{76.30}{5.190}.$
13. $\frac{1}{356.2}.$
14. $\frac{1}{9.867}.$
15. $\frac{0.02187}{3.620}.$
16. $\frac{56.80}{0.009672}.$
17. $\frac{0.0008672}{0.007650}.$
18. $\frac{89.62}{47.92}.$
19. $\frac{12.56 \times 0.004127}{46.30}.$
20. $\frac{13.42 \times 0.04169}{514.2}.$
21. $\frac{4.692 \times 314.5}{5194}.$
22. $\frac{8.964 \times 77.70}{2810}.$
23. $\frac{48.67}{31.00 \times 42.12}.$
24. $\frac{3120}{46.20 \times 0.004137}.$
25. $\frac{4861}{38.42 \times 91.50}.$
26. $\frac{315.2}{485.2 \times 0.07600}.$
27. $\frac{3.765 \times 2.834}{4.628 \times 7.105}.$
28. $\frac{28.97 \times 0.8236}{0.4123 \times 5.678}.$
29. $\frac{3.821 \times 0.007650}{0.05162 \times 6.423}.$
30. $\frac{46.32 \times 9460}{4135 \times 67.32}.$
31. $\frac{31.67 \times 4263}{4081 \times 623.1}.$
32. $\frac{4868 \times 6762}{81.81 \times 463.4}.$
33. $(3.462)^2.$
34. $(81.46)^5.$
35. $(31.42)^3 \times (0.3146)^2.$
36. $(4.672)^5 \times (51.32)^2.$

37. $(0.3762)^2 \times (5.612)^3 \times (0.01234)^2$. 38. $(0.4127)^3 \times (3.127)^2 \times (0.3456)^6$.
 39. $(576.2)^{\frac{1}{2}}$. 40. $\sqrt[3]{76.24}$.
 41. $\sqrt{673.5}$. 42. $\sqrt[3]{0.01325}$.
 43. $\sqrt{0.008962}$. 44. $\sqrt{3.769}$.
 45. $\sqrt[3]{0.8790}$. 46. $\sqrt[3]{7853}$.
 47. $(7.953)^{\frac{3}{5}}$. 48. $\sqrt[11]{0.08957}$.
 49. $\sqrt{45.62} \times \sqrt[3]{63.42}$. 50. $(0.4509)^{\frac{1}{2}} \times (6.723)^{\frac{1}{3}}$.
 51. $(86.61)^{\frac{3}{4}}$. 52. $(0.9476)^{\frac{8}{5}}$.
 53. $(7.953)^{\frac{3}{5}}$. 54. $(0.004321)^{\frac{1}{2}}$.
 55. $(3.415)^{0.2}$. 56. $(45.67)^{0.3}$.
 57. $(69.37)^{0.7}$. 58. $(2.314)^{0.9}$.
 59. $(4.639)^{1.2}$. 60. $(3147)^{2.1}$.
 61. $(3.460)^{\frac{1}{3}} \times (672.3)^{\frac{1}{4}}$. 62. $\left(\frac{35.72}{40.10}\right)^4$.
 63. $\left(\frac{31.21}{40.70}\right)^3$. 64. $\frac{\sqrt[3]{675.2}}{\sqrt{519.3}}$.
 65. $\frac{\sqrt[5]{672.2} \times 35.10}{\sqrt[3]{416.2}}$. 66. $\sqrt{\frac{86.42}{\sqrt[3]{56.82}}}$.
 67. $\sqrt[4]{\frac{0.6732}{3.146}}$. 68. $\sqrt[5]{\frac{586.2}{(31.25)^3}}$.
 69. $\sqrt[3]{\frac{(57.20)^2}{(31.42)^3}}$. 70. $\sqrt[5]{\frac{486.2 \times 0.1569}{(21.20)^2}}$.
 71. $\frac{85.21 \times \sqrt[3]{4651}}{\sqrt{46.82} \times 6.230}$. 72. $\frac{91.04 \times \sqrt{31.20} \times \sqrt[3]{4162}}{418.2 \times 45.92}$.
 73. $\frac{48.19 \times \sqrt{56.02}}{431.6 \times \sqrt[3]{46.25} \times \sqrt{16.34}}$. 74. $\frac{38.49 \times \sqrt{516.2}}{\sqrt[3]{41.35} \times 915.6}$.
 75. $\sqrt[3]{\frac{48.19 \times 312.5}{59.26 \times \sqrt{91.32}}}$. 76. $\sqrt{\frac{35.20 \times 0.4692}{615.3}}$.
 77. $\sqrt{\frac{0.008150 \times 0.08532}{0.01234 \times \sqrt[3]{0.09156}}}$. 78. $\sqrt[3]{\frac{51.31 \times 89.27}{418.2 \times 56.81}}$.
 79. $\sqrt{\frac{0.08152 \times 1.953}{95.27}}$. 80. $\sqrt{\frac{0.8531}{9.327}} \sqrt[3]{\frac{518.2}{61.52}}$.

9-14. Accuracy in Logarithmic Computations. All the numbers in the computations of the previous section were accurate to four significant figures. Therefore, by the rules of Sec. 1-15, the results of these multiplications and divisions will be accurate to four digits. We have seen that the ordinary logarithmic computation with four-place tables gives precisely this accuracy. The following examples illustrate the use of

logarithms in computations in which the question of the accuracy of the result arises.

Example 1. Compute $\frac{3.464 \times 9.8765}{46.574}$.

Assuming that the numbers given are accurate in accordance with the agreements of Sec. 1-13, we see that there is one four-digit number and that there are two five-digit numbers. We can expect a result accurate to four digits. In order to use four-place logarithms, it will be convenient, without affecting the accuracy materially, to round off all the numbers to four digits and then to compute as usual.

$$N = \frac{3.464 \times 9.876}{46.57}$$

NUMBERS	MANTISSAS
3.464	0.5396
9.876	0.9946
<i>Numerator</i>	11.5342 - 10
46.57	1.6681
<i>N</i>	9.8661 - 10

$$N = 0.7347.$$

We can assure ourselves that no major error has occurred during the computation by computing $\frac{3.46 \times 9.88}{46.6}$ by the slide rule. The result is 0.734.

Example 2. Compute $\frac{0.462 \times 9.567}{8.4163}$.

Since there is a three-digit number in this computation, we can expect a result accurate to at most three digits. It is convenient, then, if logarithms are to be used, to round off all the numbers to three digits before computing:

$$N = \frac{0.462 \times 9.57}{8.42}$$

EXERCISES

Compute each of the following quantities by logarithms, assuming that each number given has the number of significant digits specified by the usual agreements.

1. $67.24 \times 81.576 \times 95.6786$.

2. $3.7654 \times 5.672 \times 0.31563$.

3. $4.63 \times 0.05162 \times 2.31$.

4. $3.25 \times 4.62 \times 6.1576$.

5. 4.76×0.0008143 .

6. $673.2 \times 0.089164 \times 0.6843$.

7. $\frac{69.25 \times 0.084673}{0.12345}$.

8. $\frac{5.632 \times 6.4786}{30.042}$.

9. $\sqrt{\frac{0.08951 \times 6.3251}{48.632}}$.

10. $\sqrt{\frac{659.34}{0.06430 \times 9.5673}}$.

$$11. \sqrt[3]{\frac{67.31 \times 46.843}{946.878}}.$$

$$13. 46.27 \times 19.5 \times 6.4218.$$

$$15. 47.2 \times 0.008926 \times 0.096432.$$

$$17. \sqrt[3]{6.42 \times 9.862}.$$

$$19. \sqrt[5]{46.8 \times 69.5}.$$

$$21. \sqrt{\frac{61.4}{14.64 \times 6.32}}.$$

$$12. 36.2 \times 4.356 \times 19.2.$$

$$14. 35.1 \times 0.00689 \times 0.068120.$$

$$16. \sqrt{56.8 \times 97.31 \times 0.0810}.$$

$$18. \sqrt[4]{5.673 \times 0.00915}.$$

$$20. \sqrt[3]{\frac{60.5 \times 81.34}{1080}}.$$

$$22. \sqrt[3]{\frac{47.8}{19.6 \times 0.0846}}.$$

23. The wavelength of a resonant circuit is given by

$$\lambda = 1884\sqrt{LC} \text{ meters,}$$

where L is the inductance in microhenries and C is the capacitance in microfarads. In a certain circuit the inductance was measured and found to be 5.0 microhenries. The capacitance was 0.54 microfarad. Find the wavelength of the circuit at resonance.

24. The approximate inductance of a single-layer solenoid is given by the formula $L = \frac{0.03948a^2n^2}{b} K$ where L is the inductance in microhenries, a the radius in centimeters of the coil, n the number of turns, b the length in centimeters of the coil, and K is a function of $\frac{2a}{b}$. The length of a certain coil is 5 cm. and its diameter 1 cm. Let $K = 0.5255$. Find the approximate inductance of the coil if it is wound with 200 turns.

25. The time required for a bomb released from an airplane to reach the ground is given by the formula $T = \sqrt{\frac{2S}{g}}$ where T is the time in seconds, S the height in feet of the plane above ground, and g is the acceleration due to gravity which is 32.16 ft. per sec. per sec. If a plane is at a height of 2.14 miles, what is the elapsed time in seconds from the release of the bomb till it strikes?

26. Horsepower in the case of the prony brake is given by the expression $H = \frac{2\pi PLN}{33,000}$ where P is the pressure in pounds at the end of the lever arm, L is the length in feet of the lever arm, and N is the speed in revolutions per minute. In a given engine test, the pressure was 84.9 lb., the lever arm had a length of 3.15 ft., and the speed was 148 revolutions per minute. What horsepower was developed by the engine?

9-15. Hints about Logarithmic Computations. The negative exponent is defined by the relation

$$a^{-b} = \frac{1}{a^b}.$$

We can use this definition to compute numbers with negative exponents by logarithms as shown in the following example.

Example 1. Compute $(0.6192)^{-2}$.

$$N = (0.6192)^{-2} = \frac{1}{(0.6192)^2}$$

$$\log N = \log 1 - 2 \log 0.6192.$$

NUMBERS	LOGARITHMS	
1	0	10.0000 - 10
0.6192	9.7918 - 10	
$(0.6192)^2$	19.5836 - 20	9.5836 - 10
N		0.4164
$N = 2.609.$		

We have defined logarithms only for positive numbers. However, they can be used for computations with negative numbers as well, for in any operation with negative numbers involving multiplication, division, raising to powers, or extraction of roots, the sign of the result can be determined independent of any calculation, and then the absolute value of the result can be found by logarithms. The following example will illustrate.

Example 2. Compute $N = \sqrt[3]{\frac{(-0.8162)(-4.673)}{-5.160}}.$

Since each of the three factors in the radicand is negative, the radicand itself is negative; and since the cube root of a negative number is negative, the result is negative. The result is, then, the negative of

$$M = \sqrt[3]{\frac{0.8162 \times 4.673}{5.160}},$$

which we shall compute as usual.

NUMBERS	LOGARITHMS	
0.8162	9.9118 - 10	
4.673	0.6696	
<i>Numerator</i>	10.5814 - 10	
5.160	0.7126	
<i>Radicand</i>	9.8688 - 10	29.8688 - 30
M		9.9563 - 10
$M = 0.9042,$		

and

$$N = -0.9042.$$

Since there is no formula for $\log (M + N)$ in terms of $\log M$ and $\log N$, computations involving sums or differences must be found arithmetically, as indicated in the following example.

Example 3. Compute $(0.8921)^{-\frac{1}{2}} + (1.231)^3$.

Each term must be computed separately, and the results added.

$$(0.8921)^{-\frac{1}{2}} = \frac{1}{(0.8921)^{\frac{1}{2}}}$$

$$\log (0.8921)^{-\frac{1}{2}} = \log 1 - \frac{1}{2} \log (0.8921)$$

$$\log (1.231)^3 = 3 \log 1.231.$$

NUMBERS	LOGARITHMS	
1		10.0000 - 10
0.8921	9.9504 - 10	
$(0.8921)^{\frac{1}{2}}$	4.9752 - 5	9.9752 - 10
		<hr/> 0.0248

$$(0.8921)^{-\frac{1}{2}} = 1.059$$

NUMBERS	LOGARITHMS
1.231	0.0903
$(1.231)^3$	0.2709

$$(1.231)^3 = 1.866.$$

The final result is:

$$1.059 + 1.866 = 2.925.$$

The identity $a^2 - b^2 = (a + b)(a - b)$ can be used to advantage in certain logarithmic computations, as indicated in the following example.

Example 4. Compute $N = \sqrt{(45.63)^2 - (31.57)^2}$ by logarithms.

Using the identity

$$\sqrt{(45.63)^2 - (31.57)^2} = \sqrt{(45.63 + 31.57)(45.63 - 31.57)},$$

we obtain

$$N = \sqrt{77.20 \times 14.06}.$$

This expression is easily computed by logarithms, for

$$\log N = \frac{1}{2} [\log 77.20 + \log 14.06].$$

NUMBERS	LOGARITHMS
77.20	1.8876
14.06	1.1480
	<hr/> 3.0356
N	1.5178
$N = 32.95.$	

EXERCISES

Compute each of the following by logarithms. Obtain results accurate to as many significant digits as possible, assuming that the quantities are accurate only to the given number of significant digits.

1. $(6.573)^{-2}$.
2. $(6.573)^{-\frac{1}{2}}$.
3. $(5.123)^{-3}$.
4. $(0.4127)^{-2}$.
5. $(0.05137)^{-5}$.
6. $(95.37)^{-\frac{1}{4}}$.
7. $(0.5678)^{-\frac{3}{2}}$.
8. $(5.673)^{-1.2}$.
9. $(0.5181)^{-0.6}$.
10. $(0.00006612)^{-\frac{1}{2}}$.
11. $(0.513)^{-2}$.
12. $(5.21)^{-3}$.
13. $(6.2)^{-2}$.
14. $(5.67)^{-\frac{1}{2}}(2.316)^{-2}$.
15. $(6.312)^{-\frac{4}{5}} \times (81.57)^{\frac{1}{2}}$.
16. $(315)^{-\frac{1}{3}} \times 65.27$.
17. $(0.3296)^{-0.3} \times (9541)^{-\frac{1}{3}}$.
18. $\frac{(81.4)^{-3} \times (19.67)^{\frac{5}{3}}}{(954.0)^{-1}}$.
19. $\frac{(46.3)^{-1} \times 957}{18.4}$.
20. $(-16.43) \times (-47.69)$.
21. $(-8.732)(-9.591)(-157.2)$.
22. -67.3×96.1 .
23. $\sqrt[3]{-687.3}$.
24. $\sqrt{(-69.2)(-81.5)}$.
25. $(-569.3)^3$.
26. $(-569.4)^2$.
27. $(-47.81)^2$.
28. $(-6.95)^5$.
29. $\sqrt[5]{-9.516}$.
30. $(-6.891)^{-\frac{1}{3}}$.
31. $(-957.3)^{\frac{1}{7}}$.
32. $-\sqrt{68.43}$.
33. $\left(-\frac{81.43}{0.0967}\right)^3$.
34. $\sqrt[5]{(-6.815)^2}$.
35. $\sqrt{\frac{(-67.35)^3}{-6994}}$.
36. $\sqrt[3]{\frac{65.2}{(-8.712)^2}}$.
37. $\sqrt{\frac{(-6.782) \cdot (9421)^3}{-567.3}}$.
38. $\sqrt[3]{856 \cdot (-6.73)^5}$.
39. $(418.3)^2 + (619.1)^2$.
40. $(9.812)^3 - (1.423)^{-2}$.
41. $(56.2)^{\frac{3}{4}} + 61.3$.
42. $14.2(81.3 + 57.2^2)$.
43. $(81.37)^{-\frac{1}{3}} + (9.25)^{-\frac{1}{2}}$.
44. $(9.356)^5 - (8.219)^4$.
45. $\frac{21.5}{(61.3)^2 - (5.93)^2}$.
46. $\frac{31.6}{(8.15)^{\frac{1}{3}} + (6.23)^{\frac{2}{3}}}$.
47. $\frac{(61.73)^2 + (98.5)^{\frac{3}{2}}}{418.4}$.
48. $\frac{\sqrt[5]{6148} + \sqrt[7]{95.63}}{615.4}$.
49. $\frac{(89.67)^{-\frac{1}{3}}}{69.43}$.
50. $\sqrt{(61.52)^2 - (21.59)^2}$.
51. $\sqrt{(0.8912)^2 - (0.06412)^2}$.
52. $\sqrt{(95.23)^2 - (61.47)^2}$.
53. $\sqrt{(81.21)^2 - (31.23)^2}$.
54. $\sqrt{(4.96)^2 - (3.21)^2}$.
55. $\sqrt[3]{(56.81)^2 - (18.23)^2}$.
56. $\sqrt{(61.59)^2 - (42.22)^2}$.

57. $\sqrt[3]{(519)^2 - (418)^2}$.

58. $\sqrt{(147)^2 - (112)^2}$.

59. $\sqrt[3]{(149)^2 - (356)^2}$.

60. $\sqrt[3]{(18.43)^2 - (61.59)^2}$.

61. $51.6\sqrt{(61.2)^2 - (51.3)^2}$.

62. $45.2 \cdot 61.5 \cdot \sqrt[3]{(18.1)^2 - (36.2)^2}$.

63. $37.59\sqrt[3]{(51.67)^2 - (67.42)^2}$.

64. $\sqrt[5]{(6187)^2 - (5162)^2}$.

65.
$$\frac{61.59 \cdot 48.6 \cdot \sqrt{(48.63)^2 - (56.92)^2}}{78,950}$$
.

9-16. Solution of Right Triangles with the Use of Logarithms. Logarithms can be used to simplify computations involving trigonometric functions. In this section we will use logarithms to perform computations like those considered in Sec. 4-15.

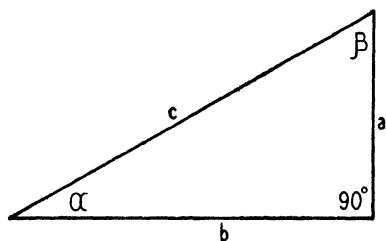


FIG. 9-3.

From Fig. 9-3 we see that

$$(1) \quad a = c \sin \alpha.$$

From this relation it follows that:

$$(2) \quad \log a = \log c + \log \sin \alpha.$$

In order to compute $\log \sin \alpha$, we could first find $\sin \alpha$ in the table of trigonometric functions and then find the logarithm of the number $\sin \alpha$. This procedure would necessitate using the table of trigonometric functions and the table of logarithms. To avoid the inconvenience of using two tables, we have at our disposal a table in which the logarithms of the trigonometric functions are given. In the Appendix, Table 4 is a table of this kind. For example, from Table 3, the table of natural functions, we find that

$$\sin 40^\circ = 0.6428$$

and from the table of logarithms

$$\log \sin 40^\circ = \log 0.6428 = 9.8081 - 10.$$

However, this last value can be found directly from Table 4. It should be noted that the -10 of the negative characteristics of logarithms of

trigonometric functions is omitted from Table 4 to save space. It will have to be supplied, of course, in computations.

Example 1. In a right triangle $a = 44.74$ and $\alpha = 52.35^\circ$. Find the remaining parts of the triangle.

As in Sec. 4-15, we have the following formulas.

$$b = a \cot \alpha, c = \frac{a}{\sin \alpha}; \beta = 90^\circ - \alpha.$$

Taking the logarithms of the first two expressions, we have

$$\log b = \log a + \log \cot \alpha,$$

$$\log c = \log a - \log \sin \alpha.$$

The logarithmic computation is shown below.

NUMBERS	LOGARITHMS	
$a = 44.74$	1.6507	(Table 2)
$\cot \alpha = \cot 52.35^\circ$	9.8873 - 10	(Table 4)
$b = 34.52$	1.5380	(Table 2)
$a = 44.74$	11.6507 - 10	(Table 2)
$\sin \alpha = \sin 52.35^\circ$	9.8986 - 10	(Table 4)
$c = 56.51$	1.7521	(Table 2)

Hence

$$b = 34.52, c = 56.51, \beta = 37.65^\circ.$$

These computations can be checked by means of the Pythagorean theorem in the form

$$a = \sqrt{c^2 - b^2} = \sqrt{(c + b)(c - b)}.$$

Since

$$c = 56.51 \quad \text{and} \quad b = 34.52,$$

we have:

NUMBERS	LOGARITHMS
$c + b = 91.03$	1.9592
$c - b = 21.99$	1.3422
$(c + b)(c - b)$	3.3014
$\sqrt{(c + b)(c - b)}$	1.6507

The logarithm 1.6507 found for $\sqrt{c^2 - b^2}$ is equal to the value of $\log a$ found in the preceding computations and hence we have a check on the work of this example.

Example 2. Compute the coordinates of the terminal point of a vector \mathbf{A} whose magnitude $A = 57.25$, direction angle $\alpha = 123.28^\circ$, and which has its initial point at the origin.

From Sec. 6-5, we have

$$x_A = A \cos \alpha, y_A = A \sin \alpha;$$

$$\log |x_A| = \log A + \log |\cos \alpha|;$$

$$\log |y_A| = \log A + \log |\sin \alpha|.$$

Since $\sin 123.28^\circ = \sin 56.72^\circ$ and $\cos 123.28^\circ = -\cos 56.72^\circ$, we have:

$$|\sin 123.28^\circ| = \sin 56.72^\circ \quad \text{and} \quad |\cos 123.28^\circ| = \cos 56.72^\circ.$$

The computation follows.

NUMBERS	LOGARITHMS	NUMBERS	LOGARITHMS
$A = 57.25$	1.7578	$A = 57.25$	1.7578
$ \cos \alpha = \cos 56.72^\circ$	9.7394 - 10	$ \sin \alpha = \sin 56.72^\circ$	9.9222 - 10
$ x_A = 31.42$	1.4972	$ y_A = 47.87$	1.6800

Since x_A is negative and y_A positive,

$$x_A = -31.42, y_A = 47.87.$$

Since

$$A^2 = x_A^2 + y_A^2,$$

$$y_A^2 = A^2 - x_A^2 = (A + x_A)(A - x_A)$$

and

$$|y_A| = \sqrt{(A + x_A)(A - x_A)}.$$

The computations follow:

$$A = 57.25, x_A = -31.42.$$

NUMBERS	LOGARITHMS
$A - x_A = 88.67$	1.9478
$A + x_A = 25.83$	1.4121
$(A + x_A)(A - x_A)$	3.3599
$ y_A = \sqrt{(A + x_A)(A - x_A)}$	1.6800

Since $\log |y_A| = 1.6800$ here as well as in the first computation, we have a check upon our work.

In computations of this kind, where lengths are measured to four significant digits and angles to the nearest hundredth of a degree, four-place logarithms give results to the same degree of accuracy.

EXERCISES

Solve the following right triangles and check. Give your results to the degree of accuracy of the given data.

- $c = 165.2, \alpha = 27.61^\circ$.
- $c = 0.2643, \beta = 63.81^\circ$.
- $a = 4682, \alpha = 15.76^\circ$.
- $a = 79,430, \beta = 5.21^\circ$.
- $b = 0.05196, \alpha = 73.82^\circ$.
- $b = 1.316, \beta = 12.35^\circ$.
- $a = 4.184, b = 5.744$.
- $a = 79,630, c = 91,180$.
- $b = 6.73 \times 10^{12}, c = 7.18 \times 10^{12}$.
- $a = 0.00023, b = 0.0037$.
- $c = 162, \alpha = 32.1^\circ$.
- $c = 0.361, \beta = 67.2^\circ$.
- $a = 18, \alpha = 48^\circ$.
- $a = 56.2, \beta = 19.4^\circ$.

Each vector **A** given below has its initial point at the origin. If A is the magnitude of **A** and α its direction angle, find by using logarithms the coordinates of the terminal point of **A**. Give your results to the degree of accuracy of the given data.

- | | |
|---|--|
| 15. $A = 87.42, \alpha = 38.61^\circ$. | 16. $A = 2387, \alpha = 157.62^\circ$. |
| 17. $A = 0.3571, \alpha = 157.62^\circ$. | 18. $A = 48.69, \alpha = 65.26^\circ$. |
| 19. $A = 659.2, \alpha = 305.67^\circ$. | 20. $A = 0.08167, \alpha = 236.52^\circ$. |
| 21. $A = 5.63, \alpha = 36.8^\circ$. | 22. $A = 89.7, \alpha = 208.2^\circ$. |
| 23. $A = 0.0612, \alpha = 179.2^\circ$. | 24. $A = 2.8, \alpha = 189^\circ$. |
| 25. $A = 3.2, \alpha = 302^\circ$. | 26. $A = 95.7, \alpha = 165.7^\circ$. |

Find the magnitude A and direction angle α of the vector **A** whose initial point is at the origin and whose terminal point is (x_A, y_A) . Give your results to the degree of accuracy of the given data.

- | | |
|-----------------------------------|---------------------------------------|
| 27. $x_A = 4.867, y_A = -3.216$. | 28. $x_A = -9.345, y_A = -8.412$. |
| 29. $x_A = 108.2, y_A = 306.8$. | 30. $x_A = -0.06792, y_A = 0.05123$. |
| 31. $x_A = 6.93, y_A = -3.21$. | 32. $x_A = -5.27, y_A = -8.67$. |
| 33. $x_A = 3.85, y_A = 4.92$. | 34. $x_A = -0.28, y_A = 0.57$. |

Using the method described in Sec. 6-5, find the sum of the vectors in each exercise. Give your results to the degree of accuracy of the given data.

- | | |
|---|---|
| 35. A : $A = 65.47, \alpha = 36.29^\circ$.
B : $B = 48.62, \beta = 109.57^\circ$. | 36. A : $A = 48.69, \alpha = 118.69^\circ$.
B : $B = 96.43, \beta = 215.37^\circ$. |
| 37. A : $A = 615.2, \alpha = 45.32^\circ$.
B : $B = 895.1, \beta = 350.19^\circ$. | 38. A : $A = 65.3, \alpha = 47.8^\circ$.
B : $B = 36.2, \beta = 137.2^\circ$. |
| 39. A : $A = 486, \alpha = 56.2^\circ$.
B : $B = 592, \beta = 318.9^\circ$. | 40. A : $A = 0.0512, \alpha = 128.3^\circ$.
B : $B = 0.0316, \beta = 208.5^\circ$. |
| 41. A : $A = 937, \alpha = 36.4^\circ$.
B : $B = 156, \beta = 142.5^\circ$.
C : $C = 886, \gamma = 232.8^\circ$. | 42. A : $A = 48.6, \alpha = 59.2^\circ$.
B : $B = 39.2, \beta = 118.6^\circ$.
C : $C = 51.6, \gamma = 240.1^\circ$. |
| 43. A : $A = 0.03167, \alpha = 63.25^\circ$.
B : $B = 0.05129, \beta = 149.67^\circ$.
C : $C = 0.03256, \gamma = 252.37^\circ$. | 44. A : $A = 5.361, \alpha = 0.00^\circ$.
B : $B = 7.594, \beta = 166.82^\circ$.
C : $C = 3.962, \gamma = 270.00^\circ$.
D : $D = 4.111, \delta = 305.76^\circ$. |

45. Find the height of a tower if the length of its shadow is measured as 165.6 ft. at a moment when the elevation of the sun is 49.85° .

46. An observation balloon 825 ft. above a battery observes that the angle of depression of an enemy battery is 5.64° . What is the distance between the two batteries?

47. A certain plane can climb at an angle of 25.5° while making 215 m.p.h. How long will it take for the plane to reach an altitude of 12,500 ft.?

48. A radio tower is on top of a building. From a point 3250 ft. from the building, the angles of elevation of the top of the building and the top of the tower are 3.42° and 5.16° respectively. How high are the building and the tower?

Additional exercises can be formed by selecting problems from Secs. 4-15, 6-4, and 6-5 for logarithmic computation.

9-17. Logarithms to Other Bases than 10. Suppose it is required to find $\log_a N$ when the available table of logarithms has the base b . By the definition of logarithms

$$N = a^{\log_a N}.$$

Then

$$\begin{aligned}\log_b N &= \log_b (a^{\log_a N}) \\ &= (\log_a N)(\log_b a).\end{aligned}$$

From this last relation, dividing by $\log_b a$, we have

$$\log_a N = \frac{\log_b N}{\log_b a}$$

or

$$(1) \quad \log_a N = \frac{1}{\log_b a} \log_b N.$$

The special case of this relation in which we are primarily interested is the one in which $a = e$ and $b = 10$, where $e = 2.71828 \dots$. We have then

$$(2) \quad \log_e N = \frac{1}{\log_{10} e} \log_{10} N.$$

Since $\log_{10} e = 0.4343$, $\frac{1}{\log_{10} e} = 2.3026$ and (2) can be written

$$(3) \quad \log_e N = 2.3026 \log_{10} N.$$

Using (3), we can thus compute logarithms to the base e from a table of logarithms to the base 10. From (2), we have

$$(4) \quad \log_{10} N = 0.4343 \log_e N.$$

Using the conventions of writing $\log_{10} N$ as $\log N$, and $\log_e N$ as $\ln N$, (3) and (4) can be written

$$(5) \quad \ln N = 2.3026 \log N,$$

$$(6) \quad \log N = 0.4343 \ln N.$$

The values of $\log N$ are found on most slide rules and, therefore, formula (5) can be used to find $\ln N$ with the slide rule alone without tables.

The so-called log log scales which are found on certain types of slide rules permit us to find directly the natural logarithm of a number without any computations.

EXERCISES

Using Table 2 in the Appendix, evaluate each of the following to four significant figures. Verify the first three figures by the slide rule.

- | | | |
|----------------------|--------------------|------------------|
| 1. $\ln 64.30$. | 2. $\ln 46.29$. | 3. $\ln 6.423$. |
| 4. $\ln 8.915$. | 5. $\ln 0.08640$. | 6. $\ln 9.812$. |
| 7. $\ln 0.0004126$. | 8. $\ln 8142$. | 9. $\ln 95.67$. |
| 10. $\ln 9.812$. | | |

Find N in each of the following to three significant figures.

- | | | |
|------------------------|------------------------|------------------------|
| 11. $\ln N = 2.301$. | 12. $\ln N = 5.609$. | 13. $\ln N = 0.412$. |
| 14. $\ln N = 3.218$. | 15. $\ln N = -2.301$. | 16. $\ln N = -0.509$. |
| 17. $\ln N = -5.623$. | 18. $\ln N = -0.823$. | |

9-18. Exponential Equations. An equation in which the unknown occurs in an exponent is called an **exponential equation**. Thus, $2^x = 7$, $5^{x^2-1} = 31$ are examples of exponential equations.

There is no general method for solving exponential equations. Many exponential equations, for example,

$$2^x + 3^x = 10,$$

cannot be solved by methods studied in this book. There is one case, however, which may be solved easily by taking the logarithm of each member. This case is the one in which the equations have the form

$$a^x = b,$$

where a and b are given and x is to be found. Taking the logarithm of each member, we obtain

$$x \log a = \log b,$$

and hence

$$x = \frac{\log b}{\log a}.$$

Example 1. Solve $2^x = 30$ for x .

Take the logarithm of each member and obtain

$$\log 2^x = \log 30,$$

$$x \log 2 = \log 30,$$

and

$$x = \frac{\log 30}{\log 2} = \frac{1.4771}{0.3010}.$$

If two or three place accuracy is desired, the last division can be performed on the slide rule. When greater accuracy is wanted, we use logarithms and obtain:

NUMBERS	LOGARITHMS
1.4771	10.1694 - 10
0.3010	9.4786 - 10
4.907	0.6908

The answer, to four significant figures, is 4.907.

Example 2. Solve $3^{x-2} = 12$ for x .

Taking the logarithm of each member and proceeding as in the previous example, we obtain

$$\log 3^{x-2} = \log 12,$$

$$(x - 2) \log 3 = \log 12,$$

$$x - 2 = \frac{\log 12}{\log 3} = \frac{1.0792}{0.4771} = 2.262,$$

$$x = 4.262.$$

EXERCISES

Solve the following equations.

1. $10^x = 20$.

2. $3^x = 50$.

3. $5^x = 0.7$.

4. $0.8^x = 0.3$.

5. $12^x = 30$.

6. $0.63^x = 8.1$.

7. $1.05^{-x} = 0.562$.

8. $15^{3x} = 75(3^x)$.

9. $\frac{1.03^{x-1}}{0.03} = 8.892$.

10. $1.04^x = 2$.

11. $1.04^{x+10} = 1.08^x$.

12. $2^x = 5$.

13. $1.5^{x-5} = 1$.

14. $2^{3x} \cdot 5^{2x-1} = 4^x \cdot 3^{x+1}$.

15. $e^x = 50$ ($e = 2.718$).

16. $e^{-5x} = 0.5$.

9-19. The Slide Rule and Logarithmic Paper. The construction and use of the Mannheim slide rule are based on the results which have been established in this chapter.

Suppose that the distance between the two index marks on the C or D scales is L . Then L is the length of the scale. The location of the marks 1, 2, 3 \dots 10 is found as follows. 1 is located at the left index mark. The point marked 2 is found so that the length from 1 to 2 is $L \log 2$. In the same way, the length from 1 to 3 is $L \log 3$, the length from 1 to 4 is $L \log 4$ and so on.

Thus, if the left index of the C scale is set at a point, marked with p on the D -scale, and a point, marked with q , is located on the C -scale, then the distance from the left index of the D -scale to the point q on the C -scale is

$$\begin{aligned} L \log p + L \log q &= L(\log p + \log q) \\ &= L \log pq. \end{aligned}$$

Therefore, the mark on the D -scale opposite q on the C -scale is pq . Thus, the slide rule is simply a mechanical device for adding and subtracting logarithms.

The scales A and B are constructed in like manner, with a basic length which is one-half the length of the scales C or D . The distance from the index to a certain number, say 5, on scale A is half as great as the distance from the index to the mark 5 on scale D , and, therefore, corresponds to the number $\sqrt{5}$ on scale D .

Thus *logarithmic scales*, on which the points, 1, 2, 3 . . . are not marked at equal distances from each other, but at distances from the origin of the scale which are proportional to $\log 1$, $\log 2$, $\log 3$, etc., are the basis of the slide rule. A little experimentation will soon reveal how the other scales on any given rule are constructed.

Scales of the same kind can be used in order to simplify the construction of graphs of certain functions. Graphs are usually plotted on cross-section paper on which horizontal and vertical lines are printed at equidistant intervals, so that the whole area is covered with small squares. On the so-called *semi-logarithmic paper*, the scale on the x -axis (or y -axis) is replaced by a logarithmic scale and the vertical lines are printed in distances corresponding to this scale, but the horizontal (or vertical) lines are at equidistant intervals as on ordinary cross section paper.

On *logarithmic paper*, both systems of lines are printed corresponding to a logarithmic scale. A few examples will show how to use these papers.

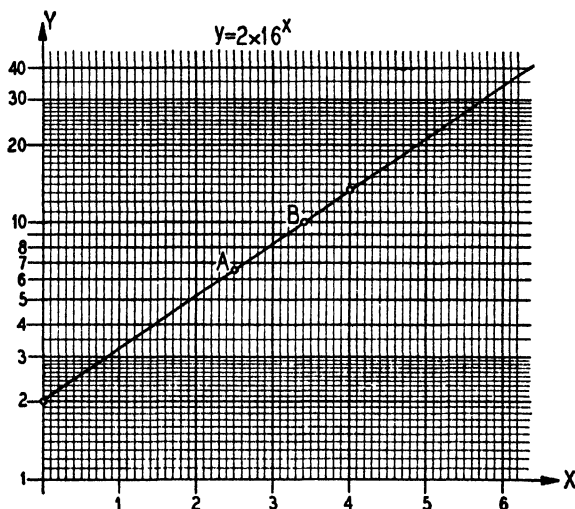


FIG. 9-4.

Example 1. Construct a graph of the function

$$(1) \quad y = 2 \times 1.6^x.$$

This equation can be written in the equivalent form

$$\log y = \log 2 + x \log 1.6;$$

and, if $\log y = u$,

$$(2) \quad u = 0.3010 + 0.2041x.$$

Since u is a linear function of x , the graph of (2) is therefore a straight line, which can be constructed easily. In order to locate the values of y on the graph without

computations, a logarithmic scale is used on the vertical axis, so that the numbers 2, 3, 4 ... are located at the distances $u = \log 2$, $u = \log 3$, $u = \log 4 \dots$.

The most convenient way to plot the graph is to find two of its points. If $x = 0$, $y = 2$; and if $x = 4$, $y = 2 \times 1.6^4 = 2 \times 6.55 = 13.10$. The graph is plotted on semi-logarithmic paper in Fig. 9-4.

This graph can now be used to solve certain problems. If $y = 2 \times 1.6^{2.5}$ has to be computed, point A of the graph gives approximately the answer $y = 6.5$. If we desire the value of x such that $2 \times 1.6^x = 10$, point B of the graph shows that $x = 3.4$ is the answer.

Example 2. Plot a graph corresponding to the equation $x^2 y^3 = 1$.

Computing the logarithms on both sides of this equation, we have $2 \log x + 3 \log y = \log 1 = 0$. This is a linear equation between $\log x$ and $\log y$. Therefore

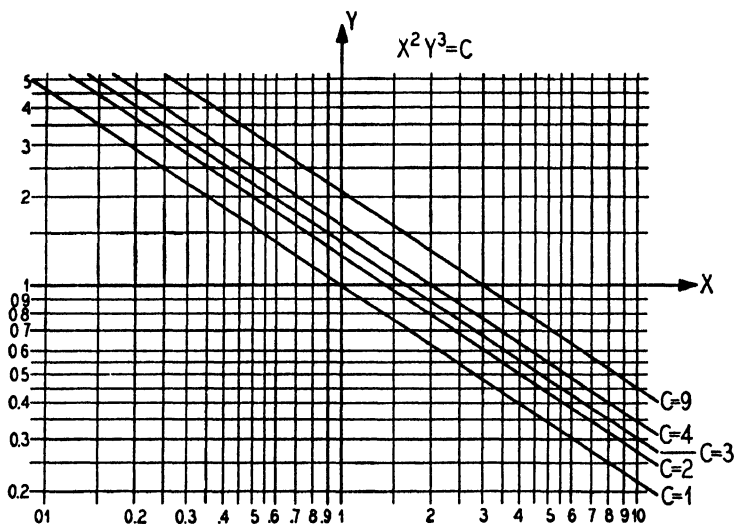


FIG. 9-5.

if we plot $\log y$ against $\log x$ we obtain a straight line. We plot this graph by plotting x and y on logarithmic scales.

In order to plot the graph, two sets of corresponding values of x and y are used. For example:

$$x_1 = 1, y_1 = 1;$$

$$x_2 = 4, y_2 = \sqrt[3]{\frac{1}{16}} = 0.397.$$

The graph is plotted in Fig. 9-5.

It is very easy to construct a system of graphs corresponding to the equations $x^2 y^3 = c$ for different values of c . From the corresponding equation $2 \log x + 3 \log y = \log c$ it can be inferred that, on logarithmic paper, all these graphs are parallel straight lines. They are easily constructed, observing that for $y = 1$, $x = \sqrt{c}$.

In Fig. 9-5 the graphs corresponding to the values $c = 1, 2, 3, 4, 9$ are plotted.

EXERCISES

Plot the graphs connected with the following equations on logarithmic or semi-logarithmic paper.

1. $u = 3.7 \times 10^v$.

2. $u = 5 \times 2.72^{1.6v}$.

3. $y = \frac{100}{2^x}$.

4. $p \times 2.72^{0.059} = 10$.

5. $x = 0.78 \cdot y^2$.

6. $pv^{1.41} = 1$.

7. $400\sqrt{LC} = 1$.

8. $M = \frac{20}{2.72^{3n}}$.

9. Construct a graphical table for the equation $xy = C$ for the integral values of C from -10 to $+10$.

10. Make the same construction as in Exercise 9 for the equation $x = \frac{C}{y^2}$.

9-20. Engineering Applications of Logarithms. Relationships of the type $y = \log x$ are frequently encountered in science, thus giving the logarithm particular significance in certain engineering problems. For example, the response of the human ear to sound is proportional to the logarithm of the power producing the sound.

The power used to produce a sound A is called the power level P_A of the sound and is measured in convenient power units, usually in watts. It is difficult to define a unit for the loudness of a sound; it is easier to compare the loudness of two sounds and to introduce a unit for the difference in loudness between the sounds. This unit is the **decibel** and is defined as follows.

If P_A and P_B are the power levels of two sounds A and B , then the difference in loudness between the sounds A and B , measured in decibels (denoted by *db.*) is $10 \log \frac{P_A}{P_B}$. This value is often called **decibel gain or loss**.

Thus the number of decibels corresponding to a power ratio $\frac{P_A}{P_B} = 10 : 1$ is $10 \log \frac{10}{1} = 10$. In this way, the following table can be constructed.

POWER RATIO	DB.	POWER RATIO	DB.
0.1 : 1	-10	10 : 1	10
1 : 1	0	100 : 1	20
2 : 1	3	1000 : 1	30

Experiment shows that a difference of 1 decibel is the smallest change in sound intensity which the ear can detect. The usefulness of the decibel in engineering problems is shown by the illustrative problem below.

Example 1. If the power output of a public address system were increased from 20 to 22 watts, would the change in loudness be discernible to the ear?

Finding the decibel gain:

$$\begin{aligned}\text{Decibel gain} &= 10 \log \frac{22}{20} \\ &= 10 \log 1.1 \\ &= 10 \cdot 0.041 = 0.41 \text{ db.}\end{aligned}$$

The change is less than 1 decibel and is not apparent to the ear.

The number of decibels corresponding to a given power ratio can be used to characterize this power ratio. In electrical circuits, it may also be used to measure current or voltage ratios. For this purpose the formula $P = I^2 R = \frac{E^2}{R}$ is used where I is the current, E the voltage, and R the resistance. The expression for the number of decibels corresponding to the power ratio $\frac{P_A}{P_B}$ may then be rewritten as follows.

$$(1) \quad \text{Decibel gain (or loss)} = 10 \log \frac{P_A}{P_B} = 10 \log \frac{I_A^2 R_A}{I_B^2 R_B} = 10 \log \frac{\frac{E_A^2}{R_A}}{\frac{E_B^2}{R_B}}.$$

If the resistances R_A and R_B are equal, these expressions become

$$(2) \quad \text{Decibel gain (or loss)} = 10 \log \frac{I_A^2}{I_B^2} = 20 \log \frac{I_A}{I_B},$$

$$(3) \quad \text{Decibel gain (or loss)} = 10 \log \frac{E_A^2}{E_B^2} = 20 \log \frac{E_A}{E_B}.$$

Since the decibel gain or loss depends on either a power, current, or voltage *ratio* in the above cases, the units in which the quantities are given are immaterial as long as they are the same in both cases.

Example 2. A motion picture sound amplifier delivers a voltage of 80 volts at a frequency of 800 cycles and 20 volts at a frequency of 90 cycles across a constant resistance. What is the decibel loss in the amplifier between these two frequencies?

The decibel loss of an amplifier delivering various voltages across a constant resistance is given by the expression (3). The decibel loss which occurs in the amplifier between the two frequencies is then given by

$$\text{Decibel loss} = 20 \log \frac{E_A}{E_B} = 20 \log \frac{80}{20} = 20 \log 4 = 20 \cdot 0.602 = 1.204 \text{ db.}$$

Example 3. The resistance of a tungsten lamp filament varies with the temperature and is given by the relationship $\frac{R_1}{R_2} = \left(\frac{T_1}{T_2}\right)^{1.3}$ where R_1 is the resistance in

ohms of the filament at room temperature T_1 , measured in degrees Kelvin, and R_2 and T_2 are the operating resistance and temperature. A given lamp was measured at a room temperature of 293° Kelvin and found to have a resistance of 24 ohms. What is the operating temperature at a measured operating resistance of 348 ohms?

The operating temperature in terms of the other factors may be obtained from the given equation $\left(\frac{T_1}{T_2}\right)^{1.2} = \frac{R_1}{R_2}$.

By inversion

$$\left(\frac{T_2}{T_1}\right)^{1.2} = \frac{R_2}{R_1} \quad \text{or} \quad \frac{T_2}{T_1} = \left(\frac{R_2}{R_1}\right)^{\frac{1}{1.2}},$$

whence

$$T_2 = T_1 \left(\frac{R_2}{R_1}\right)^{\frac{1}{1.2}}.$$

Substituting the specific values given in the example, we obtain,

$$T_2 = 293 \left(\frac{348}{24}\right)^{\frac{1}{1.2}} = 293(14.5)^{\frac{1}{1.2}}.$$

Solving by logarithms,

NUMBERS	LOGARITHMS
14.5	1.1614
$14.5^{\frac{1}{1.2}}$	0.9678
293	2.4669
$T_2 = 2721$	<u>3.4347</u>

The operating temperature T_2 of the filament is therefore 2721° Kelvin.

EXERCISES

1. A loudspeaker requires 1.2 watts to produce proper volume output in a room. If the power supplied is 1.8 watts, how many decibels have been added to the original output?

2. If the output varies 4 db. above and 3 db. below the original output of 1.2 watts in Problem 1, between what power levels does the amplifier operate?

3. The ratio of power in speech may vary as much as 250 to 1. What is this range in decibels?

4. The undistorted power output of a 45 type tube is 2.0 watts and that of the 2A3 is 3.5 watts. How many decibels greater is the output of the 2A3 tube?

5. The 6F6 pentode tube delivers 3.2 watts with a plate voltage of 250 volts. When the plate voltage is increased to 285 volts, the power output is increased to 4.8 watts. How many decibels increase in power does this represent?

6. A resistance carries a current of 42 milliamperes. If the current is reduced to 8 milliamperes what is the change in decibels?

7. A single 45 type tube delivers 2.0 watts of output. When two such tubes are used in a push-pull circuit the output is increased to 12.0 watts. What change in decibels does this represent?

8. How much must the output of an 8.0-watt amplifier be increased if the output is to be raised 3.4 db.?

9. The energy loss in iron due to repeated magnetizations, such as occurs in transformer cores, is given by the Steinmetz hysteresis equation $W = \eta B^{1.6}$ where W is the loss in energy in ergs in 1 cu. cm. of iron for each cycle of magnetization, and η is the hysteresis coefficient which for a particular kind of iron has a value of 2.5×10^{-3} . The maximum magnetic induction density in maxwells per square centimeter is designated by B . In testing a sample of iron the energy loss per cubic centimeter was measured to be 2150 ergs. If the hysteresis coefficient is as given above, determine the maximum magnetic induction density.

10. The cold resistance of a tungsten lamp filament was measured at a room temperature of 295° Kelvin and found to be 30 ohms. The temperature of the filament under normal operating conditions was determined and found to be 2610° Kelvin. What is the resistance of the filament under operating conditions?

PROGRESS REPORT

The work involved in numerical computations is so great that the value of methods by which calculations may be simplified cannot be overestimated. Logarithms, defined as the exponents to which a base must be raised in order to obtain given numbers, permit us to transform complicated computations into simpler ones, thus saving energy and time. This chapter was devoted to the examination of the properties of logarithms, to the explanation of tables of logarithms, and to the discussion of the use of logarithmic tables for numerical computations.

Two systems of logarithms were mentioned, the common logarithms and the natural logarithms, and it was shown that there is a constant ratio between the logarithms of any number in the two systems.

It was explained how the construction of the slide rule is based on logarithms and how the idea of logarithmic scales is used on logarithmic papers to permit us to simplify the graphs of many curves.

Logarithms are not only a tool for numerical computation but also are used to describe many relations in science. As an example, the comparing of power levels by decibels was discussed.

CHAPTER 10

THE FUNDAMENTAL RELATIONS OF TRIGONOMETRY

In Chapter 7 identical and conditional equations were discussed. Both types of equations occur frequently in trigonometry where the variables involved are angles and functions of angles. Most of this chapter will be devoted to the discussion of trigonometric identities, that is, trigonometric relations which are true for all values of the variables (angles) involved. At the end of this chapter, a few conditional trigonometric equations will be discussed.

Trigonometric identities have many uses in mathematics. Most important is their use in the simplification of trigonometric expressions and formulas which occur in various engineering applications.

10-1. The Fundamental Trigonometric Identities. There are several trigonometric identities which are so frequently used that they are commonly known as the **fundamental identities**. These are

$$\begin{aligned} \sin \theta &= \frac{1}{\csc \theta}, & \csc \theta &= \frac{1}{\sin \theta}. \\ \cos \theta &= \frac{1}{\sec \theta}, & \sec \theta &= \frac{1}{\cos \theta}. \end{aligned} \quad (1)$$

$$\tan \theta = \frac{1}{\cot \theta}, \quad \cot \theta = \frac{1}{\tan \theta}.$$

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta}. \\ \cot \theta &= \frac{\cos \theta}{\sin \theta}. \end{aligned} \quad (2)$$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1. \\ 1 + \tan^2 \theta &= \sec^2 \theta. \\ 1 + \cot^2 \theta &= \csc^2 \theta. \end{aligned} \quad (3)$$

All these relationships are proved by using the original definitions of the trigonometric functions as given in Chapter 4. For example, the

first three identities of group (1) are proved immediately from the definitions in the following manner (see Fig. 4-6 of Chapter 4):

$$\frac{1}{\csc \theta} = \frac{1}{\frac{r}{y}} = \frac{y}{r} = \sin \theta.$$

$$\frac{1}{\sec \theta} = \frac{1}{\frac{r}{x}} = \frac{x}{r} = \cos \theta.$$

$$\frac{1}{\cot \theta} = \frac{1}{\frac{x}{y}} = \frac{y}{x} = \tan \theta.$$

The other three identities of (1) are proved in the same manner.

Similar proofs apply to the identities of group (2). For example,

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} = \tan \theta.$$

In order to prove the third set of identities, we again use the fundamental definitions, this time in connection with the Pythagorean theorem. For every angle θ the sides of the reference triangle satisfy the relation

$$(4) \quad y^2 + x^2 = r^2.$$

Dividing both sides of the equation by r^2 , we obtain

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1 \quad \text{or} \quad \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1.$$

Now, substituting the definitions $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$ in the above equation, we obtain the first identity of (3), namely,

$$\sin^2 \theta + \cos^2 \theta = 1.$$

This relation is true for every angle θ .

The second and third identities are proved in a similar manner, dividing (4) by x^2 and y^2 respectively.

Since the fundamental identities hold for all values of θ , they are frequently useful in the simplification of trigonometric expressions.

Consequently, it is of great value for the student of applied mathematics to be able to manipulate and use these identities as the occasion demands.

Example 1. In a certain problem, to find a length L when given an angle θ and another length a , an engineer developed a formula

$$L = \frac{a^2 \sin^2 \theta + a^2 \cos^2 \theta}{\csc \theta \tan \theta}.$$

Can he, by use of identities, simplify this formula?

Factoring the numerator, and substituting in the denominator the identities

$$\csc \theta = \frac{1}{\sin \theta}, \quad \tan \theta = \frac{\sin \theta}{\cos \theta}$$

we obtain

$$L = \frac{a^2(\sin^2 \theta + \cos^2 \theta)}{\frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta}}.$$

But from (3) of this section $\sin^2 \theta + \cos^2 \theta = 1$. Therefore

$$L = \frac{a^2 \cdot 1}{1 \cdot \frac{1}{\cos \theta}} = a^2 \cos \theta.$$

Example 2. Simplify

$$\frac{1 + \cos \beta}{\sin \beta} + \frac{\sin \beta}{1 + \cos \beta}.$$

Adding the two fractions by finding the common denominator, we obtain

$$\frac{1 + 2 \cos \beta + \cos^2 \beta + \sin^2 \beta}{\sin \beta(1 + \cos \beta)}.$$

Upon substituting the identity $\sin^2 \beta + \cos^2 \beta = 1$, this becomes

$$\frac{2 + 2 \cos \beta}{\sin \beta(1 + \cos \beta)}.$$

Factoring the numerator and substituting $\frac{1}{\sin \beta} = \csc \beta$, we get

$$\frac{2(1 + \cos \beta)}{\sin \beta(1 + \cos \beta)} = \frac{2}{\sin \beta} = 2 \csc \beta.$$

Example 3. Express in terms of $\cos \alpha$:

$$K = \frac{1 - 2 \sin^2 \alpha}{1 + \tan^2 \alpha}.$$

Substituting $\sin^2 \alpha = 1 - \cos^2 \alpha$ (derived from $\sin^2 \alpha + \cos^2 \alpha = 1$) and $\sec^2 \alpha = 1 + \tan^2 \alpha$, we obtain

$$K = \frac{1 - 2(1 - \cos^2 \alpha)}{\sec^2 \alpha} = \frac{2 \cos^2 \alpha - 1}{\sec^2 \alpha}.$$

But $\sec^2 \alpha = \frac{1}{\cos^2 \alpha}$. Therefore

$$K = \frac{2 \cos^2 \alpha - 1}{\frac{1}{\cos^2 \alpha}} = 2 \cos^4 \alpha - \cos^2 \alpha.$$

Since the procedure depends on the type of result desired, there are no general instructions which can be given for the manipulation of trigonometric expressions. At first the student cannot do much more than proceed by trial and error. Gradually, after some experience has been gained, a certain intuition may develop. There is no necessarily correct method of procedure. Often several methods will yield the same result.

EXERCISES

Test each of the fundamental identities for the particular angle given.

- | | | | |
|---------------------------|----------------------------|---------------------------|------------------------------|
| 1. $\theta = 30^\circ$. | 2. $\theta = 300^\circ$. | 3. $\theta = 45^\circ$. | 4. $\theta = 140^\circ$. |
| 5. $\theta = 330^\circ$. | 6. $\theta = 76^\circ$. | 7. $\theta = 161^\circ$. | 8. $\theta = -25^\circ$. |
| 9. $\theta = 90^\circ$. | 10. $\theta = 180^\circ$. | 11. $\theta = 0^\circ$. | 12. $\theta = 12.21^\circ$. |

By use of the fundamental identities express each of the six trigonometric functions in terms of the given function, assuming θ to be in the first quadrant.

- | | | |
|---------------------|---------------------|---------------------|
| 13. $\sin \theta$. | 14. $\cos \theta$. | 15. $\tan \theta$. |
| 16. $\cot \theta$. | 17. $\sec \theta$. | 18. $\csc \theta$. |

Simplify each of the following trigonometric expressions as much as possible.

- | | |
|--|---|
| 19. $\sin \theta \cot \theta \sec \theta$. | 20. $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$. |
| 21. $\frac{E_1 \cos \theta}{2} + \frac{E_2}{2 \sec \theta}$. | 22. $\frac{\cos \theta}{\sec \theta} - \sin \theta \csc \theta$. |
| 23. $\frac{I_m \sec A}{1 + \tan^2 A}$. | 24. $\sin^2 A(1 + \tan^2 A)$. |
| 25. $\frac{2 \csc \theta - 1}{\cot \theta}$. | 26. $\frac{\tan 45^\circ + \tan^2 A}{\cot 45^\circ + \cot^2 A}$. |
| 27. $\frac{\cos^2 \beta}{1 - \cos^2 \beta}$. | 28. $\frac{E_1 - \cos z_1}{\cos z_1} - E_1 \sec z_1$. |
| 29. $\cos A \sqrt{\sec^2 A - \left(2 - \sin \frac{\pi}{2}\right)^2}$. | 30. $\cos^4 \theta - \sin^4 \theta + 2 \sin^2 \theta$. |
| 31. $\sec x - \frac{\sin^2 x + \sin x + \cos^2 x}{\cos x}$. | 32. $\frac{2 \sin \beta}{1 + \cos \beta} + \frac{1 + \cos \beta}{\frac{1}{2} \sin \beta}$. |

$$33. \frac{\sqrt{\sec^2 \theta - 1}}{\sqrt{1 - \sin^2 \theta}}.$$

$$34. \frac{E_1 \sin \frac{3\pi}{2}}{E_2 \sin \beta (\tan \beta + \cot \beta)}.$$

$$35. \tan \alpha \sec \alpha - \frac{\csc^2 \alpha - 1 - \csc (\pi + \alpha)}{\cot^2 \alpha}.$$

$$36. \frac{\sin z + 2 \sin z \cos z}{2 - 2 \sin^2 z + \cos z}.$$

$$37. \sec^2 \omega t - \frac{(\sec \omega t - 1) \tan^2 \omega t}{\sec \omega t + 1}.$$

$$38. \frac{(1 + \cos t \tan t) \sec^2 t}{1 + (\tan t + \sec t)^2}.$$

$$39. \frac{\tan \theta}{\csc \theta} + \cos \theta.$$

$$40. \frac{\cos \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{\cos \theta} - 2 \tan \theta.$$

Prove the fundamental identities.

$$41. \csc \theta = \frac{1}{\sin \theta}.$$

$$42. \sec \theta = \frac{1}{\cos \theta}.$$

$$43. \cot \theta = \frac{1}{\tan \theta}.$$

$$44. \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

$$45. 1 + \tan^2 \theta = \sec^2 \theta.$$

$$46. 1 + \cot^2 \theta = \csc^2 \theta.$$

10-2. Some Trigonometric Formulas. Following the fundamental identities are three groups of important identities known as the **addition formulas**, the **double-angle formulas**, and the **half-angle formulas**.

The Addition Formulas. We shall accept without proof the following formulas:

$$(1) \quad \begin{aligned} \sin (A + B) &= \sin A \cos B + \cos A \sin B. \\ \sin (A - B) &= \sin A \cos B - \cos A \sin B. \end{aligned}$$

$$(2) \quad \begin{aligned} \cos (A + B) &= \cos A \cos B - \sin A \sin B. \\ \cos (A - B) &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

$$(3) \quad \begin{aligned} \tan (A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B}. \\ \tan (A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}. \end{aligned}$$

The Double-Angle Formulas. If the two angles considered in (1) and (2) are equal, (i.e., if $A = B$), then the addition formulas (1) and (2) become respectively

$$\begin{aligned} \sin (A + A) &= \sin A \cos A + \cos A \sin A, \\ \cos (A + A) &= \cos A \cos A - \sin A \sin A. \end{aligned}$$

or

$$(4) \quad \begin{aligned} \sin 2A &= 2 \sin A \cos A. \\ \cos 2A &= \cos^2 A - \sin^2 A. \end{aligned}$$

The Half-Angle Formulas. Two more expressions can be derived from the second of formulas (4) by the substitutions $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$. The results of these substitutions are

$$\begin{aligned} (5) \quad & 2 \sin^2 A = 1 - \cos 2A. \\ & 2 \cos^2 A = 1 + \cos 2A. \end{aligned}$$

Formulas (5) are true for all values of A . They remain true, therefore, if A is replaced by $\frac{A}{2}$ and $2A$ by A .

$$\begin{aligned} (6) \quad & 2 \sin^2 \frac{A}{2} = 1 - \cos A. \\ & 2 \cos^2 \frac{A}{2} = 1 + \cos A. \end{aligned}$$

Using (6), it is easy to compute $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of A .

$$\begin{aligned} (7) \quad & \sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}. \\ & \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}. \end{aligned}$$

The sign $+$ is chosen in these formulas if $\sin \frac{A}{2}$ (or $\cos \frac{A}{2}$) is positive; the sign $-$ is chosen if $\sin \frac{A}{2}$ (or $\cos \frac{A}{2}$) is negative.

The above formulas are often useful in the manipulation of trigonometric expressions. Corresponding formulas for the other trigonometric functions are given in exercises of this section; the proofs are left to the student.

Various examples using the formulas (1)–(7) of this section will now be shown.

Example 1. Evaluate $\sin 90^\circ$ using $90^\circ = 60^\circ + 30^\circ$.

Substituting $A + B = 60^\circ + 30^\circ$ in the addition formula of (1),

$$\sin 90^\circ = \sin (60^\circ + 30^\circ) = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ,$$

and evaluating,

$$\sin 90^\circ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

which is the correct value.

Example 2. Given $\sin A = \frac{7}{8}$ and $\cos B = -\frac{1}{4}$, where both A and B are in the second quadrant, evaluate $\cos(A - B)$ and $\tan(A + B)$.

The formulas to be used are

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

However, these require the evaluation of other functions of A and B , which is easily accomplished by forming the triangles of reference in the proper quadrants (Fig. 10-1).

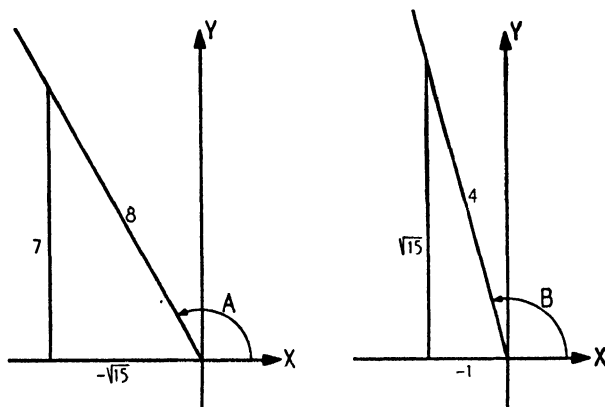


FIG. 10-1.

Now, substituting the proper values,

$$\cos(A - B) = \left(-\frac{\sqrt{15}}{8}\right)\left(-\frac{1}{4}\right) + \left(\frac{7}{8}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{\sqrt{15}}{4 \cdot 8} (1 + 7) = \frac{\sqrt{15}}{4},$$

$$\tan(A + B) = \frac{\left(-\frac{7}{\sqrt{15}}\right) + \left(-\frac{\sqrt{15}}{1}\right)}{1 - \left(-\frac{7}{\sqrt{15}}\right)\left(-\frac{\sqrt{15}}{1}\right)} = \frac{\frac{-7 - 15}{\sqrt{15}}}{1 - 7} = \frac{22}{\sqrt{15}} \cdot \frac{1}{6} = \frac{11\sqrt{15}}{45}.$$

Example 3. Express $\sin 6x$ in terms of functions of (a) $3x$, (b) $12x$.
Substituting in the first of formulas (4) and (7),

$$(a) \sin 6x = 2 \sin 3x \cos 3x,$$

$$(b) \sin 6x = \pm \sqrt{\frac{1 - \cos 12x}{2}}.$$

Example 4. Given that $\cos 30^\circ = \frac{\sqrt{3}}{2}$; find (a) $\cos 60^\circ$, (b) $\cos 15^\circ$.

By the formulas of this section

$$\begin{aligned} (a) \cos 60^\circ &= 2 \cos^2 30^\circ - 1 \\ &= 2 \left(\frac{\sqrt{3}}{2}\right)^2 - 1 = \frac{3}{2} - 1 = \frac{1}{2}, \end{aligned}$$

$$(b) \cos 15^\circ = +\sqrt{\frac{1 + \cos 30^\circ}{2}} = +\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{3}}.$$

Why was the plus sign chosen before the radical?

Example 5. Show that

$$E_1 E_2 \sin \omega t \cos \omega t - E_1 E_2 \cos (\omega t + 90^\circ) = \frac{E_1 E_2}{2} [\sin 2\omega t + 2 \sin \omega t].$$

Multiplying the first term of the left-hand side by $\frac{2}{2}$ and expanding the second term by the proper formula, we obtain

$$E_1 E_2 \left[\frac{2 \sin \omega t \cos \omega t}{2} - \{ \cos \omega t \cos 90^\circ - \sin \omega t \sin 90^\circ \} \right].$$

Substituting $\sin 2\omega t$ for $2 \sin \omega t \cos \omega t$ and evaluating, we get

$$E_1 E_2 \left[\frac{\sin 2\omega t}{2} - 0 + \sin \omega t \right] = \frac{E_1 E_2}{2} [\sin 2\omega t + 2 \sin \omega t].$$

EXERCISES

Using the fundamental identities and formulas (1)–(7) of this section, prove the following identities or formulas.

$$1. \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

$$2. \cot (A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

$$3. \cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

$$4. \frac{1}{\sec (A \pm B)} = \frac{\csc A \csc B \mp \sec A \sec B}{\csc A \csc B \sec A \sec B}.$$

$$5. \frac{1}{\csc (A \pm B)} = \frac{\sec A \csc B \pm \csc A \sec B}{\csc A \csc B \sec A \sec B}.$$

$$6. \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$7. \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$$

$$8. \cos 2A = \frac{\csc^2 A - \sec^2 A}{\csc^2 A \sec^2 A}.$$

$$9. \sin 2A = \frac{2}{\sec A \csc A}.$$

$$10. \tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}.$$

$$11. \tan A = \frac{1 - \cos 2A}{\sin 2A}.$$

$$12. \tan A = \frac{\sin 2A}{1 + \cos 2A}.$$

13. Develop a formula for $\cot A$ in terms of functions of the angle $2A$.

14. Develop a formula for $\sec A$ in terms of functions of the angle $2A$.

15. Develop a formula for $\csc A$ in terms of functions of the angle $2A$.

16-30. For each of the identities in Exercises 1-15, determine the values of the unknown angles for which the equality fails.

By the addition formulas (1) and (2) find, without tables, the sine, cosine, and tangent of:

$$31. 60^\circ, \text{ using } 60^\circ = 30^\circ + 30^\circ.$$

$$33. 75^\circ.$$

$$35. 90^\circ.$$

$$37. -345^\circ.$$

$$39. -\frac{\pi}{12}.$$

$$32. 60^\circ, \text{ using } 60^\circ = 90^\circ - 30^\circ.$$

$$34. 15^\circ.$$

$$36. 195^\circ.$$

$$38. \left(\pi - \frac{\pi}{4}\right).$$

$$40. \frac{11\pi}{12}.$$

Expand and simplify.

$$41. \cos\left(\frac{3\pi}{2} - \beta\right).$$

$$43. \sin(30^\circ - A).$$

$$45. \cos\left(\alpha + \frac{\pi}{3}\right).$$

$$47. \cos(-A) = \cos(0^\circ - A).$$

$$49. \tan(-A).$$

$$42. \sin\left(\frac{\pi}{2} + \theta\right).$$

$$44. \sin(A - 30^\circ).$$

$$46. \tan\left(\frac{5\pi}{6} + \phi\right).$$

$$48. \sin(-A).$$

$$50. \sin[\pi + (\alpha + \beta)].$$

Evaluate $\sin(A + B)$, $\cos(A + B)$, $\sin(A - B)$, and $\cos(A - B)$ in each of the following cases.

$$51. \text{ Given } \cos A = \frac{1}{3}, \cos B = \frac{1}{2}, A \text{ and } B \text{ in the first quadrant.}$$

$$52. \text{ Given } \sin A = \frac{4}{5}, \cos B = \frac{5}{13}, A \text{ in second, } B \text{ in first quadrant.}$$

$$53. \text{ Given } \tan A = \frac{1}{2}, \cot B = 3, \text{ both acute angles.}$$

$$54. \text{ Given } \sec A = 4, \sin B = -\frac{1}{4}, \sin A \text{ negative, } \cos B \text{ negative.}$$

$$55. \text{ Given } \sin A = \frac{3}{5}, \cos B = -\frac{1}{3}, A \text{ and } B \text{ in same quadrant.}$$

$$56. \text{ Given } \cot A = -\frac{2}{3}, \sec B = \frac{\sqrt{13}}{4}, 180^\circ < A < 360^\circ, 0^\circ < B < 270^\circ.$$

$$57. \text{ Given } \sin A = \frac{2}{3}, \cos B = \frac{3}{5}, A \text{ and } B \text{ in same quadrant.}$$

$$58. \text{ Given } \cot A = \frac{\sqrt{7}}{2}, \csc B = \frac{1}{2}, 90^\circ < A < 360^\circ, 90^\circ < B < 360^\circ.$$

$$59. \text{ Given } \csc A = -1, \csc B = -2, \cos B > 0.$$

$$60. \text{ Given } \tan A = 2, \sin B = \frac{1}{3}, A \text{ and } B \text{ in first quadrant.}$$

In Exercises 61-70 express the sine and cosine of the given angle in terms of functions of an angle (a) half as large (b) twice as large.

61. 40° .

Example:

$$(a) \sin 40^\circ = 2 \sin 20^\circ \cos 20^\circ, \\ \cos 40^\circ = \cos^2 20^\circ - \sin^2 20^\circ.$$

$$(b) \sin 40^\circ = \sqrt{\frac{1 - \cos 80^\circ}{2}}, \\ \cos 40^\circ = \sqrt{\frac{1 + \cos 80^\circ}{2}}.$$

62. 142° .

63. 200° .

64. 4θ .

65. $6x$.

66. $\frac{\alpha}{2}$.

67. π .

68. $\frac{7\pi}{8}$.

69. 18θ .

70. $\frac{3}{2}A$.

Using the double-angle formulas find the sine and cosine of

71. 90° .

72. 60° .

73. 120° .

74. 360° .

75. 270° .

76. 0° .

Using the half-angle formulas find the sine and cosine of

77. 15° .

78. 180° .

79. 30° .

80. 45° .

81. 90° .

82. 0° .

Find $\sin 2A$, $\cos 2A$, $\sin \frac{A}{2}$, $\cos \frac{A}{2}$.

83. Given $\sin A = \frac{1}{2}$, A in second quadrant.

84. Given $\cos A = \frac{3}{4}$, A in fourth quadrant.

85. Given $\cot A = -\frac{1}{4}$, $\sin A > 0$.

86. Given $\tan A = -4$, A in third quadrant.

87. Given $\sec A = -3$, A in third quadrant.

88. Given $\csc A = \frac{1}{2}$, $450^\circ < A < 540^\circ$.

89. Given $\cos A = -1$, $0^\circ < A < 360^\circ$.

90. Given $\sin A = \frac{1}{2}$, A in first quadrant.

In each of the following exercises transform the expression on the left-hand side of the equal sign into that on the right-hand side. These are transformations used in electrical engineering.

$$91. \frac{R(I_m \sin \alpha)^2}{\pi} = \frac{RI_m^2}{2\pi} (1 - \cos 2\alpha).$$

$$92. \frac{E_m \sin \alpha \cos \theta}{\beta - \alpha} - \frac{E_m \cos \alpha \sin \theta}{\alpha - \beta} = \frac{E_m}{\beta - \alpha} \sin (\alpha + \theta).$$

$$93. \frac{EI \cos \theta}{2\pi} - \frac{EI \cos (2\alpha + \theta)}{2\pi} = \frac{EI}{2\pi} [\cos \theta - \cos 2\alpha \cos \theta + \sin 2\alpha \sin \theta].$$

94. Given $p = ei$, where $e = E_m \sin \omega t$ and $i = I_m \sin \omega t$. Show that

$$p = \frac{E_m I_m}{2} (1 - \cos 2\omega t).$$

95. If $P_r = EI \cos \theta$, $P_x = EI \sin \theta$, show that

$$\sqrt{P_r^2 + P_x^2} = EI.$$

96. Show that

$$E_m[\sin \omega t - \sin (\omega t + 90^\circ)] = \sqrt{2} E_m \sin (\omega t - 45^\circ).$$

Hint. From Chapter 4, $\sin (\omega t + 90^\circ) = \cos \omega t$.

97. Show that

$$\begin{aligned} E_m \sin \omega t - E_m \sin (\omega t + 240^\circ) &= E_m \left[\frac{3}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t \right] \\ &= \sqrt{3} E_m \sin (\omega t + 30^\circ). \end{aligned}$$

98. Show that $E_m \sin (\omega t + 120^\circ) - E_m \sin \omega t = \sqrt{3} E_m \sin (\omega t + 150^\circ)$.

99. Show that $E_m \sin (\omega t + 240^\circ) - E_m \sin (\omega t + 120^\circ) = \sqrt{3} E_m \sin (\omega t + 270^\circ)$.

100. Show that $E_m \sin (\omega t - 120^\circ) = E_m \sin (\omega t + 240^\circ)$.

101. Show that $E_m \sin (\omega t - 240^\circ) = E_m \sin (\omega t + 120^\circ)$.

102. Show that

$$\begin{aligned} [A \sin (\alpha + \theta_1)][B \sin (\alpha + \theta_2)] &= AB \left[\frac{1 - \cos 2\alpha}{2} \cos \theta_1 \cos \theta_2 + \right. \\ &\quad \left. \frac{\sin 2\alpha}{2} \cos \theta_1 \sin \theta_2 + \frac{\sin 2\alpha}{2} \cos \theta_2 \sin \theta_1 + \frac{1 + \cos 2\alpha}{2} \sin \theta_1 \sin \theta_2 \right]. \end{aligned}$$

10-3. Other Trigonometric Formulas. The following formulas are also used in mathematics and engineering for the manipulation of trigonometric expressions.

The product formulas,

$$\sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)],$$

$$(1) \quad \sin A \cos B = \frac{1}{2} [\sin (A - B) + \sin (A + B)],$$

$$\cos A \cos B = \frac{1}{2} [\cos (A - B) + \cos (A + B)],$$

are useful in expressing products of sines and cosines in terms of sums of those functions.

The proofs follow readily from the addition formulas. For example, to prove the first of the above identities, the addition formula

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

is subtracted from

$$\cos (A - B) = \cos A \cos B + \sin A \sin B,$$

giving

$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B$$

which may be solved for the desired result:

$$\sin A \sin B = \frac{1}{2}[\cos (A - B) - \cos (A + B)].$$

In similar fashion the other product identities can be readily derived by adding or subtracting the proper pair of formulas in (1) and (2) of Sec. 10-2.

The sum formulas,

$$\sin A + \sin B = 2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right),$$

$$\sin A - \sin B = 2 \cos \left(\frac{A + B}{2} \right) \sin \left(\frac{A - B}{2} \right),$$

$$\cos A + \cos B = 2 \cos \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right),$$

(2)

$$\cos A - \cos B = -2 \sin \left(\frac{A + B}{2} \right) \sin \left(\frac{A - B}{2} \right),$$

$$\sin A + \cos B = 2 \sin \left(\frac{A - B}{2} + 45^\circ \right) \cos \left(\frac{A + B}{2} - 45^\circ \right),$$

$$\sin A - \cos B = 2 \cos \left(\frac{A - B}{2} + 45^\circ \right) \sin \left(\frac{A + B}{2} - 45^\circ \right),$$

are utilized when it is necessary to change a sum of sines and cosines into a product of similar functions. These formulas can be derived from the product formulas discussed in the first part of this section.

Example 1. Express $\sin 7y \cdot \cos \frac{y}{2}$ as a sum of trigonometric functions.

Using the second formula of (1), and taking $A = 7y$, $B = \frac{y}{2}$,

$$\begin{aligned} \sin 7y \cdot \cos \frac{y}{2} &= \frac{1}{2} \left[\sin \left(7y - \frac{y}{2} \right) + \sin \left(7y + \frac{y}{2} \right) \right] \\ &= \frac{1}{2} \left[\sin \frac{13y}{2} + \sin \frac{15y}{2} \right]. \end{aligned}$$

Example 2. Express $\left[\cos \left(\omega t - \frac{\pi}{2} \right) - \cos \omega t \right]$ as a product of trigonometric functions.

For the difference of cosines the fourth identity of (2) is used. Taking $A = \left(\omega t - \frac{\pi}{2}\right)$, $B = \omega t$,

$$\begin{aligned}\cos\left(\omega t - \frac{\pi}{2}\right) - \cos \omega t &= -2 \sin\left(\frac{2\omega t - \frac{\pi}{2}}{2}\right) \sin\left(\frac{-\frac{\pi}{2}}{2}\right) \\ &= -2 \sin\left(\omega t - \frac{\pi}{4}\right) \sin\left(-\frac{\pi}{4}\right),\end{aligned}$$

and since $\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$,

$$\cos\left(\omega' - \frac{\pi}{2}\right) - \cos \omega t = \sqrt{2} \sin\left(\omega t - \frac{\pi}{4}\right).$$

An alternative method for achieving the final result would have been the substitution $\sin \omega t = \cos\left(\frac{\pi}{2} - \omega t\right) = \cos\left(\omega t - \frac{\pi}{2}\right)$ in the given expression, resulting in $\sin \omega t - \cos \omega t$. This could then have been expressed as a product by the last identity of group (2).

EXERCISES

Derive the formulas:

1. $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)].$
2. $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)].$
3. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)].$
4. $\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right).$
5. $\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right).$
6. $\sin A + \cos B = 2 \sin\left(\frac{A - B}{2} + 45^\circ\right) \cos\left(\frac{A + B}{2} - 45^\circ\right).$
7. $\sin A - \cos B = 2 \cos\left(\frac{A - B}{2} + 45^\circ\right) \sin\left(\frac{A + B}{2} - 45^\circ\right).$

Write the following as sums, using (1) of Sec. 10-3.

- | | |
|---|--|
| 8. $\sin 30^\circ \cos 50^\circ.$ | 9. $\sin 70^\circ \sin 40^\circ.$ |
| 10. $\cos 2\theta \cos 4\theta.$ | 11. $\sin 6x \cos 3x.$ |
| 12. $\cos \frac{\theta}{2} \cos \theta.$ | 13. $\sin \frac{\pi}{7} \sin \frac{\pi}{14}.$ |
| 14. $\sin(2x - \alpha) \cos(2x + \alpha).$ | 15. $\cos \alpha t \cos(-\alpha t).$ |
| 16. $\sin \omega t \sin(\omega t + \alpha).$ | 17. $\cos(\theta + \alpha + \phi) \cos(\theta - \alpha + \phi).$ |
| 18. $\cos(6\omega t - \alpha) \sin(2\omega t + 2\alpha).$ | 19. $\sin\left(x - \frac{\pi}{6}\right) \cos\left(\frac{\pi}{6} - x\right).$ |

Write the following as products, using (2) of Sec. 10-3.

20. $\cos 60^\circ + \cos 40^\circ$. 21. $\sin \frac{\pi}{8} - \sin \frac{2\pi}{3}$.
22. $\sin \theta + \sin \frac{\theta}{4}$. 23. $\cos 2\alpha - \sin 4\alpha$.
24. $\cos \frac{3x}{2} - \cos 2x$. 25. $\sin 20\omega + \cos 30\omega$.
26. $\cos (2\alpha - 60^\circ) + \cos (2\alpha + 60^\circ)$. 27. $\sin (\omega t - \alpha) - \sin (\alpha - 3\omega t)$.
28. $\sin (2\omega t + 3\theta) + \sin (2\omega t + 3\theta)$. 29. $\cos 4\omega t - \cos \left(10\omega t - \frac{7\pi}{6}\right)$.
30. $\sin (\alpha - \beta) - \cos (3\alpha - 7\beta)$.
31. $\cos \left(x + y + \frac{3\pi}{2}\right) - \sin \left(x - y - \frac{7\pi}{2}\right)$.

In each of the following exercises convert the left-hand side of each of the equalities into the right-hand side by means of the identities in this chapter, particularly those of Sec. 10-3.

32. $\sin 150^\circ + \cos 60^\circ = 8 \frac{\sin 90^\circ}{2} \frac{\cos 60^\circ}{2}$.
33. $\sin 150^\circ - \cos 60^\circ = \sqrt{3} \cos 90^\circ$.
34. $\sin 3\theta + \cos 5\theta = [\cos \theta - \sin \theta][\cos 4\theta + \sin 4\theta]$.
35. $\sin A + \sin (A + 2\pi) = -2 \sin (A + \pi)$.
36. $\frac{\cos 2x - \cos 6x}{\sin 6x - \sin 2x} = \tan 4x$.
37. $\frac{\sin 5\alpha + \sin 3\alpha}{\cos 5\alpha - \cos 3\alpha} = -\cot \alpha$.
38. $\frac{\cos 47^\circ + \cos (-133^\circ)}{\sin 47^\circ - \sin (-133^\circ)} = 0$.
39. $\frac{\sin 110^\circ \cos 20^\circ}{1 + \sin 130^\circ} = \frac{1}{2}$.
40. $\cos 191^\circ \cos 11^\circ = -\sin^2 101^\circ$.
41. $\sqrt{\cos \frac{29\pi}{22} \cos \frac{15\pi}{22}} = \cos \frac{7\pi}{22}$.
42. $\frac{\sin 10\alpha + \sin 14\alpha - \sin 4\alpha}{\sin 10\alpha + \sin 14\alpha + \sin 4\alpha} = \tan 5\alpha \cot 7\alpha$.
43. $\frac{\tan \alpha}{\cot \beta \cot \gamma} = \tan \alpha + \tan \beta + \tan \gamma$.
44. $\frac{\sec A + \tan A - 1}{\sec A + \tan A + 1} = \tan \left(\frac{1}{2} A\right)$.

Follow the above procedure in Exercises 45-50. These are manipulations used by electrical engineers.

45. If $\cos \theta_n = \frac{A_n}{\sqrt{A_n^2 + B_n^2}}$, show that if $0^\circ \leq \theta_n \leq 90^\circ$ and $B_n > 0$,
- $$A_n \sin n\alpha + B_n \cos n\alpha = \sqrt{A_n^2 + B_n^2} \sin (n\alpha + \theta_n).$$

46. If $p = ei$, where $e = E_m \sin \alpha$ and $i = I_m \sin (\alpha + \theta)$, show that

$$p = \frac{E_m I_m}{2} \cos \theta - \frac{E_m I_m}{2} \cos (2\alpha + \theta).$$

47. Given $e_1 = 110\sqrt{2} \sin \left(\omega t + \frac{\pi}{2} \right)$, $e_2 = 110\sqrt{2} \sin \omega t$, show that

$$e_1 + e_2 = 220 \sin \left(\omega t + \frac{\pi}{4} \right).$$

48. If $p = ei$, where $i = I_m \sin \alpha$ and $e = E_m \sin (\alpha + 90^\circ)$, show that

$$p = \frac{1}{2} E_m I_m \sin 2\alpha.$$

49. What will be the result in Exercise 48, if $e = E_m \sin (\alpha - 90^\circ)$?

50. Express as a product of trigonometric functions:

$$e = 10 \sin \omega t + 10 \sin (3\omega t + \theta).$$

51. In a circuit, the transferred power P is given by

$$P = \frac{E^2 Z_L \cos \theta}{[Z \cos \phi + Z_L \cos \theta]^2 + [Z \sin \phi + Z_L \sin \theta]^2}.$$

This has a maximum value when $Z_L = Z$. Show that

$$P_{\max} = \frac{E^2 \cos \theta}{2Z[1 + \cos (\phi - \theta)]}$$

by substituting $Z_L = Z$ in the formula for P .

10-4. A Useful Application of the Trigonometric Identities. In Chapter 5 it was shown that a function which is the sum or difference of two or more functions may be graphed by composition of ordinates. Various questions arise in connection with this process, particularly when the elementary functions used are periodic. One question which is of particular interest to the engineer is: "Under what conditions will a curve which is the sum or difference of sine and cosine curves again be a sine or cosine curve?"

The question may immediately be simplified by recalling from Chapter 4 that for all values of x , $\cos x = \sin \left(\frac{\pi}{2} - x \right)$. Hence *any sum or difference of sine and cosine functions may be rewritten as a sum or difference of sine functions alone.*

Example 1. The function $f(t) = \sin 2t - 4 \cos 3t + \cos \left(2t - \frac{\pi}{8} \right)$ may be rewritten with sine functions only:

$$\begin{aligned} f(t) &= \sin 2t - 4 \sin \left(\frac{\pi}{2} - 3t \right) + \sin \left[\frac{\pi}{2} - \left(2t - \frac{\pi}{8} \right) \right] \\ &= \sin 2t + 4 \sin \left(3t - \frac{\pi}{2} \right) - \sin \left(2t - \frac{5\pi}{8} \right). \end{aligned}$$

Hence, the question now is: "Under what conditions will a curve which is the sum or difference of sine curves again be a sine curve?" We shall consider two cases. First, the sum of two sine functions with unequal periods, and second, the sum of two sine functions with equal periods. It will be shown in the following example that the sum of two sine functions with unequal periods is not necessarily a sine curve.

Example 2. By composition of ordinates show that

$$y = \sin 2x + \sin x$$

is not a sine function.

Graphing $y_1 = \sin x$ and $y_2 = \sin 2x$ (Fig. 10-2) whose periods are 2π and π respectively, we see that the composite function $y = \sin x + \sin 2x$ obviously is not a sine function.

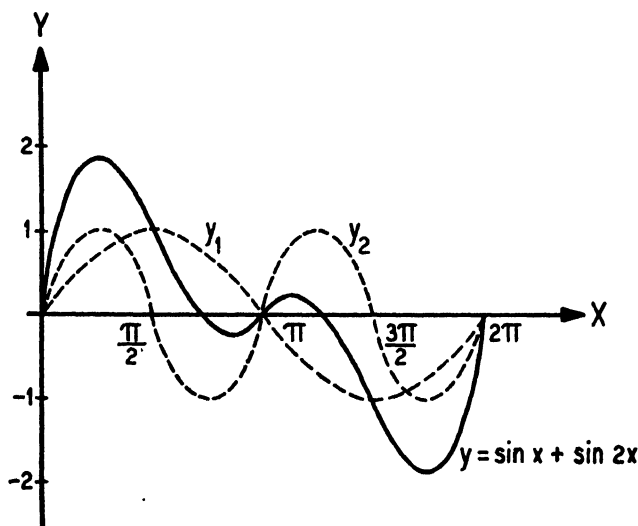


FIG. 10-2.

In the second case a very useful theorem can be stated:

The sum of the two sine functions which have the same period,

$$A_1 \sin (\omega t + \alpha_1) + A_2 \sin (\omega t + \alpha_2),$$

is again a sine function

$$A \sin (\omega t + \alpha)$$

with amplitude $A = \sqrt{M^2 + N^2}$ and initial phase α , α being an angle in standard position whose terminal side passes through (M, N) , where M and N are given by the relations

$$M = A_1 \cos \alpha_1 + A_2 \cos \alpha_2,$$

$$N = A_1 \sin \alpha_1 + A_2 \sin \alpha_2.$$

The proof of this theorem is not difficult. Consider the sum function

$$f(t) = A_1 \sin(\omega t + \alpha_1) + A_2 \sin(\omega t + \alpha_2).$$

Using the addition formula (1) of Sec. 10-2, the above sum may be expanded into the form

$$\begin{aligned} f(t) &= A_1 \sin \omega t \cos \alpha_1 + A_1 \cos \omega t \sin \alpha_1 + A_2 \sin \omega t \cos \alpha_2 + \\ &\quad A_2 \cos \omega t \sin \alpha_2 \\ &= [A_1 \cos \alpha_1 + A_2 \cos \alpha_2] \sin \omega t + [A_1 \sin \alpha_1 + A_2 \sin \alpha_2] \cos \omega t, \end{aligned}$$

and, if we set

$$\begin{aligned} (1) \quad M &= A_1 \cos \alpha_1 + A_2 \cos \alpha_2, \\ N &= A_1 \sin \alpha_1 + A_2 \sin \alpha_2, \end{aligned}$$

this may be written

$$(2) \quad f(t) = M \sin \omega t + N \cos \omega t,$$

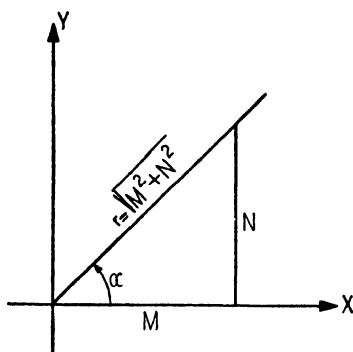


FIG. 10-3.

where M and N are determined by $A_1, A_2, \alpha_1, \alpha_2$. We can now find an angle α such that (M, N) is on its terminal side. From the triangle of reference of α (Fig. 10-3)

$$\begin{aligned} (3) \quad \sin \alpha &= \frac{N}{\sqrt{M^2 + N^2}}, \\ \cos \alpha &= \frac{M}{\sqrt{M^2 + N^2}}. \end{aligned}$$

Thus, if we multiply and divide the right-hand side of (2) by $\sqrt{M^2 + N^2}$,

$$f(t) = \sqrt{M^2 + N^2} \left[\frac{M}{\sqrt{M^2 + N^2}} \sin \omega t + \frac{N}{\sqrt{M^2 + N^2}} \cos \omega t \right],$$

and by (4)

$$f(t) = \sqrt{M^2 + N^2} [\sin \omega t \cos \alpha + \cos \omega t \sin \alpha].$$

Using the addition formula (1) of Sec. 10-2

$$f(t) = \sqrt{M^2 + N^2} \sin(\omega t + \alpha);$$

we thus obtain

$$A_1 \sin(\omega t + \alpha_1) + A_2 \sin(\omega t + \alpha_2) = \sqrt{M^2 + N^2} \sin(\omega t + \alpha)$$

where M and N are given by (1) and α is given by (3). In short, *the sum of the two given sine functions is again a sine function with amplitude $\sqrt{M^2 + N^2}$ and initial phase α .*

It is obvious that the sum of any number of sine functions with the same period is again a sine function with the same period, for by taking two functions at a time and applying the theorem just discussed the sum can finally be reduced to a single sine function.

Example 3. Express $\sin t + 2 \sin(t + \pi)$ as a sine function.

For purposes of verification, we will first perform the addition graphically by composition of ordinates. Adding the ordinates of graphs $y_1 = \sin t$ and $y_2 = 2 \sin(t + \pi)$, we obtain the graph $y = \sin t + 2 \sin(t + \pi)$, which appears to be also the graph of $y = \sin(t + \pi)$, (Fig. 10-4).

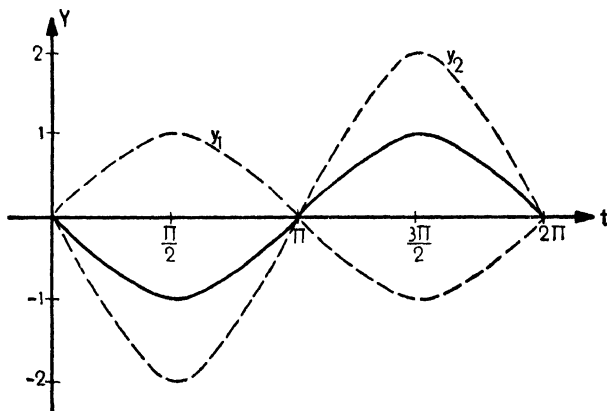


FIG. 10-4.

Let us perform this operation by the methods of this section. For this example $\omega = 1$, $A_1 = 1$, $A_2 = 2$, $\alpha_1 = 0$, $\alpha_2 = \pi$. Hence by the relations (1)

$$M = 1 \cos 0 + 2 \cos \pi = 1 - 2 = -1,$$

$$N = 1 \sin 0 + 2 \sin \pi = 0 + 0 = 0.$$

Thus $\alpha = +\pi$; also

$$A = \sqrt{M^2 + N^2} = \sqrt{1 + 0} = 1.$$

Hence, by the theorem the sum is

$$A \sin(\omega t + \alpha) = \sin(t + \pi),$$

as was predicted from the graph.

Example 4. Given that

$$e = E_{m_1} \sin \omega t + E_{m_2} \sin (\omega t - 90^\circ),$$

show that

$$e = E_m \sin (\omega t - B),$$

where

$$E_m = \sqrt{E_{m_1}^2 + E_{m_2}^2} \quad \text{and} \quad \tan B = \frac{E_{m_2}}{E_{m_1}}.$$

(Assume that $E_{m_1} \geq 0$ and $E_{m_2} \geq 0$.)

To use the theorem of this section, take $\omega = \omega$, $A_1 = E_{m_1}$, $A_2 = E_{m_2}$, $\alpha_1 = 0$, $\alpha_2 = -90^\circ$. Then by the relations (1)

$$M = E_{m_1} \cos 0^\circ + E_{m_2} \cos (-90^\circ) = E_{m_1}.$$

$$N = E_{m_1} \sin 0^\circ + E_{m_2} \sin (-90^\circ) = -E_{m_2}.$$

Therefore, in the wave $A \sin (\omega t + \alpha)$ obtained by addition,

$$A = \sqrt{M^2 + N^2} = \sqrt{E_{m_1}^2 + E_{m_2}^2},$$

and

$$\tan \alpha = \frac{N}{M} = -\frac{E_{m_2}}{E_{m_1}},$$

giving

$$e = \sqrt{E_{m_1}^2 + E_{m_2}^2} \sin (\omega t + \alpha).$$

If we take $\alpha = -B$, then

$$e = \sqrt{E_{m_1}^2 + E_{m_2}^2} \sin (\omega t - B)$$

where $\tan (-B) = -\frac{E_{m_2}}{E_{m_1}}$ or $\tan B = \frac{E_{m_2}}{E_{m_1}}$.

EXERCISES

Express as a single sine function:

1. $\sin x + 2 \sin x$.
2. $\sin x + 2 \cos x$.
3. $2 \sin \omega t + 3 \cos \omega t$.
4. $\cos \left(\omega t - \frac{\pi}{4} \right) + 2 \cos \omega t$.
5. $\sin (x - 30^\circ) + 3 \sin (x - 60^\circ)$.
6. $3 \cos (x - 30^\circ) - \frac{3}{2} \sin (x + 30^\circ)$.
7. $3.7 \sin \left(x - \frac{\pi}{8} \right) - 4.3 \sin \left(x + \frac{\pi}{7} \right)$.
8. $2.1 \sin \left(x + \frac{3\pi}{4} \right) + 1.8 \sin \left(x - \frac{\pi}{8} \right)$.
9. $100 \sin (x - 46.2^\circ) - 110 \cos (x - 41^\circ)$.
10. $11 \cos (\omega t + 2) + \sin (\omega t - 3.75)$.
11. $2 \sin \left(\omega t + \frac{\pi}{4} \right) + 2 \sin \left(\omega t - \frac{3\pi}{4} \right)$.
12. $7.3 \sin (x - 17.8^\circ) + 8.5 \cos (x + 11.0^\circ)$.
13. $21.9 \sin (x - 22.2^\circ) - 32.2 \sin (x + 11.1^\circ)$.

$$14. 62.2 \sin (\omega t - 16^\circ) + 51 \cos \left(\frac{\omega t}{2} - 8^\circ \right).$$

$$15. 3 \sin (\omega t + 37.1^\circ) + 6 \sin (3\omega t - 14.2^\circ).$$

$$16. 10 \cos \left(\omega t - \frac{7\pi}{18} \right) - \cos \left(\omega t - \frac{11\pi}{18} \right).$$

$$17. 10.2 \sin (\omega t - 28.1^\circ) - 9.1 \sin (\omega t + 11.8^\circ).$$

$$18. 1.7 \sin (\omega t - 12.2^\circ) + 2.2 \cos (\omega t - 17.6^\circ).$$

$$19. 2 \sin \omega t - 3 \cos (\omega t - 20^\circ) + 3 \sin (\omega t + 20^\circ).$$

$$20. 2 \sin \omega t + 3 \sin (\omega t - 30^\circ) + 4 \sin (\omega t - 60^\circ).$$

21. Given that

$$e_0 = I_m R \sin \omega t + I_m X_L \cos \omega t.$$

Show that

$$e_0 = I_m Z \sin (\omega t + \theta)$$

$$\text{where } Z = \sqrt{R^2 + X_L^2} \quad \text{and} \quad \tan \theta = \frac{X_L}{R}.$$

10-5. Trigonometric Equations. In trigonometry as in algebra there exist identical equalities and conditional equalities, the former being true for all values of the angles involved, the latter being true for only certain specific angles. For example, suppose that power in a particular circuit is given experimentally by

$$P = 110 \cos \theta,$$

and it is desired to find for what values of θ the power will be 55. The conditional equation

$$55 = 110 \cos \theta \quad \text{or} \quad \cos \theta = \frac{55}{110} = 0.5$$

is obtained. This is certainly true only for certain values of θ , namely,

$$\theta = \frac{\pi}{3}, \quad \frac{5\pi}{3}, \quad \frac{7\pi}{3}, \quad \frac{11\pi}{3}, \text{ etc.}$$

There are many types of trigonometric equations. Frequently the equations are of such a nature that the methods of algebra may be applied directly.

Example 1. Find all values of θ , $0^\circ \leq \theta < 360^\circ$, which satisfy the equation

$$3 \sin \theta + 2 \sin \theta = 4.$$

This is solved in much the same way as the algebraic equation $3x + 2x = 4$. Combining the terms on the left-hand side,

$$5 \sin \theta = 4,$$

and

$$\sin \theta = \frac{4}{5} = 0.8000.$$

Then from the tables

$$\theta = 53.13^\circ, 126.87^\circ.$$

Example 2. Find all values of x , $0 \leq x < 360^\circ$, for which $\cos 3x = 0.8910$.

Since $\cos 3x$ is not equal to $3 \cos x$, we cannot here divide both sides of the equation by 3. Instead, with the aid of the table we find the possible values for the angle $3x$, noting that to obtain values of x between 0 and 360° we must find values of $3x$ between 0 and $3 \cdot 360^\circ = 1080^\circ$. Thus

$$3x = 27^\circ, 333^\circ, 387^\circ, 693^\circ, 747^\circ, 1053^\circ,$$

and

$$x = 9^\circ, 111^\circ, 129^\circ, 231^\circ, 249^\circ, 351^\circ.$$

Example 3. Find all values of θ , $0^\circ \leq \theta < 360^\circ$, for which $2 \sin^2 \theta + 3 \sin \theta = 2$.

This equation is solved in the same way that the equation $2x^2 + 3x = 2$ would be solved in algebra. Thus, subtracting 2 from both sides,

$$2 \sin^2 \theta + 3 \sin \theta - 2 = 0.$$

Factoring, we obtain

$$(2 \sin \theta - 1)(\sin \theta + 2) = 0.$$

Now, since the product of the two factors may be zero only when one of the factors is zero, this equation will be true if either

$$2 \sin \theta - 1 = 0$$

or

$$\sin \theta + 2 = 0.$$

From these conditions

$$\sin \theta = \frac{1}{2}, \text{ and therefore } \theta = 30^\circ, 150^\circ,$$

or

$$\sin \theta = -2, \text{ which is impossible.}$$

Hence the permissible values of θ are 30° and 150° .

Frequently, before algebraic methods can be used, the form of an equation must be changed by use of the identities already discussed in this chapter. No general procedure can be given which will be successful in all cases. In many equations which involve several different trigonometric functions, it is often expedient to write the equation so that only one function is involved.

Example 4. Find all values of θ , $0^\circ \leq \theta < 360^\circ$, which satisfy the equation $\sin \theta = \csc \theta$.

To write the equation in terms of only one trigonometric function, substitute the identity $\csc \theta = \frac{1}{\sin \theta}$. Thus

$$\sin \theta = \frac{1}{\sin \theta},$$

and

$$\sin^2 \theta = 1.$$

Transposing,

$$\sin^2 \theta - 1 = 0,$$

and factoring,

$$(\sin \theta - 1)(\sin \theta + 1) = 0.$$

If each factor is set equal to zero,

$$\sin \theta = 1, \text{ and therefore } \theta = 90^\circ,$$

$$\sin \theta = -1, \text{ and therefore } \theta = 270^\circ.$$

Thus $\theta = 90^\circ, 270^\circ$ are the only angles between 0° and 360° for which $\sin \theta = \csc \theta$.

In many cases, the method of solving a trigonometric equation must be suggested by the solver's intuition. Each problem may require a new method of attack. The following examples will serve as illustrations.

Example 5. Find $0^\circ \leq \theta < 360^\circ$ such that $\sin 2\theta = \cos \theta$.

Transposing,

$$\sin 2\theta - \cos \theta = 0,$$

and substituting $2 \sin \theta \cos \theta = \sin 2\theta$,

$$2 \sin \theta \cos \theta - \cos \theta = 0.$$

Factoring the left-hand side,

$$\cos \theta (2 \sin \theta - 1) = 0.$$

Thus

$$\cos \theta = 0,$$

$$2 \sin \theta - 1 = 0, \text{ and therefore } \sin \theta = \frac{1}{2},$$

from which

$$\theta = 90^\circ, 270^\circ \text{ and } \theta = 30^\circ, 150^\circ.$$

Thus the solution to our problem is $\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$.

Example 6. Find $0^\circ \leq x < 360^\circ$ such that $2 \sin x - 3 \cos x = -2$.

There are two methods by which this equation may be solved.

Method 1. Substituting the identity $\sin x = \pm \sqrt{1 - \cos^2 x}$, we have

$$\pm 2\sqrt{1 - \cos^2 x} = 3 \cos x - 2.$$

Squaring both sides

$$4 - 4 \cos^2 x = 9 \cos^2 x - 12 \cos x + 4,$$

from which

$$13 \cos^2 x - 12 \cos x = 0.$$

Factoring

$$\cos x (13 \cos x - 12) = 0,$$

whence

$$\cos x = 0, \quad \cos x = \frac{12}{13} = 0.9231,$$

giving

$$x = 90^\circ, 270^\circ \text{ and } x = 22.61^\circ, 337.39^\circ.$$

However since both sides of the equation were squared, extraneous roots may have been introduced. Upon checking,

$$x = 22.61^\circ \text{ and } 270^\circ$$

are the only roots which satisfy the original equation.

Method 2. Since the left-hand side of the equation written as

$$2 \sin x + 3 \sin (x - 90^\circ)$$

is the sum of two sine functions with the same period, the method of the previous section may be used to write this as a single sine function. Here $\omega = 1$, $A_1 = 2$, $A_2 = 3$, $\alpha_1 = 0^\circ$, $\alpha_2 = -90^\circ$, and hence

$$M = 2 \cos 0^\circ + 3 \cos (-90^\circ) = 2,$$

$$N = 2 \sin 0^\circ + 3 \sin (-90^\circ) = -3.$$

Thus for the sum function

$$A = \sqrt{M^2 + N^2} = \sqrt{4 + 9} = \sqrt{13},$$

$$\tan \alpha = \frac{-3}{2}, \text{ and therefore } \alpha = 303.69^\circ,$$

and substituting the sum function $\sqrt{13} \sin (x + 303.69^\circ)$ in the left-hand side of the given equation for $2 \sin x - 3 \cos x$, we have:

$$\sqrt{13} \sin (x + 303.69^\circ) = -2.$$

Now, dividing both sides by $\sqrt{13}$,

$$\sin (x + 303.69^\circ) = -\frac{2}{\sqrt{13}} \quad \text{or} \quad -\frac{2\sqrt{13}}{13} = -0.5547.$$

Thus the angle $x + 303.69^\circ = 213.69^\circ, 326.31^\circ$, and hence

$$x = \begin{cases} 213.69^\circ - 303.69^\circ = -90^\circ = 270^\circ, \\ 326.31^\circ - 303.69^\circ = 22.62^\circ. \end{cases}$$

EXERCISES

In the following exercises, find all the angles greater than or equal to 0° and less than 360° for which the given equation is true.

1. $\sin x = 0.4540$.

2. $\cos x = -0.8990$.

3. $2 \sin \theta = -1.7642$.

4. $1.7 \tan \alpha = 6.7641$.

5. $2 \cos x + 3 \cos x = -4.7162$.

6. $0.5 \cot x + 0.5 \tan (90^\circ - x) = 1.0000$.

7. $3.1 \sin x - 0.8445 = 4.3 \sin x$.

8. $\sec \phi = -2.7777$.

9. $\frac{0.8187}{\csc x} = 0.6800$.

10. $3 \sin \beta = -6.4980$.

11. $\sin 2x = 0.3264$.

12. $\cos 3y = 0.5000$.

13. $\sin \left(\frac{x}{3} \right) = 0.7428$.

14. $\tan \left(\frac{3\theta}{4} \right) = -2.0000$.

15. $\frac{\cos \left(\frac{x}{2} \right)}{2} = -0.1992$.

16. $\frac{7 \sin 1.8t}{8} = 0.7403$.

17. $2 \sin 2x + 3 \sin 2x = 4.8015$.
18. $\sin \left(\frac{2t}{5} \right) = 10 \cos \left(\frac{2t}{5} \right)$.
19. $2 \sec \left(\frac{\theta}{3} \right) = 2.81$.
20. $3 \cos 2x = 86.4900$.
21. $(\sin x - \frac{1}{2})(\sin x + \frac{1}{2}) = 0$.
22. $\sin^2 \theta - \frac{1}{18} = 0$.
23. $\cos^2 \theta - 1 = 0$.
24. $\cos^2 y - \frac{3}{4} = 0$.
25. $(2 \cos x - \sqrt{3})(2 \cos x + \sqrt{2}) = 0$.
26. $4 \cos^2 x - 4\sqrt{3} \cos x + 3 = 0$.
27. $6 \tan^2 x + 12 = 17 \tan x$.
28. $6 \sin x + 8 \sin^2 x = 2$.
29. $6 \sec^2 \alpha - 6 \sec \alpha = 12$.
30. $\sin^2 \alpha - 0.6723 = 0$.
31. $\sin^2 \alpha + 0.6723 = 0$.
32. $\sin \theta - \frac{1}{\sin \theta} = 0$.
33. $4 \cos \theta = \sec \theta$.
34. $\cos^2 x - \sin x - 1 = 0$.
35. $17 - 13 \sin \theta = 24 \cos^2 \theta$.
36. $\sin \theta (2 + 3 \tan \theta) = 0$.
37. $5 \tan x \sin x - 10 \tan x = 0$.
38. $\frac{7 \cos x}{\csc x} + 2 \sin x = 0$.
39. $2 \sec \theta \tan \theta + 10 = 4 \sec \theta + 5 \tan \theta$.
40. $\tan \theta - 2 \sin \theta = 0$.
41. $\cos 2\theta - 2 \sin 4\theta = 0$.
42. $4 \sin 3\beta - 3 \cos 3\beta = 4$.
43. $2 \sin^2 \alpha + \frac{2 \sin \alpha}{\csc \alpha} = 4 - 4 \cos^2 \alpha$.
44. $3 \sin 2x \cos 3x - 3 \cos 2x \sin 3x = -2$.
45. $\cos \frac{\theta}{2} \cos \theta - \sin \frac{\theta}{2} \sin \theta = 0.7142$.
46. $\cos 2t + 3 \cos t + 2 = 0$.
47. $\cos 6x - 3 \cos 3x + 2 = 0$.
48. $\cos^2 x - \sin^2 x = -\cos x$.
49. $\sin \theta \cos 2\theta \tan 3\theta = 0$.
50. $2 \sin^2 \frac{x}{2} + \cos x = 1$.
51. $4 \sin 3\beta - 3 \cos 3\beta = 4$.
52. $\sqrt{3} \sin x - \cos x = 1$.
53. $\sin 3x + \sin x = 0$.
54. $\sin (3x - \pi) - \sin x = 0$.
55. $\cos 3x = \sin x + \cos x$.
56. $\sin \theta + \sin 5\theta = \cos 2\theta$.
57. $1.76 \sin 2x - 2.42 \cos 2x = 6.71$.
58. $10.2 \sin \frac{x}{2} + 14.0 \sin \frac{x}{2} = 8.8$.

PROGRESS REPORT

In this chapter two kinds of trigonometric equations were considered, identities and conditional equations. The identities were used to simplify or change the form of many trigonometric expressions, and their usefulness in this connection for applications was illustrated. Finally, several common types of conditional trigonometric equations were solved.

CHAPTER 11

THE OBLIQUE TRIANGLE

In Chapter 4 we discussed the use of the trigonometric functions in finding unknown sides and angles of a right triangle. However, a right triangle is a special type of triangle, and it is obvious that many problems will arise in which the triangle involved is not a right triangle. A triangle not a right triangle is called an **oblique triangle**. In this chapter we shall develop the **law of sines** and the **law of cosines**, two relations which hold for *any* triangle. These relations permit the solution of any triangle problem in which enough parts are given to determine the triangle. The law of cosines can also be used to find the magnitude of the vector which is the sum of two given vectors. Finally, we shall consider the inverse trigonometric functions.

11-1. The Law of Sines. Throughout this chapter we shall label the sides of a triangle a , b , and c ; the opposite angles will be labeled respectively α , β , and γ ; and the vertices of these angles will be labeled respectively A , B , and C . This is done in Fig. 11-1.

The law of sines states that *in any triangle the sides are proportional to the sines of the opposite angles*. Expressed symbolically:

$$(1) \qquad \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

This law is readily verified. Consider any oblique triangle ABC (Fig. 11-1). In the figure we have the only two possible types of

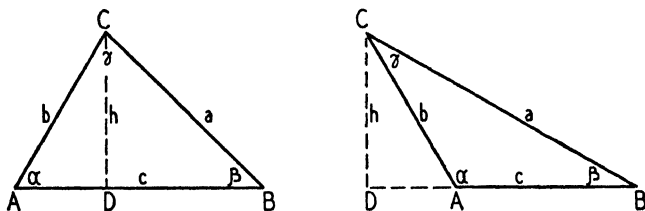


FIG. 11-1.

oblique triangles, the first with three acute angles, the second with one obtuse and two acute angles. The steps in the proof below are valid for both triangles.

From the vertex C draw the altitude h , denoting by D the point at which h meets the opposite side or its projection. Then

$$\sin \alpha = \frac{h}{b} \quad \text{or} \quad h = b \sin \alpha,$$

$$\sin \beta = \frac{h}{a} \quad \text{or} \quad h = a \sin \beta.$$

From these two equalities

$$a \sin \beta = b \sin \alpha.$$

Dividing both sides by the quantity $\sin \beta \sin \alpha$, we obtain

$$(2) \quad \frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

Similarly, the relation

$$(3) \quad \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

may be verified by using the altitude from vertex A to the opposite side. When (2) and (3) are combined, the law of sines as stated in (1) is obtained.

11-2. The Law of Cosines. The law of cosines states that *in any triangle the square of any side is equal to the sum of the squares of the other two sides less twice the product of those sides and the cosine of their included angle*. Expressed symbolically, we have the three forms:

$$(1) \quad a^2 = b^2 + c^2 - 2bc \cos \alpha,$$

$$(2) \quad b^2 = a^2 + c^2 - 2ac \cos \beta,$$

$$(3) \quad c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

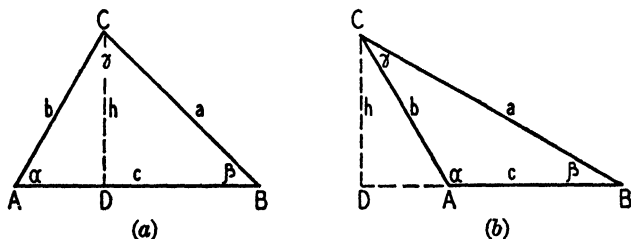


FIG. 11-2.

Consider any oblique triangle ABC (Fig. 11-2). From the vertex C in each triangle draw the altitude h , denoting by D the point at which h meets the opposite side or its projection.

Now for both cases

$$a^2 = h^2 + (DB)^2,$$

and

$$h^2 = b^2 - (DA)^2.$$

Substituting for h^2 from the second relation into the first,

$$(4) \quad a^2 = b^2 - (DA)^2 + (DB)^2.$$

Equality (4) applies to both triangles. The remainder of the proof must be carried out separately for each figure. In

FIG. 11-2a

$$DB = c - DA$$

Substituting for DB in (4),

$$a^2 = b^2 - (DA)^2 + (c - DA)^2$$

or

$$a^2 = b^2 + c^2 - 2c(DA).$$

From the figure,

$$DA = b \cos \alpha$$

whence

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

FIG. 11-2b

$$DB = c + DA$$

Substituting for DB in (4),

$$a^2 = b^2 - (DA)^2 + (c + DA)^2$$

or

$$a^2 = b^2 + c^2 + 2c(DA).$$

From the figure,

$$\begin{aligned} DA &= b \cos (\angle DAC) \\ &= b \cos (180^\circ - \alpha) \\ &= -b \cos \alpha \end{aligned}$$

whence

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

Thus in both cases we have the result given in (1). Relations (2) and (3) may be obtained in like manner by using other altitudes of the triangles.

11-3. The Classification of Oblique Triangle Problems. A triangle has three sides and three angles, a total of six parts. When any three are given, at least one of which is a side, the triangle can be constructed by means of a ruler and compass. When three angles are given, the triangle is not uniquely determined. For more details see the Appendix.

In plane geometry, the ruler and compass serve as the tools for constructing a triangle if any three parts, at least one of which is a side, are given. From the constructed triangle, we can find the missing parts by measurement. In trigonometry the law of sines and the law of cosines are the tools for computing the missing parts of the triangle. Thus

1. In geometry we *construct*; in trigonometry we *compute*.

2. The tools of geometry are *ruler and compass*; the tools of trigonometry are *formulas*.

To use the law of sines, any two of the three ratios in (1) of Sec. 11-1 may be selected to give an equation; for example,

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}.$$

Any such equation contains four parts of the triangle, in this case a , c , α , γ . Therefore, if any three of these parts are known, the fourth may be found by solving the equation. In like manner, each form of the cosine law involves four parts of the triangle. For example,

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

involves b , a , c , β . Here again, if any three of these parts are known, the fourth may be found by solving the equation. Thus, in general, if three parts of a triangle are known and it is desired to find a fourth part, the proper procedure is to select a form of the sine law or the cosine law which contains the four parts mentioned. After substitution of the known values in the pertinent equation, the unknown element can be found.

In order to systematize the procedures for solving triangles, we shall classify all oblique triangle problems into the following four cases.

Case 1. Given two angles and any side.

Case 2. Given two sides and the included angle.

Case 3. Given two sides and an angle opposite one.

Case 4. Given three sides.

In some problems the given parts may be of such a nature that no triangle is possible. In other problems the given parts determine one or two triangles. In many instances these possibilities can be recognized after a sketch of the triangle has been carefully made with a ruler and a protractor. Hence, before attempting to solve a problem by means of trigonometry, it is advisable to construct the possible figure or figures from the given parts. Finer criteria, to be used when a sketch is unsatisfactory, will be given as the various cases are discussed.

The computations may be performed in any convenient manner. If only two- or three-figure accuracy is required, a slide rule having trigonometric scales is most satisfactory. For greater accuracy, logarithms should be used. The computations in the examples of this chapter will be performed with logarithms.

There are various methods of checking the accuracy of computations of this kind. Measurement of the unknown sides and angles in the carefully drawn sketch is usually enough of a check, although it will detect

only major errors. Other more accurate methods of checking these computations are discussed in books on trigonometry, but they will not be considered here.

11-4. Case 1: Given Two Angles and Any Side. This case may be summarized in the following way.

(a) *Number of triangles:*

(1) *None if the sum of the two given angles is greater than or equal to 180° .*

(2) *One if the sum of the two given angles is less than 180° .*

(b) *Method of solution: Law of sines.*

The number of triangles. For the discussion of this case and the geometrical construction, see the Appendix.

The method of solution. When the sum of the two given angles is less than 180° , the difference between 180° and this sum gives the third angle. With the three angles of the triangle known, each of the unknown sides can be found by means of the sine law relation involving an unknown side and the known side.

Example. Given an oblique triangle in which $\alpha = 77.00^\circ$, $\beta = 72.00^\circ$, and $c = 10.20$. Find γ , a , b .

The triangle is sketched in Fig. 11-3. To find γ we have at once

$$\gamma = 180.00^\circ - (77.00^\circ + 72.00^\circ) = 180.00^\circ - 149.00^\circ$$

whence

$$\gamma = 31.00^\circ.$$

Now to find b and a we use

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \quad \text{and} \quad \frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

whence

$$b = \frac{c \sin \beta}{\sin \gamma} = \frac{10.20 \sin 72.00^\circ}{\sin 31.00^\circ},$$

$$a = \frac{c \sin \alpha}{\sin \gamma} = \frac{10.20 \sin 77.00^\circ}{\sin 31.00^\circ}.$$

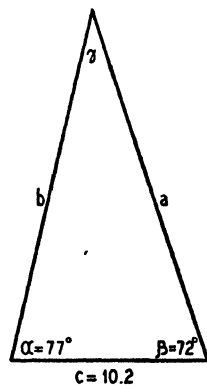


FIG. 11-3.

Performing the computations by logarithms we have the following:

NUMBERS	LOGARITHMS
10.20	1.0086
$\sin 72.00^\circ$	9.9782 - 10
$10.20 \sin 72.00^\circ$	10.9868 - 10
$\sin 31.00^\circ$	9.7118 - 10
b	1.2750

$$b = 18.83$$

NUMBERS	LOGARITHMS
10.20	1.0086
$\sin 77.00^\circ$	9.9887 - 10
$10.20 \sin 77.00^\circ$	10.9973 - 10
$\sin 31.00^\circ$	9.7118 - 10
a	1.2855

$$a = 19.30$$

That these results are correct can be checked roughly by measuring the figure.

When, as in this example, lengths are given accurate to four significant figures, and angles to the nearest hundredth of a degree, computation by four-place logarithms gives results of the same degree of accuracy. When lengths are given accurate to three figures and angles to the nearest tenth of a degree, the logarithmic computations can be carried out with a four-place table precisely as in the example above and the answers rounded off to the same degree of accuracy. Thus in the example, if the data given were $\alpha = 77.0^\circ$, $\beta = 72.0^\circ$, and $c = 10.2$, the work would be the same, but the result would be rounded off and given as $\gamma = 31.0^\circ$, $b = 18.8$, and $a = 19.3$. A similar remark can be made if the data are given to the two significant figures and the nearest degree. The slide rule can, of course, be used if the desired degree of accuracy is three significant figures and the nearest tenth of a degree, or less. (See Sec. 4-3.)

EXERCISES

Solve the following triangles for the three unknown parts. Give results which are as accurate as the data will permit, assuming that they are accurate in accordance with the agreements about significant digits.

1. $\alpha = 21^\circ$, $\beta = 82^\circ$, $a = 11$.
2. $\alpha = 47.1^\circ$, $\beta = 53.3^\circ$, $b = 72.7$.
3. $\beta = 39.9^\circ$, $\gamma = 11.8^\circ$, $b = 2.35$.
4. $\alpha = 111.6^\circ$, $\gamma = 44.8^\circ$, $c = 16.5$.
5. $\gamma = 77.2^\circ$, $\alpha = 13.4^\circ$, $a = 2.22$.
6. $\beta = 100.0^\circ$, $\gamma = 48.4^\circ$, $c = 0.476$.
7. $\alpha = 6.2^\circ$, $\beta = 8.4^\circ$, $c = 21.7$.
8. $\alpha = 71.74^\circ$, $\gamma = 79.18^\circ$, $b = 16.73$.
9. $\gamma = 46.85^\circ$, $\beta = 31.47^\circ$, $a = 1764$.
10. $\beta = 106.7^\circ$, $\gamma = 76.1^\circ$, $a = 11.15$.

In each of the following triangles find the unknown part listed. Give results which are as accurate as the data will permit.

11. $\alpha = 22^\circ$, $\gamma = 14^\circ$, $c = 172$, $\beta =$.
12. $\beta = 108.2^\circ$, $\alpha = 44.3^\circ$, $a = 11.2$, $b =$.
13. $\gamma = 73.4^\circ$, $\beta = 43.4^\circ$, $b = 122$, $c =$.
14. $\beta = 121.2^\circ$, $\gamma = 27.2^\circ$, $\alpha = 31.6^\circ$, $a =$.
15. $\beta = 121.2^\circ$, $\gamma = 27.2^\circ$, $b = 28.4$, $a =$.
16. $\beta = 121.2^\circ$, $\gamma = 27.2^\circ$, $a = 743$, $c =$.
17. $\alpha = 44.7^\circ$, $\beta = 52.3^\circ$, $c = 4.26$, $b =$.
18. $\gamma = 102.3^\circ$, $\alpha = 61.7^\circ$, $c = 137.8$, $a =$.
19. $\beta = 14.20^\circ$, $\gamma = 9.72^\circ$, $b = 0.4467$, $c =$.
20. $\alpha = 113.46^\circ$, $\gamma = 20.98^\circ$, $b = 0.4467$, $a =$.

21. State the law of sines, assuming that one of the angles of the triangle is a right angle.

22. State the law of cosines, assuming that one of the angles of the triangle is a right triangle.

11-5. Case 2: Given Two Sides and the Included Angle. In this case we have the following.

- (a) *Number of triangles: One.*
- (b) *Method of solution: Law of cosines.*

The number of triangles. If the angle included between the two given sides is less than 180° , a single triangle is determined and can be easily constructed (see the Appendix).

The method of solution. The third side can be found by the law of cosines. Then with the three sides known the law of sines can be used to determine the two unknown angles. For example, if a , b , and γ are known, c can be found by the cosine law in the form

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Knowing a , c , and γ , we can find α by the sine law; or knowing b , c , and γ , we can find β by the sine law. Of course, α and β can also be found by using other forms of the cosine law, but the sine law is more convenient for computations. In general, whenever either the law of sines or the law of cosines can be used, we shall use the former, for the law of sines is much better adapted for logarithmic or slide rule computations. An advantage of computing *both* α and β by the sine law is that the relation $\alpha + \beta + \gamma = 180^\circ$ can then be used as a check on the accuracy of the computations.

Example Given an oblique triangle in which $a = 12.4$, $c = 20.7$, $\beta = 26.0^\circ$. Find α , γ , b .

First we make an accurate sketch (Fig. 11-4). Using the law of cosines to find b , we write,

$$b^2 = a^2 + c^2 - 2ac \cos \beta.$$

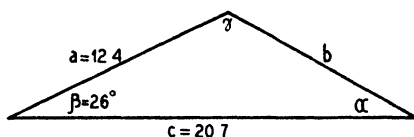


FIG. 11-4.

The computation carried out to three significant numbers gives .

$$a^2 = 12.4^2 = 154, \quad c^2 = 20.7^2 = 428.$$

The term $2ac \cos \beta$ is computed by logarithms:

NUMBERS	LOGARITHMS
2	0.3010
$a = 12.4$	1.0934
$c = 20.7$	1.3160
$\cos \beta = \cos 26.0^\circ$	9.9537 - 10
$2ac \cos \beta = 461$	<u>2.6641</u>

From the computed values

$$b^2 = 154 + 428 - 461 = 121,$$

$$b = 11.0.$$

To find α and γ we use the law of sines.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}, \quad \frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$$

whence

$$\sin \alpha = \frac{a \sin \beta}{b} = \frac{12.4 \sin 26.0^\circ}{11.0},$$

$$\sin \gamma = \frac{c \sin \beta}{b} = \frac{20.7 \sin 26.0^\circ}{11.0}.$$

The corresponding logarithmic computation gives:

NUMBERS	LOGARITHMS	NUMBERS	LOGARITHMS
12.4	1.0934	20.7	1.3160
$\sin 26.0^\circ$	<u>9.6418 - 10</u>	$\sin 26.0^\circ$	<u>9.6418 - 10</u>
$12.4 \sin 26.0^\circ$	10.7352 - 10	$20.7 \sin 26.0^\circ$	10.9578 - 10
11.0	<u>1.0414</u>	11.0	<u>1.0414</u>
$\sin \alpha$	9.6938 - 10	$\sin \gamma$	9.9164 - 10

$$\alpha = 29.6^\circ$$

$$\gamma = 124.4^\circ$$

In finding γ from $\log \sin \gamma$, there is some question as to whether γ is acute or obtuse, that is, whether $\gamma = 55.6^\circ$ or $\gamma = 180^\circ - 55.6^\circ = 124.4^\circ$, since both have the same sine and log sine. However, Fig. 11-4 shows that γ is obtuse.

To check, $\alpha + \beta + \gamma = 29.6^\circ + 26.0^\circ + 124.4^\circ = 180.0^\circ$, whence the computations are accurate.

When the angle in question is very nearly a right angle, it might be difficult to determine from the figure whether to choose the acute or obtuse value of the angle. In such cases, *solve for the angle opposite the shortest side first. It will always be an acute angle.*

EXERCISES

Solve the following triangles for the three unknown parts, giving results which are as accurate as the data will permit.

- $\alpha = 21.4^\circ, b = 10, c = 12.$
- $\beta = 44.8^\circ, a = 20, c = 30.$
- $\alpha = 74.3^\circ, b = 11, c = 14.$
- $\gamma = 33^\circ, b = 22, a = 19.$
- $\beta = 112^\circ, c = 42, a = 76.$
- $\alpha = 89^\circ, c = 89, b = 43.$
- $\gamma = 72.6^\circ, a = 0.473, b = 0.818.$
- $\alpha = 132.2^\circ, b = 1.88, c = 5.71.$
- $\beta = 13.21^\circ, c = 0.4261, a = 0.9824.$
- $\alpha = 53.5^\circ, c = 1.82, b = 6.49.$

In each of the following triangles determine the fourth part listed, giving results which are as accurate as the data will permit.

- $\alpha = 72^\circ, b = 15, c = 21, a = \quad .$
- $\beta = 32^\circ, a = 12, c = 30, b = \quad .$
- $\gamma = 123.7^\circ, a = 172, b = 94.2, c = \quad .$
- $\beta = 88.1^\circ, c = 0.243, a = 1.73, \gamma = \quad .$

15. $\gamma = 66.2^\circ$, $b = 1250$, $a = 1076$, $c =$.
 16. $\gamma = 37.63^\circ$, $a = 11.80$, $b = 15.26$, $\alpha =$.
 17. $\alpha = 82.2^\circ$, $c = 0.431$, $b = 1.21$, $\beta =$.
 18. $\beta = 54.12^\circ$, $c = 1.716$, $a = 4.810$, $\alpha =$.
 19. $\alpha = 14^\circ$, $b = 1039$, $c = 2321$, $\gamma =$.
 20. $\beta = 47.2^\circ$, $a = 21.8$, $c = 18.2$, $b =$.

11-6. Case 3. Given Two Sides and an Angle Opposite One. In this case the given parts may form two, one, or no triangles, depending on the relative size and position of the parts. When a figure will not give the number of triangles possible, the table given below may be used. The case in which there are two triangles is often referred to as the **ambiguous case**.

Consider a triangle in which a , b , and α are given and the problem is to determine the other parts. Using the law of sines to find angle β we have

$$\sin \beta = \frac{b}{a} \sin \alpha.$$

We shall now use this expression, $\sin \beta$, to enumerate the various cases arising from this problem:

(a) *Number of triangles.* Assuming that a , b , and α are given, we have the following possibilities.

When $\alpha \geq 90^\circ$		When $\alpha < 90^\circ$		
If $a > b$, there is <i>one</i> triangle and β is acute.	If $a \leq b$, there is <i>no</i> triangle.	If $a \geq b$, there is <i>one</i> triangle and β is acute.	If $a < b$, and	
			If $\sin \beta < 1$, there are <i>two tri-</i> <i>angles</i> , β being acute in one and obtuse in the other.	If $\sin \beta = 1$, there is <i>one triangle</i> and $\beta = 90^\circ$.

(b) *Method of solution: Law of sines.* When parts other than a , b , α are given, the table must be interpreted with α replaced by the given angle, a by the side opposite, and b by the second given side.

Number of triangles. We shall illustrate geometrically the various possibilities given in the above table for $\alpha < 90^\circ$. A similar discussion for the case $\alpha \geq 90^\circ$ is left for the student.

The construction must be performed in the following way (Fig. 11-5).

(1) Draw the given angle α .

(2) Lay off length b on one side of α .

(3) Draw an arc of radius a from the end point C of b . The result depends on the relative lengths of a and b . The different possibilities are plotted in Fig. 11-5 in order to indicate clearly how the number of solutions depends on the relative sizes of a and b .

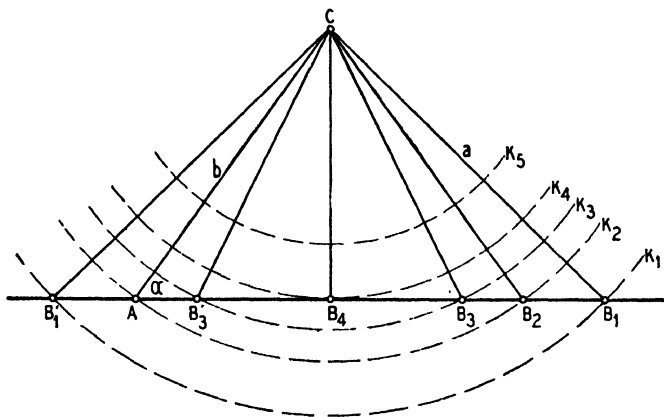


FIG. 11-5.

The case $a > b$ is represented by the circle K_1 . This circle meets the horizontal side of the angle α in B_1 and B_1' . The triangle AB_1C is the only triangle formed with the parts a , b , α .

If the length a is chosen so that $a = b$, the circle K_2 is obtained. In this case, the one triangle AB_2C is formed by the given parts.

If $a < b$, three different cases are possible, corresponding to the circles K_3 , K_4 , and K_5 . The circle K_3 meets the horizontal side of α in the two points B_3 and B_3' , and the two triangles AB_3C and $AB_3'C$ are formed from the given parts. If a is sufficiently small, a circle like K_5 is found which does not meet the horizontal side of α , and there is no triangle. There is also an intermediate case in which the circle has only a contact point with the straight line. As shown by the circle K_4 , only one triangle (triangle AB_4C) is obtained in this case.

Method of solution. When two sides and an angle opposite one are given, the remaining parts of the triangle can be computed by the use of the law of sines, as illustrated in the following examples.

Example 1. Solve the triangle $\beta = 44^\circ$, $b = 17$, $a = 10$, assuming that a has two significant digits.

From a sketch of the triangle (Fig. 11-6) we can conclude that only one triangle is possible and α is acute. To find α , we have by the law of sines

$$\sin \alpha = \frac{a}{b} \sin \beta = \frac{10}{17} \sin 44^\circ.$$

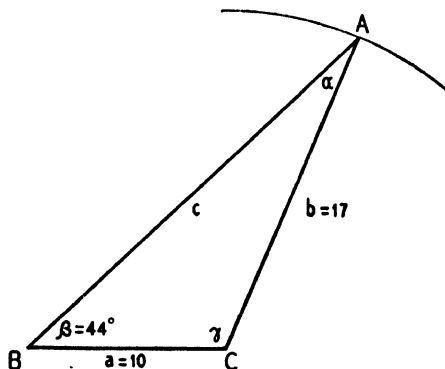


FIG. 11-6.

Using logarithmic computations we have:

NUMBERS	LOGARITHMS
10	1.0000
$\sin 44^\circ$	9.8418 - 10
$10 \sin 44^\circ$	10.8418 - 10
17	1.2304
$\sin \alpha$	9.6114 - 10

$$\alpha = 24^\circ$$

To find the angle γ we have:

$$\gamma = 180^\circ - (44^\circ + 24^\circ) = 180^\circ - 68^\circ$$

whence

$$\gamma = 112^\circ.$$

Finally, to obtain c we write

$$c = \frac{b \sin \gamma}{\sin \beta} = \frac{17 \sin 112^\circ}{\sin 44^\circ}.$$

NUMBERS	LOGARITHMS
17	1.2304
$\sin 112^\circ$	9.9672 - 10
$17 \sin 112^\circ$	11.1976 - 10
$\sin 44^\circ$	9.8418 - 10
c	1.3558

$$c = 23$$

Example 2. Solve the triangle $\gamma = 52^\circ$, $c = 5.1$, $a = 8.2$.

From the sketch (Fig. 11-7), we can conclude that there is no solution. We shall now sketch this conclusion computationally with the aid of the table of various cases given above. In this example, $\gamma < 90^\circ$ and $c < a$, so it is necessary to evaluate

$$\sin \alpha = \frac{a}{c} \sin \gamma = \frac{8.2}{5.1} \sin 52^\circ = 1.27 > 1,$$

which shows that there is no solution and no such triangle is possible.

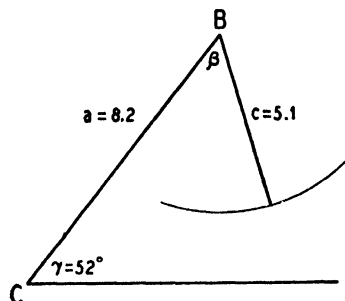


FIG. 11-7.

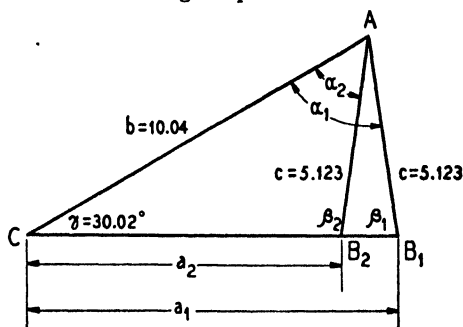


FIG. 11-8.

Example 3. Given $\gamma = 30.02^\circ$, $c = 5.123$, $b = 10.04$. Find the other parts of the triangle.

Making the sketch (Fig. 11-8), we find that there are two such triangles. Using the law of sines we have

$$\sin \beta = \frac{b}{c} \sin \gamma = \frac{10.04}{5.123} \sin 30.02^\circ.$$

Computing:

NUMBERS	LOGARITHMS
10.04	1.0017
$\sin 30.02^\circ$	9.6993 - 10
$10.04 \sin 30.02^\circ$	10.7010 - 10
5.123	0.7095
$\sin \beta$	9.9915 - 10

Since there are two triangles, there are two angles β such that $\log \sin \beta = 9.9915 - 10$.

$$\beta_1 = 78.70^\circ$$

$$\beta_2 = 101.30^\circ$$

For the third angle α_1 we have

$$\begin{aligned} \alpha_1 &= 180.00^\circ - (\gamma + \beta_1) \\ &= 180.00^\circ - 108.72^\circ \\ \alpha_1 &= 71.28^\circ. \end{aligned}$$

For the third angle α_2 we have

$$\begin{aligned} \alpha_2 &= 180.00^\circ - (\gamma + \beta_2) \\ &= 180.00^\circ - 131.32^\circ \\ \alpha_2 &= 48.68^\circ. \end{aligned}$$

Finally, to find a_1 we write

$$a_1 = \frac{c \sin \alpha_1}{\sin \gamma} = \frac{5.123 \sin 71.28^\circ}{\sin 30.02^\circ}.$$

Finally, to find a_2 we write

$$a_2 = \frac{c \sin \alpha_2}{\sin \gamma} = \frac{5.123 \sin 48.68^\circ}{\sin 30.02^\circ}.$$

Computing:

NUMBERS	LOGARITHMS
5.123	0.7095
$\sin 71.28^\circ$	9.9764 - 10
5.123 $\sin 71.28^\circ$	10.6859 - 10
$\sin 30.02^\circ$	9.6993 - 10
a_1	0.9866

$$a_1 = 9.696$$

Hence the triangle is:

$$\alpha_1 = 71.28^\circ, a_1 = 9.696$$

$$\beta_1 = 78.70^\circ, b = 10.04$$

$$\gamma = 30.02^\circ, c = 5.123$$

Computing:

NUMBERS	LOGARITHMS
5.123	0.7095
$\sin 48.68^\circ$	9.8757 - 10
5.123 $\sin 48.68^\circ$	10.5852 - 10
$\sin 30.02^\circ$	9.6993 - 10
a_2	0.8859

$$a_2 = 7.690$$

Hence the triangle is:

$$\alpha_2 = 48.68^\circ, a_2 = 7.690$$

$$\beta_2 = 101.30^\circ, b = 10.04$$

$$\gamma = 30.02^\circ, c = 5.123$$

That these results are correct may be checked roughly by measuring the figure.

EXERCISES

In each of the following exercises, determine the number of triangles possible and find the missing parts, giving results as accurate as the given data will permit.

- $\alpha = 32.3^\circ, a = 10.7, b = 10.7.$
- $\alpha = 143^\circ, a = 184, b = 161.$
- $a = 0.092, \beta = 55^\circ, b = 0.131.$
- $c = 1.90, b = 1.87, \gamma = 78.8^\circ.$
- $b = 641, c = 635, \gamma = 78.8^\circ.$
- $\beta = 48.3^\circ, a = 0.21, b = 0.17.$
- $\alpha = 62.7^\circ, a = 38.2, c = 41.2.$
- $c = 22.2, \gamma = 27.1^\circ, b = 51.5.$
- $c = 17.31, a = 17.31, \alpha = 98.72^\circ.$
- $b = 102.3, a = 108.2, \beta = 102.24^\circ.$

In each of the following exercises find the fourth part listed, giving results as accurate as the given data will permit. There may be two, one, or no possible values.

- $\alpha = 95.1^\circ, b = 107, a = 142, \beta =$.
- $c = 12.0, a = 12.3, \alpha = 85.2^\circ, \gamma =$.
- $c = 1.21, \beta = 143.4^\circ, b = 0.87, \gamma =$.
- $\beta = 72.6^\circ, b = 2.28, a = 1.84, \alpha =$.
- $\gamma = 57.3^\circ, c = 24.4, a = 22.1, \alpha =$.
- $\gamma = 30^\circ, b = 200, c = 100, a =$.
- $\beta = 76.84^\circ, c = 0.5121, b = 0.5041, a =$.
- $b = 22.1, \gamma = 57.3^\circ, c = 20.6, \alpha =$.
- $a = 0.4800, c = 0.5120, \alpha = 76.84^\circ, \beta =$.
- $c = 22.1, b = 18.2, \beta = 57.3^\circ, a =$.

11-7. Case 4. Given Three Sides. For this case we have:

(a) *Number of triangles:*

(1) *None, if the sum of any two sides is less than or equal to the third side.*

(2) *One in any other case.*

(b) *Method of solution: Law of cosines.*

Number of triangles. From plane geometry we know that in any triangle the sum of the lengths of any two sides is greater than the length

of the other side. Hence if the three sides given do not satisfy this condition, no triangle is possible. If the three sides given do satisfy this condition, one triangle is determined. The geometrical construction of this triangle is found in the Appendix.

Method of solution. Since none of the angles are known, it is first necessary to use the law of cosines to find one angle. Then a second angle may be found by the law of sines, and the third angle by subtracting the sum of the first two from 180° .

An angle of a triangle is always smaller than 180° . From this fact it follows that when the cosine of an angle is given, the angle can be uniquely determined. When the cosine is positive, the angle is acute; when the cosine is negative, the angle is obtuse. In following the procedure given above, it is most convenient to find the obtuse angle first, if the triangle contains such an angle. The remaining two angles are acute, and to find them we can use the law of sines conveniently, without any fear of the ambiguity mentioned in Sec. 11-5. When using the law of cosines, therefore, *the largest angle (that is, the angle opposite the largest side) should be found first.*

Example. Given $a = 7.30$, $b = 10.1$, $c = 12.7$. Find α , β , γ .

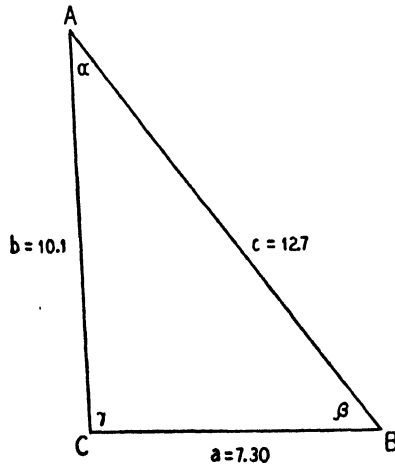


FIG. 11-9.

The sketch of the triangle is given in Fig. 11-9. Since c is the longest side, γ is the largest angle, and we find it first by the law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \gamma,$$

which gives us

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(7.30)^2 + (10.1)^2 - (12.7)^2}{2(7.30)(10.1)}.$$

Performing the computations we obtain

$$\cos \gamma = -0.0406,$$

$$\gamma = 180^\circ - 87.7^\circ,$$

$$\gamma = 92.3^\circ.$$

To find β , we use the law of sines,

$$\sin \beta = \frac{b \sin \gamma}{c} = \frac{10.1 \sin 92.3^\circ}{12.7}.$$

NUMBERS	LOGARITHMS
10.1	1.0043
$\sin 92.3^\circ$	9.9996 - 10
$10.1 \sin 92.33^\circ$	11.0039 - 10
12.7	1.1038
$\sin \beta$	9.9001 - 10

$$\beta = 52.6^\circ$$

Finally,

$$\alpha = 180^\circ - (\alpha + \beta) = 180^\circ - (92.3^\circ + 52.6^\circ),$$

$$\alpha = 35.1^\circ.$$

EXERCISES

Find the angles of each of the following triangles, giving results as accurate as the given data will permit.

1. $a = 1.23$, $b = 4.16$, $c = 3.71$.
2. $a = 231$, $b = 412$, $c = 202$.
3. $a = 0.492$, $b = 1.013$, $c = 0.763$.
4. $a = 27.2$, $b = 26.8$, $c = 35.8$.
5. $a = 1651$, $b = 2218$, $c = 1298$.
6. $a = 8.23$, $b = 9.71$, $c = 9.06$.
7. $a = 48.6$, $b = 51.3$, $c = 72.8$.
8. $a = 87.12$, $b = 105.03$, $c = 87.12$.
9. $a = 102$, $b = 76.2$, $c = 21.2$.
10. $a = 0.213$, $b = 1.43$, $c = 0.863$.

In each of the following exercises find the angle listed, giving results as accurate as the given data will permit.

11. $a = 2.6$, $b = 4.8$, $c = 3.5$, $\alpha =$.
12. $a = 1.82$, $b = 0.82$, $c = 1.10$, $\gamma =$.
13. $a = 0.843$, $b = 1.121$, $c = 0.922$, $\gamma =$.
14. $a = 87.2$, $b = 52.2$, $c = 79.2$, $\beta =$.
15. $a = 154$, $b = 138$, $c = 92$, $\alpha =$.
16. $a = 17.2$, $b = 23.3$, $c = 19.8$, $\beta =$.
17. $a = 32.4$, $b = 76.1$, $c = 11.8$, $\gamma =$.
18. $a = 112.2$, $b = 76.8$, $c = 201.2$, $\beta =$.

11-8. Application to Vector Analysis. The methods of solving oblique triangle problems permit simple solutions of many problems which would require involved procedures if the methods of solving right triangles

were used. Any problem which can be set up in terms of triangles may now be solved, provided, of course, that enough parts are known.

The solution of oblique triangles places at our disposal a second method applicable to some of the operations with vectors which were discussed in Chapter 6. The problems of finding the resultant of two given vectors (Sec. 6-3) and of finding the components of a given vector in two given directions (Sec. 6-4) may be carried out by solving triangles, as shown in the following illustrations.

Example 1. Find the resultant of the two vectors

$$\underline{D}: D = 10.0, \delta = 31.0^\circ; \underline{E}: E = 15.0, \epsilon = 296.0^\circ,$$

where the italicized capital letters denote the magnitudes of the vectors and the corresponding Greek letters denote the direction angles.

The graphical solution of this problem is given in Fig. 11-10. The vectors \underline{D} and \underline{E} with their resultant $\underline{L} = \underline{D} + \underline{E}$ form an oblique triangle. Denote the angles

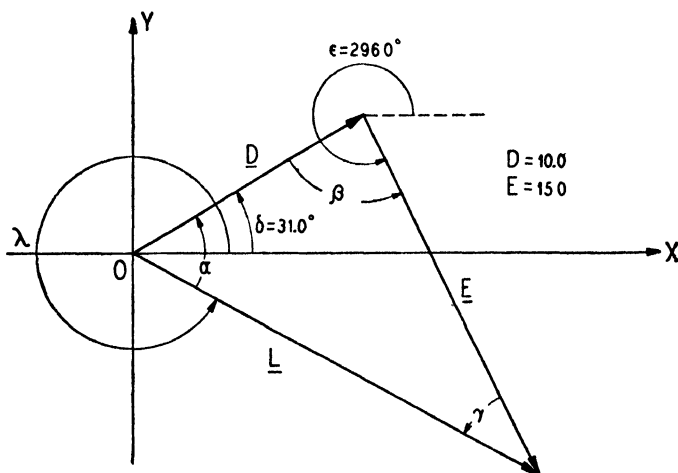


FIG. 11-10.

of this triangle by α , β , and γ . Our object is to find the magnitude and direction angle of the vector \underline{L} , that is, the length L and the angle $\alpha - \delta$.

From the triangle, we have

$$\beta = \epsilon - (180.0^\circ + \delta) = 296.0^\circ - 180.0^\circ - 31.0^\circ,$$

$$\beta = 85.0^\circ.$$

To find L we use the law of cosines and obtain

$$L^2 = D^2 + E^2 - 2DE \cos \beta = 100 + 225 - 300 \cos 85.0^\circ = 299,$$

$$L = 17.3.$$

To find α we use the law of sines and obtain

$$\sin \alpha = \frac{E \sin \beta}{L} = \frac{15.0 \sin 85.0^\circ}{17.3}.$$

NUMBERS	LOGARITHMS
15.0	1.1761
$\sin 85.0^\circ$	$9.9983 - 10$
$15.0 \sin 85.0^\circ$	$11.1744 - 10$
17.3	1.2380
$\sin \alpha$	$9.9364 - 10$

Since from the figure α is obviously an acute angle, we have

$$\alpha = 59.7^\circ.$$

Therefore

$$\alpha - \delta = 59.7^\circ - 31.0^\circ = 28.7^\circ,$$

and if we denote by λ the direction angle of \mathbf{L} , we have then

$$\lambda = 360.0^\circ - 28.7^\circ = 331.3^\circ.$$

Thus $\mathbf{L} = \mathbf{D} + \mathbf{E}$ is the vector given by

$$\mathbf{L}: L = 17.3, \lambda = 331.3^\circ.$$

Example 2. Resolve the vector $\mathbf{L}: L = 200, \lambda = 63^\circ$ into two components \mathbf{D} and \mathbf{E} whose direction angles are 0° and 122° respectively. Assume that L is given to two significant figures.

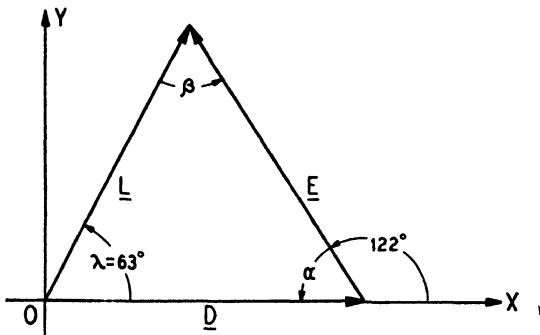


FIG. 11-11.

We first construct the diagram shown in Fig. 11-11. From the figure and the law of sines

$$E = \frac{L \sin \lambda}{\sin \alpha} = \frac{200 \sin 63^\circ}{\sin 58^\circ} = 210.$$

Since

$$\beta = 180 - (\alpha + \lambda) = 180^\circ - 121^\circ = 59^\circ,$$

we have also

$$D = \frac{L \sin \beta}{\sin \alpha} = \frac{200 \sin 59^\circ}{\sin 58^\circ} = 202.$$

Hence the components of \mathbf{L} are

$$\mathbf{D}: D = 202, \delta = 0^\circ; \mathbf{E}: E = 210, \epsilon = 122^\circ.$$

✓ EXERCISES

In each problem find the magnitude and direction angle of the resultant of the two given vectors. State your results as accurately as the given data will permit.

1. $A: A = 12, \alpha = 23^\circ; B: B = 16, \beta = 81^\circ.$
2. $A: A = 7.8, \alpha = 115^\circ; B: B = 11.2, \beta = 73^\circ.$
3. $A: A = 122, \alpha = 63^\circ; B: B = 183, \beta = 231^\circ.$
4. $A: A = 0.44, \alpha = 312^\circ; B: B = 0.71, \beta = 10^\circ.$
5. $A: A = 2.61, \alpha = 227^\circ; B: B = 7.82, \beta = 213^\circ.$
6. $A: A = 17.6, \alpha = 312^\circ; B: B = 28.2, \beta = 270^\circ.$
7. $A: A = 1420, \alpha = 152^\circ; B: B = 2000, \beta = 98^\circ.$
8. $A: A = 18.87, \alpha = 7.2^\circ; B: B = 31.26, \beta = 112.5^\circ.$
9. $A: A = 66.2, \alpha = 206.3^\circ; B: B = 141.3, \beta = 251.8^\circ.$
10. $A: A = 137, \alpha = 374^\circ; B: B = 284, \beta = 292^\circ.$
11. $A: A = 1.34, \alpha = 80.1^\circ; B: B = 6.80, \beta = 246.2^\circ.$
12. $A: A = 165, \alpha = 114.5^\circ; B: B = 371, \beta = 342.2^\circ.$
13. $A: A = 72.3, \alpha = 132.3^\circ; B: B = 211, \beta = 194.0^\circ.$
14. $A: A = 0.82, \alpha = 219.3^\circ; B: B = 1.23, \beta = 336.3^\circ.$
15. $A: A = 55.2, \alpha = 8.6^\circ; B: B = 460.1, \beta = 177.3^\circ.$
16. $A: A = 7.81, \alpha = 348.8^\circ; B: B = 13.34, \beta = 191.7^\circ.$

In each problem find the components of the given vector in the directions (a) 10° , 112° , (b) 60° , 330° , and (c) 90° , 240° . State results which are as accurate as the given data will permit.

- | | |
|---|--|
| 17. $A: A = 625, \alpha = 48^\circ.$ | 18. $A: A = 55.2, \alpha = 23.7^\circ.$ |
| 19. $A: A = 1420, \alpha = 90^\circ.$ | 20. $A: A = 28.2, \alpha = 98^\circ.$ |
| 21. $A: A = 47.8, \alpha = 300^\circ.$ | 22. $A: A = 1.34, \alpha = 236^\circ.$ |
| 23. $A: A = 0.718, \alpha = 61^\circ.$ | 24. $A: A = 7.83, \alpha = 200^\circ.$ |
| 25. $A: A = 1.23, \alpha = 336.3^\circ.$ | 26. $A: A = 284, \alpha = 292^\circ.$ |
| 27. $A: A = 31.26, \alpha = 112.5^\circ.$ | 28. $A: A = 7.81, \alpha = 348.8^\circ.$ |
| 29. $A: A = 1076, \alpha = 45^\circ.$ | 30. $A: A = 10,540, \alpha = 156^\circ.$ |
| 31. $A: A = 18.87, \alpha = 7.2^\circ.$ | 32. $A: A = 0.82, \alpha = 180^\circ.$ |

✓ 33. Two men at points A and B , 1400 ft. apart on the south bank of a river, sight a point C on the north bank of the river. If angle $CAB = 84.2^\circ$ and angle $CBA = 82.7^\circ$, how far are A and B from C ?

✓ 34. To find the distance of a lighthouse A from a point C on shore, a base line CB , 2500 ft. long, is laid off the shore, and the angle $ACB = 82.7^\circ$, $ABC = 44.2^\circ$. Find the distance CA .

✓ 35. A and B are points at opposite ends of a lake. At a point C , 4000 ft. from A and 10,820 ft. from B , the line AB subtends an angle of 152° . How long is the lake?

✓ 36. An engineer wishes to build a trestle from point A to point B across a swamp. At a point C which is 5420 ft. from A and 7480 ft. from B the angle ACB is 146° . How long must the trestle be?

✓ 37. A man at point A observes a point O across a lake and a point C to the left of O and the lake. The angle OAC is 38.7° . He then walks directly to C measuring the distance as 2760 ft., and from C to O measuring that distance as 1640 ft. Find the length of the lake from A to O .

38. At a point A , a man finds the angle of elevation of a certain mountain peak to be 12.2° . If he walks 1500 ft. on a level plain toward the peak, the angle of elevation becomes 18.6° . How high is the peak?

39. An observer in a balloon 600 ft. above the ground sights a tower. The angles of depression for the top and bottom of the tower are 38.2° and 42.5° respectively. How high is the tower?

40. A plane flies from A to B , a distance of 52 miles, and then from B to C , a distance of 41.7 miles. If at C the angle BCA is 21.5° , how far is C from A ?

41. The angle of intersection of two straight highways at point A is 67.8° . Along one road the distance to a town B is 20 miles, and along the other the distance to a point C is 25 miles. Assuming all the highways are straight, how far is C from B ?

42. If town A is 13.2 miles from B and 28.2 miles from C , and if C is 21.6 miles from B , find the angles at which straight roads joining the three towns will intersect.

43. A farmer has a triangular field with sides 2642, 1820, and 942 ft. long. What is the area of the field?

44. A vertical antenna 100 ft. tall is to be erected at a point P on a sloping street which makes an angle of 10° with the horizontal. If the antenna is to be supported by guy wires fastened 10 ft. below its top, how far down the street must a 150-ft. guy wire be secured?

45. In Exercise 44, determine how long the guy wire which is secured up the hill must be if it is to make the same angle with the antenna.

46. Two ships A and B are 2300 ft. apart. Both sight a buoy O near a reef. They measure angles $OAB = 86.8^\circ$ and $OBA = 79.3^\circ$. What are the distances of O from A and B ?

47. An observer A is stationed 7840 ft. from another observer B . If the two observers simultaneously sight an airplane C , and find the angles $CAB = 76.2^\circ$ and $CBA = 68.7^\circ$, what is the distance between the airplane and B ?

48. If a tender leaves a ship at an angle of 38° , how far will the tender be from the ship in 30 minutes if both travel on straight lines, the tender averaging 25 knots and the ship 15 knots?

49. A scouting party travels 6400 ft. from its base B to a point C . At C it turns and travels 3000 ft. to another point A . At A it observes that angle $BAC = 12.6^\circ$. How far is the party from its base?

50. An observer can see two guns, one at A and one at B . He observes that the time interval between the flash and the report of the gun is 5 seconds for A and 3 seconds for B . If the angle at his eye subtended by the guns is 72° , find the distance from B to A . Sound travels at approximately 1100 ft. per second.

51. An attacking force is located at a point A , 10 miles from objective O . The plan is to divide into three groups. The first group is to go to the left at an angle of 42° with the line AO for a distance of 17 miles, at which point it will turn and come directly at O . The second group is to proceed directly toward O from A . The third group is to proceed to the right at an angle of 37° with OA for a distance of 20 miles, at which point it will turn and advance directly toward O . Find the angle through which the first and third groups must turn and the average speed each must make to reach O in $1\frac{1}{2}$ hours.

52. A mountain top is observed by two surveyors A and B who are 10,000 ft. apart and at the same height above sea level. To surveyor A the horizontal angle between the mountain top and surveyor B is 60.12° , and the angle of elevation of the mountain top is 25.2° . At B the horizontal angle between surveyor A and the mountain top is 56.6° . Find the height of the mountain above the level of the surveyors.

53. An airplane is observed simultaneously from two stations 10,500 ft. apart. At station *A* the horizontal angle of the plane and station *B* is 73.8° , and the angle of elevation of the plane is 18.7° . At *B* the horizontal angle between *A* and the aircraft is 48.6° . How high is the plane?

54. In the previous exercise, if the aircraft stays at the same height and one minute later the observer *A* finds that the angle of elevation is 12.9° and the new line of sight makes a horizontal angle of 108° with the old line of sight, how fast is the plane traveling?

55. An airplane travels 200 m.p.h. in a straight line bearing 122° . At the end of 2.75 hours how far south and how far east of the starting point will the plane be?

56. A speedboat travels 72 miles on a line bearing 47.3° and then 54 miles on a line bearing 242.2° . How far and in what direction must the boat travel to get back to its starting point?

57. A plane traveling at 200 m.p.h. flies for 17 minutes on a line bearing 126.2° and then for 32 minutes on a line bearing 91.8° . How far and in what direction must the plane fly to get back to its base?

58. A ship going directly north observes at 1 o'clock a lighthouse at 13.7° . Two hours later the bearing of the lighthouse from the ship is 143.8° . If the ship travels 15 m.p.h., find the distance of the ship from the lighthouse at the time of the second observation.

59. A ship observes two lighthouses *A* and *B* which are known to be 46 miles apart. The bearing of *A* is 86.7° , of *B*, 44.2° . One-half hour later the bearing of *A* is 138.6° , of *B*, 82.1° . How fast and in what direction is the boat sailing?

60. A train starts from a station *A* and goes 410 miles to a city *B*. The bearing of *B* from *A* is 144° . Then the train is to proceed to a city *C*, 200 miles from *B*. From *C*, it returns to its starting point, a distance of 256 miles. What is the bearing of *C* from *B* and of *A* from *C*?

61. A submarine is at rest 10 miles outside a harbor, and the bearing of the harbor from the sub is 84° . An enemy ship sails from the port at 15 miles per hour along a line bearing 248° . At what speed and in what direction should the sub go to contact the ship after the ship has gone 9 miles?

Solve the following exercises by using vectors. Give all results to one decimal place.

62. Two forces, one of 110 lb., the other of 72 lb., act simultaneously on an object. The angle between their directions is 37° . Find the resultant and the angle it makes with each force.

63. A 2.7-lb. force and a 6.3-lb. force are pulling on a weight in such a way that the angle between their directions is 106.3° . Find the resultant force and the angle it makes with each force.

64. Two forces are acting simultaneously on an object *P*. The angle between the forces is 132° . If one of the forces is 200 lb. and the resultant force is 350 lb., find the second force and the angle that the resultant force makes with each component.

65. If a force of 640 lb. is acting with an unknown force at an angle of 13° to produce a resultant force of 800 lb., find the magnitude of the unknown force and the angle that the resultant makes with each of the other forces.

66. A man pulls a wagon in such a way that he exerts a force of 88 lb. at an angle of 34° with the horizontal. Find the horizontal and vertical components of this force.

67. By means of a pulley, a rope fastened to a weight of 300 lb. makes an angle of 27° with the vertical. How much force must be exerted on the rope to lift the weight from the ground?

68. Two horses are pulling on a wagon, the first exerting 300 lb. in a direction which makes an angle of 30° with straight ahead, the second exerting 350 lb. at an angle of 26° with straight ahead. If they are pulling on opposite sides of the straight ahead direction, what will be the magnitude and direction of the resultant force?

69. It is known that a 250-lb. force will be required to move a certain object straight ahead. If two men, the first of whom can exert 175 lb. and the second 145 lb., wish to exert exactly 250 lb. straight ahead, at what angle should each pull on the object?

70. A man finds that the force required to move his lawn mower is 80 lb. (exerted along the handle) and that the handle makes an angle of 48° with the horizontal. If he adjusts the handle to make an angle of 38° with the horizontal, how much force must be exerted along the handle to make the mower move?

71. A weight of 220 lb. is hanging at the end of a cable. What is the magnitude of the horizontal force that must be exerted on the weight to hold it in such a position that the rope will make an angle of 18.7° with the vertical?

72. A car weighing 1900 lb. is standing on a 20° slope. Neglecting friction, how much force would have to be exerted to prevent the car from rolling downhill?

73. A boy is just able to push a 50-lb. wagon up a slope. If he is exerting a force of 20 lb., what is the angle of elevation of the slope?

74. If a force of 86 lb. exerted parallel to an inclined plane is required to prevent a 150-lb. weight from sliding down the plane (disregarding friction), what is the angle that the plane makes with the horizontal?

75. A certain derrick consists of a boom which makes an angle of 36° with the vertical and a steel cable which extends from the weight vertically to the upper end of the boom and then horizontally to a windlass. Find the compression in the boom if the weight being lifted is 2000 lb.

76. A derrick boom which makes an angle of 27° with the vertical has a horizontal supporting cable. Find the compression in the boom if the weight being lifted is 1470 lb.

77. In still air a plane can fly 140 m.p.h. If the plane encounters a head wind of 30 m.p.h. that makes an angle of 13.5° with the desired line of flight, at what angle with the intended line of flight must the pilot head the plane? Under such circumstances how long will it take the plane to fly 745 miles?

78. In still air a certain pursuit plane can fly 210 m.p.h. On a day when a 14 m.p.h. wind blows in a direction which makes an angle of 216° with the north, what must be the direction of the plane in order to fly directly south?

79. In the above problem what must be the direction of the plane in flying to a base which has a bearing of 227° from the plane's home field?

80. On a clear day with no wind a bomber reaches an objective O from its base B in 3 hours. The bearing of O from B is 122° , and the distance is 540 miles. Two days later there is a south wind of 26 m.p.h. In what direction and how long must the bomber fly to reach O from B ?

81. A man wishes to cross a river from A to a point B 4 miles upstream. The rate of his boat in still water is 10 m.p.h. and the rate of the river is 3 m.p.h. If the river is $\frac{1}{2}$ mile wide, at what angle must he travel with the banks of the river? How far will he travel and how long will it take him?

82. Three boats X , Y , Z are at a point A on the east bank of a river, in which current flows south at the rate of 1.6 m.p.h. All three boats have a speed of 12.5 m.p.h. in still water. These boats are to meet at an island B , 4675 ft. directly across the river from A . X is to proceed to a point 2000 ft. directly upstream from B and then come down to the island. Y is to proceed directly across to B , and Z is to proceed to a point 1500 ft. downstream from B and then approach the island from the south. At what angle with the river bank must each boat direct its course? If it is desired that all three boats reach the island at midnight, at what time must each set out?

11-9. The Inverse Trigonometric Functions. In algebra when a relation $y = f(x)$ is solved for x in terms of y , obtaining say $x = g(y)$, then $g(y)$ is called the **inverse function** of $f(x)$. Conversely, $f(x)$ may be called the inverse of $g(y)$. For example, consider $y = 2x + 3$. This may

be solved for x to give $x = \frac{y-3}{2}$. Then $x = \frac{y-3}{2}$ is called the inverse

of $y = 2x + 3$, and $y = 2x + 3$ may be called the inverse of $x = \frac{y-3}{2}$.

(See Sec. 3-5.)

The equation

$$(1) \quad x = \sin y$$

determines a value of x for every value of y . In other words, (1) gives x as a function of y . It can also be used to determine y as a function of x .

For instance, if $x = \frac{1}{2}$ is given, the corresponding values of y are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, etc.

In this way y is a function of x when x and y are related by (1). In order to express this function, we say that

$$(2) \quad y \text{ is an angle whose sine is } x.$$

This statement is abbreviated by

$$(3) \quad y = \arcsin x.$$

Thus the inverse function of (1) is given by (3). The relation (3) is preferably read by the defining phrase (2), but the phrases *y is equal to the arc sine of x* and *y is equal to the inverse sine of x* are also in common use.

Some textbooks use the notation $\sin^{-1} x$ in place of $\arcsin x$. This notation, however, is not recommended because $\sin^{-1} x$ may be mistaken for $(\sin x)^{-1} = \frac{1}{\sin x}$.

Example 1. Evaluate $\arcsin \frac{1}{2}$.

If we set $y = \arcsin \frac{1}{2}$, by definition this means that

$$\sin y = \frac{1}{2},$$

and hence

$$y = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ etc.,}$$

or

$$\arcsin \frac{1}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ etc.}$$

Example 2. Evaluate $\arcsin 0.6157$.

As above, set $\theta = \arcsin 0.6157$. Then by the definition,

$$\sin \theta = 0.6157.$$

Therefore

$$\theta = 38^\circ, 142^\circ, \text{ etc.,}$$

or

$$\arcsin 0.6157 = 38^\circ, 142^\circ, \text{ etc.}$$

The same notations apply to the other five trigonometric functions when written in the inverse form. Thus the accompanying table lists, in the two right-hand columns, equivalent notations for expressing the relation given in the left-hand column.

RELATIONSHIP EXPRESSED	DIRECT NOTATION	INVERSE NOTATION
y is the angle whose sine is x	$x = \sin y$	$y = \arcsin x$
y is the angle whose cosine is x	$x = \cos y$	$y = \arccos x$
y is the angle whose tangent is x	$x = \tan y$	$y = \arctan x$
y is the angle whose cotangent is x	$x = \cot y$	$y = \text{arccot } x$
y is the angle whose secant is x	$x = \sec y$	$y = \text{arcsec } x$
y is the angle whose cosecant is x	$x = \csc y$	$y = \text{arccsc } x$

From Examples 1 and 2 it is evident that the inverse function

$$(5) \quad y = \arcsin x$$

has for one value of x many values for y . Although to a given angle y there corresponds only one value for the sine x , to a given sine value x there correspond infinitely many angles y . This is also true for the other functions. It is convenient to select one of the values from the multiple values of $\arcsin x$ as the principal value. We denote this principal value by capitalizing the initial letter of the function, e.g., $\text{Arcsin } x$ or $\text{Sin}^{-1} x$. Similarly we select a single principal value from the multiple values of each of the other inverse trigonometric functions.

The principal values of $\arcsin x$, $\arccos x$, and $\arctan x$ are defined by the following table.

FUNCTION	PRINCIPAL VALUE LIES IN THE INTERVAL
$y = \arcsin x$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$	$0 \leq y \leq \pi$
$y = \arctan x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$	$0 < y < \pi$

That the restriction of y to the interval given in this table for each function defines a single value can be seen readily from the graphs of the trigonometric functions given in Chapter 5. Since the principal values of $\operatorname{arcsec} x$ and $\operatorname{arccsc} x$ are not used until the calculus, they will not be defined here.

Example 3. Find the value of $y = \operatorname{Arcsin} \frac{1}{2}$, $y = \operatorname{Arctan} 1$, $y = \operatorname{Arccos} \left(-\frac{\sqrt{3}}{2} \right)$.

The notation indicates that principal values are desired. Rewriting each of the functions in direct notation and finding the angle in the proper interval, we have:

$$\begin{array}{lll} y = \operatorname{Arcsin} \frac{1}{2} & \sin y = \frac{1}{2}, & y = \frac{\pi}{6}; \\ y = \operatorname{Arctan} 1, & \tan y = 1, & y = \frac{\pi}{4}; \\ y = \operatorname{Arccos} \left(-\frac{\sqrt{3}}{2} \right), & \cos y = -\frac{\sqrt{3}}{2}, & y = \frac{5\pi}{6}. \end{array}$$

Example 4. Find the values of $\arcsin 0$.

The notation indicates that all the possible values are desired. Setting $y = \arcsin 0$, and rewriting it in the form

$$\sin y = 0,$$

we have

$$y = 0, \pi, 2\pi, 3\pi, \text{ etc.}$$

EXERCISES

Solve for y .

- $x = \sin y.$
- $z = 2 \cos y.$
- $x = \sec y.$
- $x = \sec 2y.$
- $2x = \cos 2y.$
- $\frac{x}{2} = \cot 3y.$
- $x = \sin (y - \pi).$
- $\sin \frac{y}{r} = m.$
- $\frac{x-1}{2} = \csc y.$
- $\tan \left(y - \frac{\pi}{2} \right) = \frac{x-1}{2}.$
- $\frac{t}{x} = \sin 2y.$

12. $x = t \sin 2y.$

13. $\sec 7y = 7t.$

14. $a = b \sin y.$

15. $\tan y = \frac{N}{M}.$

16. $M \tan y = N.$

17. $2 \sin \frac{y}{2} = 4x - 3.$

18. $x = 3 \sin 4y.$

19. $a + 2 = \frac{1}{2} \tan (y + 1).$

20. $\frac{a+b}{c} = \frac{1}{2} \cos \frac{4(y-1)}{3}.$

Solve for x .

21. $\arcsin x = y.$

22. $\arccos x = \theta.$

23. $\arccos x = \frac{\theta}{2}.$

24. $\beta = \operatorname{arccsc} \frac{2bx}{5}.$

25. $\alpha + \theta = \arcsin \frac{x}{I_m}.$

26. $\arccos 2ax = \omega t.$

27. $\arcsin \frac{x}{a} = \alpha.$

28. $\arctan x = 8y.$

29. $\frac{\arctan x}{8} = \theta.$

30. $\frac{a^4}{8} \arcsin \frac{x}{a} = \frac{y}{4}.$

31. $\theta = \arctan \frac{Mx}{N}.$

32. $2\theta = \frac{1}{2} \arccos x.$

33. $377t = 80^\circ + \arcsin \frac{x}{141.4}.$

34. $2\theta = 4 \operatorname{arccot} \frac{3x}{2}.$

35. $\theta = \frac{a^2}{2} \arcsin \frac{x}{a} + C.$

36. $\theta = \arctan \left(\frac{x}{R} \right).$

37. $\alpha = \arctan \left(\frac{-x}{R} \right).$

38. $\theta = \arctan \left(\frac{x-x_0}{R} \right).$

39. $\phi = \frac{\pi}{2} + \arcsin \frac{2x}{5}.$

40. $\theta = \frac{2}{\sqrt{4ac-b^2}} \arctan \frac{2ax+b}{\sqrt{4ac-b^2}}.$

Evaluate the following by first rewriting as direct trigonometric functions. Find six possible values of the unknown if principal values are not indicated.

41. $y = \operatorname{Arcsin} \frac{1}{2}.$

42. $y = \operatorname{Arcsin} \left(-\frac{1}{2} \right).$

43. $y = \arcsin \left(-\frac{1}{2} \right).$

44. $\theta = \arctan \sqrt{3}.$

45. $\theta = \operatorname{Arctan} \sqrt{3}.$

46. $x = \operatorname{arcsec} 2.$

47. $y = \operatorname{arccot} 0.$

48. $\alpha = \operatorname{Arctan} \left(-\frac{\sqrt{3}}{3} \right).$

49. $\alpha = \operatorname{Arccos} \frac{\sqrt{3}}{2}.$

50. $y = \operatorname{Arccos} \left(-\frac{3}{2} \right).$

51. $t = \operatorname{Arcsin} 0.$

52. $y = \arcsin 0.$

53. $x = \arccos (-1).$

54. $2x = \arccos (-1).$

55. $5y = \arctan (-1).$

56. $y = \frac{1}{5} \arctan (-1).$

57. $(\theta - \pi) = \operatorname{Arccos} \left(-\frac{1}{2} \right).$

58. $\left(y + \frac{\pi}{3} \right) = \arcsin 1.$

59. $2 \left(y - \frac{\pi}{2} \right) = \operatorname{Arctan} 1.$

60. $\left(y + \frac{\pi}{6} \right) = 2 \operatorname{Arctan} (-1).$

$$61. \left(\theta - \frac{\pi}{4} \right) = \frac{1}{2} \arccos \left(-\frac{\sqrt{2}}{2} \right).$$

$$63. y = \pi + 6 \operatorname{Arccos} \left(\frac{1}{2} \right).$$

$$65. y = \operatorname{Arctan} (-1.376).$$

$$67. 2\theta = \operatorname{arcsec} (-1.255).$$

$$69. \operatorname{Arcsin} 0.0929 = 4x.$$

$$71. \operatorname{Arctan} (-0.5851) = \frac{3\theta}{2}.$$

$$73. 6 \operatorname{Arccos} 0.9945 = 12\alpha.$$

$$75. \operatorname{arcsin} 0.5125.$$

$$77. \operatorname{arccsc} 1.675.$$

$$79. 3 \operatorname{Arctan} 0.3378.$$

$$62. (y + 3\pi) = \frac{1}{10} \operatorname{Arcsin} \left(-\frac{\sqrt{3}}{2} \right).$$

$$64. y = \operatorname{Arcsin} 0.5807.$$

$$66. y = \operatorname{arcsec} 1.406.$$

$$68. 2y = \operatorname{Arccos} 0.8090.$$

$$70. \operatorname{arctan} 6.314 = \frac{y}{4} - 27^\circ.$$

$$72. \operatorname{Arcsin} 0.5688 - 2\pi = 3y.$$

$$74. \operatorname{Arccos} (-0.3035).$$

$$76. \operatorname{arcsec} (-2.228).$$

$$78. \operatorname{Arctan} 0.8796.$$

$$80. 62^\circ - 2 \operatorname{Arcsin} (-0.8192).$$

Evaluate the following expressions.

$$81. \operatorname{Arctan} \frac{1}{2} - \operatorname{Arctan} \frac{1}{3}.$$

$$83. \operatorname{Arctan} 0.8431 - 2 \operatorname{Arcsin} (-0.4266).$$

$$84. \sin [\operatorname{Arcsin} 0.6246].$$

$$85. \cos [\operatorname{Arccos} (-0.4633)].$$

$$87. \tan [\operatorname{Arctan} (-1)] + 2.$$

$$89. 12 \operatorname{Arcsin} (-0.2482) - \operatorname{Arccos} 0.4824.$$

$$90. \tan [\operatorname{Arctan} \frac{4}{3} - \operatorname{Arctan} \frac{1}{2}].$$

$$92. \tan \left[\frac{\pi}{3} + \frac{1}{2} \operatorname{Arcsec} \frac{5}{4} \right].$$

$$94. \operatorname{Arctan} \frac{7}{4} + \operatorname{Arctan} \left(-\frac{5}{2} \right) - 2 \frac{\pi}{3}.$$

$$82. 2 \operatorname{Arcsin} \left(-\frac{\sqrt{2}}{2} \right) + \operatorname{Arccos} 0 + 1.$$

$$86. \sec [\operatorname{Arccos} (-0.4633)].$$

$$88. \operatorname{Arcsin} 1 - \operatorname{Arcsin} (-1).$$

$$91. \cos [\operatorname{Arctan} (-\frac{1}{2}) + \operatorname{Arcsin} \frac{1}{2}].$$

$$93. \sin [(\frac{1}{2} \operatorname{Arccos} \frac{3}{5}) - 137^\circ].$$

$$95. 4 \operatorname{Arccos} \frac{8}{13} + 3(\operatorname{Arcsin} \frac{2}{5}).$$

PROGRESS REPORT

In plane geometry the ruler and compass serve as the tools for constructing a triangle if any three parts, at least one of which is a side, are given. From the constructed triangle, the missing parts are found by measurement. But the method of construction is not very accurate. To solve such problems by means of computations we introduced in this chapter the law of sines and the law of cosines, which hold for any triangle. The law of cosines was also used to solve problems involving vectors. At the end of the chapter the inverse trigonometric functions were studied.

CHAPTER 12

THE J OPERATOR

In this chapter we shall develop the operations with complex numbers and the operations with their graphical representations. We shall find that for complex numbers the graphical operations of addition and subtraction are like the addition and subtraction of vectors. Consequently, complex numbers are frequently applied to vector problems, especially in the theory of sinusoidal alternating currents. Further, the operations of multiplication and division for complex numbers can be used advantageously in the theory of alternating currents. Charles P. Steinmetz, a prominent mathematical engineer, was one of the first men to apply complex numbers extensively to the theory of alternating-current circuits. Thus complex numbers, in addition to being an important mathematical tool, are a most important theoretical and practical tool for the electrical engineer.

12-1. Imaginary Numbers. In Chapter 1 it was shown how the various operations led to the consideration of different kinds of numbers. Thus in Sec. 1-1 subtraction brought forth negative numbers while division introduced fractions. Later on in Sec. 1-3 the extraction of roots forced the consideration of irrationals which together with the integers and fractions formed the real numbers.

By reviewing Sec. 1-3 the student will remember that the irrational numbers were introduced by extracting the n th root of *positive* rational numbers and the n th (n -odd) root of *negative* rational numbers. However, the operation of taking the n th (n -even) root of *negative* numbers was carefully avoided. This last operation will now be introduced.

The extraction of a square root of a negative number like $\sqrt{-4}$ is not possible, since there is no real number which when squared will yield -4 . The square of any real number, whether positive or negative, is a positive number. The same holds true in case of the fourth, sixth, or any even-indexed root of a negative number; for example, $\sqrt[4]{-81}$ and $\sqrt[6]{-64}$. Hence, if we wish to extract the roots of all numbers, positive and negative, we must make a new definition which will introduce a new type of number into our number system.

Consider the following square roots of negative numbers:

$$\sqrt{-4}, \quad \sqrt{-9}, \quad \sqrt{-3}, -\sqrt{-16}, \quad \sqrt{-\frac{4}{25}}.$$

Let us suppose that these as yet undefined symbols obey, as far as possible, the laws of radicals given in Sec. 8-6. Hence it follows that:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4(-1)} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1} \\ \sqrt{-9} &= \sqrt{9(-1)} = \sqrt{9} \cdot \sqrt{-1} = 3\sqrt{-1} \\ \sqrt{-3} &= \sqrt{3(-1)} = \sqrt{3} \cdot \sqrt{-1} = 1.73\sqrt{-1} \\ -\sqrt{-16} &= -\sqrt{16(-1)} = -\sqrt{16} \cdot \sqrt{-1} = -4\sqrt{-1} \\ \sqrt{-\frac{4}{25}} &= \sqrt{\frac{4}{25}(-1)} = \sqrt{\frac{4}{25}} \cdot \sqrt{-1} = \frac{2}{5}\sqrt{-1}.\end{aligned}$$

The symbol $\sqrt{-1}$ appears in every one of these results.

*This symbol $\sqrt{-1}$ is called the **imaginary unit**, or **j operator**, and will be denoted by j . The fundamental property of this symbol is that*

$$j^2 = -1.$$

Using this symbol we can now write the expressions above in the form $\sqrt{-4} = j2$, $\sqrt{-9} = j3$, $\sqrt{-3} = j1.73$, $-\sqrt{-16} = -j4$, $\sqrt{-\frac{4}{25}} = j\frac{2}{5}$. Most mathematics texts write the real coefficient before the j as, for example, $5j$ and $\frac{2}{5}j$, but since it is quite common in engineering books to let the j precede, we shall follow the latter practice.

While j is called the imaginary unit, any multiple of j is called an **imaginary number**. Thus from the above examples $j2$, $j3$, $j1.73$, $-j4$, $j\frac{2}{5}$ are imaginary numbers. We now assume that imaginary numbers obey the laws of addition, subtraction, multiplication, division, and raising to a power stated in Chapter 1. Thus we have

$$\begin{aligned}j^3 &= j^2 \cdot j = -1(j) = -j, \\ j^4 &= j^2 \cdot j^2 = -1(-1) = 1.\end{aligned}$$

We can now write down a table for the powers of j , where $j = \sqrt{-1}$,

$$\begin{aligned}j &= j \\ j^2 &= -1 \\ j^3 &= -j \\ j^4 &= 1\end{aligned}$$

Since $j^4 = 1$, we also have $j^5 = j^4 \cdot j = j$, $j^6 = j^4 \cdot j^2 = j^2$, $j^7 = j^4 \cdot j^3 = j^3$, $j^8 = j^4 \cdot j^4 = j^4$, etc., and hence that the successive integral powers of j run through the cycle $j, -1, -j, 1$ given in the above table.

It was stated above that it is customary to write j instead of $\sqrt{-1}$. It should be pointed out here that mathematicians use the letter i for $\sqrt{-1}$, but electrical engineers use j to avoid confusion with current which is denoted by i . Since this book is primarily intended for the engineering student, the j notation will be employed.

The choice of the term **imaginary unit** for $\sqrt{-1}$ is unfortunate from the standpoint of an engineering student, for this quantity is not imaginary in the sense that the adjective is used in everyday language. The reason for the choice of this term lies in the historical development of mathematics; it was introduced hundreds of years ago when to extract the square root of a negative number was thought to be inconceivable. From our standpoint it might have been better to call it the **j -number**.

In dealing with square roots of negative numbers it is convenient to introduce j for $\sqrt{-1}$. This will be done in the example which follows.

Example 1.

$$\sqrt{-\frac{4}{9}} = \sqrt{\frac{4}{9}} \cdot \sqrt{-1} = j\frac{2}{3}.$$

$$\sqrt{-0.01} = \sqrt{0.01} \cdot \sqrt{-1} = j0.1.$$

$$\sqrt{-45} = \sqrt{45} \cdot \sqrt{-1} = j3\sqrt{5} = j6.71.$$

$$\sqrt{-7} = \sqrt{7} \cdot \sqrt{-1} = j\sqrt{7} = j2.65.$$

This can be condensed in the following rule.

$$\text{For } a > 0, \quad \sqrt{-a} = j\sqrt{a},$$

i.e., the square root of a negative number $\sqrt{-a}$ can be written as j times the square root of the corresponding positive number \sqrt{a} .

Multiples of j can be added like any other quantity. This is made plain by what follows.

Example 2.

$$j2 + j7 - j = j8.$$

$$j\frac{1}{3} + j\frac{1}{4} - j\frac{1}{5} = j\frac{20 + 15 - 12}{60} = j\frac{23}{60}.$$

$$\sqrt{-9} + \sqrt{-25} - \sqrt{-4} = j3 + j5 - j2 = j6.$$

$$j^25 + j8 = -j5 + j8 = j3.$$

EXERCISES

Simplify the following by expressing each number in one of the forms, j , -1 , $-j$, 1 .

1. j^2 .

2. j^4 .

3. j^5 .

4. j^3 .

5. j^7 .

6. j^6 .

7. j^{10} .

8. j^8 .

9. j^{11} .

10. j^{12} .

11. j^9 .

12. j^{15} .

Express each number in terms of j , and simplify.

13. $\sqrt{-49}$.

14. $\sqrt{-81}$.

15. $\sqrt{-\frac{9}{25}}$.

16. $-\sqrt{-64}$.

17. $-\sqrt{-\frac{9}{16}}$.

18. $-\sqrt{-121}$.

19. $\sqrt{-\frac{1}{49}}$.

20. $\sqrt{-0.01}$.

21. $\sqrt{-12}$.

22. $\sqrt{-0.09}$.

23. $\sqrt{-27}$.

24. $-\sqrt{-64}$.

25. $-\sqrt{-96}$.

26. $-\sqrt{-144}$.

27. $\sqrt{-\frac{75}{4}}$.

28. $-\sqrt{-\frac{48}{25}}$.

29. $\sqrt{-25} + \sqrt{-16}$.

30. $\sqrt{-49} - \sqrt{-4}$.

31. $\sqrt{-64} + \sqrt{-25}$.

32. $\sqrt{-\frac{25}{4}} + \sqrt{-\frac{9}{4}}$.

Using the table of square roots, write the following in terms of j .

33. $\sqrt{-2}$.

34. $\sqrt{-3}$.

35. $\sqrt{-7}$.

36. $-\sqrt{-5}$.

37. $\sqrt{-10}$.

38. $-\sqrt{-8}$.

39. $\sqrt{-45}$.

40. $-\sqrt{-18}$.

41. $-\sqrt{-6}$.

42. $\sqrt{-12}$.

43. $\sqrt{-48}$.

44. $-\sqrt{-75}$.

Simplify.

45. $j8 + j3 - j5$.

46. $j^32 + j8$.

47. $j^44 - 2$.

48. $j^47 - j^22$.

49. $j^511 - j3$.

50. $j^412 - 8$.

12-2. Complex Numbers. Real numbers were introduced early in this book. Imaginary numbers (i.e., multiples of j) were defined in the preceding section. To make the real and imaginary numbers part of one system of numbers we define new quantities in the following way.

If a and b are real numbers, $a + jb$ is called a **complex number**. The number a is called the **real part** of the complex number $a + jb$, the number jb is called the **imaginary part**. Thus $2 + j3$, $5 - j9$, $\frac{1}{3} + j\frac{2}{3}$, $2.16 - j0.09$, $3 - j2$ are examples of complex numbers.

Real and imaginary numbers are a part of the system of complex numbers. For when $b = 0$ and a is any real number, then $a + jb$ gives us the real numbers; on the other hand when $a = 0$ and b is any real number, then $a + jb$ gives us the imaginary numbers (multiples of j). We can summarize this by saying that the real numbers and the imaginary numbers are subdivisions of the complex numbers.

We have come now to the end of the road in defining new numbers. It will be well to stop and to review the main features of this travel. *Starting with the natural numbers 1, 2, 3, 4, ... which are suggested by counting:*

1. *Subtraction led to negative numbers.*

2. *Division brought forth fractions (rational numbers).*

3. Square roots of positive numbers produced irrational numbers.

4. Square roots of negative numbers produced complex numbers.

It is shown in more advanced mathematics textbooks that in working with the different operations of mathematics, no other numbers than the complex numbers are required. It can also be shown that the operations with complex numbers can be defined so that the numbers obey the fundamental laws of operation stated for real numbers in Sec. 1-8. In the remaining sections of the present chapter the student will learn how to use the complex numbers.

Complex numbers are of great importance in mathematics and in the applications of mathematics to practical problems. This is particularly true of electrical engineering where alternating currents and voltages can be handled by methods which will be developed in this chapter. In fact many problems in alternating current theory would be very difficult to solve without the use of complex numbers.

12-3. Equality of Complex Numbers. In this section and in the next few sections we shall define equality, sum, and product in such a way that the complex numbers obey the same laws that the real numbers obey. We start with a definition of equality of two complex numbers.

If two complex numbers $x + jy$ and $a + jb$ are equal, then $x = a$ and $y = b$.

Symbolically:

$$\text{If } x + jy = a + jb, \text{ then } x = a \text{ and } y = b.$$

This can also be stated by saying that complex numbers are made equal by equating their real parts and also their imaginary parts. For example, if $x + jy = 5 - j3$, then $x = 5$, and $y = -3$.

In particular, when equating a complex number to zero, the above definition becomes:

$$\text{If } a + jb = 0, \text{ then } a = 0 \text{ and } b = 0.$$

Using this definition we can solve equations containing complex quantities.

Example. What values of x and y satisfy the equation $x + y + jx - jy = 2 + j3 + j$?

Simplifying and separating reals and imaginaries by the above definition we get

$$(x + y) + j(x - y) = 2 + j4,$$

$$x + y = 2,$$

$$x - y = 4.$$

Solving these equations simultaneously by the method of Sec. 2-16, we find:

$$x = 3, y = -1.$$

The following definition is useful in working with complex quantities.

Two complex numbers such as

$$a + jb \quad \text{and} \quad a - jb$$

whose real parts are the same, and for which the coefficients of j are numerically equal but opposite in sign are called conjugate complex numbers, and each is the conjugate of the other.

$a + jb$ and $a - jb$ are conjugate.

Examples of conjugate complex numbers are given by:

$$\begin{array}{cc} 3 + j4 & \text{and} \quad 3 - j4, \\ -1 - j & \text{and} \quad -1 + j, \\ j5 & \text{and} \quad -j5 \\ 3 & \text{and} \quad 3. \end{array}$$

The last example illustrates the fact that a *real number is its own conjugate*.

Throughout this chapter it is assumed that except for $j = \sqrt{-1}$ all literal numbers are real.

EXERCISES

Find the conjugate of the following complex numbers.

- | | | | |
|---------------|------------------------|----------------|----------------|
| 1. $5 + j2$. | 2. $12 - j5$. | 3. $2a + jb$. | 4. $-2 + j$. |
| 5. $-7 - j$. | 6. $j3$. | 7. 5 . | 8. $-j$. |
| 9. -3 . | 10. $2 + 5\sqrt{-1}$. | 11. $c - jd$. | 12. $x + jy$. |

Find the values of x and y which satisfy the following equations.

- | | |
|---------------------------------------|--------------------------------------|
| 13. $x + jy = 1 + j2$. | 14. $2x + jy = 6$. |
| 15. $2x - j3y = 8 + j8$. | 16. $x + y + j(x - y) = 5 + j$. |
| 17. $2x + 7y + j(3x - 2y) = -3 - j$. | 18. $2x - 3y - j(x + y) = -2 - j9$. |

12-4. Operations with Complex Numbers. A complex number when reduced to the form $a + jb$ may be regarded as a binomial. Thus the addition, subtraction, and multiplication of complex numbers are reduced to the corresponding operations with binomials in which one term is real and the other imaginary.

Example 1. Examples of the addition and subtraction of complex numbers are given by the following:

$$(2 + j3) + (4 - j) = 2 + j3 + 4 - j = 6 + j2.$$

$$(5 - j2) - (7 - j9) = 5 - j2 - 7 + j9 = -2 + j7.$$

$$\left(\frac{1}{2} + j\frac{1}{3}\right) + \left(1 - j\frac{1}{2}\right) - \left(\frac{1}{3} - j\right) = \frac{1}{2} + j\frac{1}{3} + 1 - j\frac{1}{2} - \frac{1}{3} + j = \frac{4}{3} + j\frac{5}{6}.$$

$$(7 - j3) + (j - 7) = 7 - j3 + j - 7 = -j2.$$

Example 2. In multiplying complex numbers, the student must make use of the fact that $j^2 = -1$.

$$(2 + j5)(1 + j3) = 2 + j6 + j5 + j^2 15 = 2 + j11 - 15 = -13 + j11$$

$$(1 - j3)(-2 + j) = -2 + j7 - j^2 3 = -2 + j7 + 3 = 1 + j7.$$

We can now put the definitions of addition, subtraction, and multiplication of complex numbers in the following symbolic way:

Addition (or Subtraction): $(a + jb) + (c + jd) = (a + c) + j(b + d)$.

Multiplication: $(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$.

It is of interest to apply the above definitions of addition, subtraction, and multiplication to conjugate complex numbers $a + jb$ and $a - jb$. One thus obtains:

Sum of conjugates: $(a + jb) + (a - jb) = 2a$.

Difference of conjugates: $(a + jb) - (a - jb) = j2b$.

Product of conjugates: $(a + jb)(a - jb) = a^2 + b^2$.

Therefore the sum or the product of two conjugate imaginary numbers is a real number.

The fact that the product of two conjugate imaginary numbers is a real number leads to the following method for division of complex numbers. *Division by a complex number is performed by multiplying numerator and denominator by the conjugate of the denominator:*

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}.$$

Example 3. Perform the following divisions.

$$\frac{3 + j4}{1 + j} = \frac{3 + j4}{1 + j} \cdot \frac{1 - j}{1 - j} = \frac{3 + j - j^2 4}{1 + 1} = \frac{3 + j + 4}{2} = \frac{7}{2} + j\frac{1}{2}.$$

$$\frac{3 - j2}{8 - j6} = \frac{3 - j2}{8 - j6} \cdot \frac{8 + j6}{8 + j6} = \frac{24 + j2 + 12}{64 + 36} = \frac{36 + j2}{100} = 0.36 + j0.02.$$

The student will notice that this method for division by a complex number corresponds to the method of Sec. 8-9 for rationalizing a denominator.

The work of this section can be summarized by saying that *complex numbers may be used in algebraic operations in the same way that real numbers are used provided that j^2 is replaced by -1 .*

Remark and warning. It has been explained so far in this chapter that the square roots of negative numbers are expressed in terms of the symbol j where $j^2 = -1$. For example, the roots of the equation

$x^2 = -4$ are $x = \pm\sqrt{-4}$ or $x = \pm j2$. It should be emphasized, however, that we use the form $\pm j2$ in preference to $\pm\sqrt{-4}$. Had we used the less desirable notation $\pm\sqrt{-4}$ for the roots of the equation $x^2 = -4$, we might be tempted to take the following erroneous step in checking the root $x = \sqrt{-4}$.

$$(\sqrt{-4})^2 = \sqrt{-4} \cdot \sqrt{-4} = \sqrt{(-4)(-4)} = \sqrt{16} = 4,$$

while the given equation is $x^2 = -4$ and not $x^2 = 4$. The reason for this error is the fact that the rules for operating with square roots of positive numbers cannot be entirely extended to square roots of negative numbers. To prevent such errors, we write $\pm j2$ in place of $\pm\sqrt{-4}$ and obtain $(j2)^2 = j^2 4 = -4$, which yields the correct result.

In the light of this discussion, *the student should always introduce j in place of $\sqrt{-1}$ when working with square roots of negative real numbers.*

Example 4. Multiply $(3 - \sqrt{-16})(2 + \sqrt{-9})$.

$$\begin{aligned}(3 - \sqrt{-16})(2 + \sqrt{-9}) &= (3 - j4)(2 + j3) = 6 + j - j^2 12 \\ &= 6 + j + 12 = 18 + j.\end{aligned}$$

EXERCISES

Perform the indicated operations and simplify to the form $a + jb$.

1. $(4 - j3) + (2 - j4)$.
2. $(12 - j5) + (12 + j5)$.
3. $(3 + \sqrt{-49}) + (7 - \sqrt{-64})$.
4. $(\frac{1}{2} + j\frac{1}{3}) + (-\frac{1}{3} + j\frac{1}{2})$.
5. $(4 - j7) - (2 + j3)$.
6. $(0.6 + j5) + (0.3 - j2)$.
7. $(3 - \sqrt{-25}) + (4 + \sqrt{-9})$.
8. $(0.5 + \sqrt{-16}) + (0.3 - j3)$.
9. $(0.7 - j0.6) - (0.5 + j0.1)$.
10. $(5 - j7) - (5 + j7)$.
11. $(0.4 + j0.3) - (0.1 - j0.5)$.
12. $(0.30 - j0.25) + (0.75 + j0.75)$.
13. $(2 - j3)(2 + j3)$.
14. $(4 - j2)(4 + j2)$.
15. $(5 + j2)(5 - j3)$.
16. $(-\frac{1}{2} + j\frac{1}{6})(\frac{1}{2} - j)$.
17. $(\frac{1}{4} + j\frac{1}{3})(\frac{1}{2} - j)$.
18. $(\frac{1}{5} - j\frac{1}{3})(-\frac{1}{2} + j\frac{1}{2})$.
19. $(4 + j3)^2$.
20. $(1 - j3)^2$.
21. $(\frac{1}{6} + j)^2$.
22. $(1 + j\sqrt{3})^2$.
23. $(\frac{1}{4} + j2)^2$.
24. $(0.5 + j0.3)^2$.
25. $(2 - j3)^3$.
26. $(4 + j2)^3$.
27. $(1 - j\sqrt{2})^3$.
28. $(3 - j4)^3$.
29. $(j^4 + j^3 2 + j^2 3 + j4)^2$.
30. $(j + j^2 3 + j^3 2 + j^4)^2$.
31. $(j + j^2 + j^3)^4$.
32. $\frac{1}{1 + j}$.
33. $\frac{1}{3 + j2}$.
34. $\frac{1}{-3 + j}$.

35. $\frac{4+j}{j}$.

36. $\frac{1+j}{1-j}$.

37. $\frac{2+j3}{3-j4}$.

38. $\frac{7+\sqrt{-4}}{5-\sqrt{-36}}$.

39. $\frac{4+\sqrt{-9}}{2-\sqrt{-1}}$.

40. $\frac{2+j}{1-j} \cdot \frac{3-j2}{5+j}$.

41. $\frac{4+j}{1+j} \cdot \frac{2-j}{3-j2}$.

42. $\frac{2-j}{1+j} \cdot \frac{1-j2}{3-j4}$.

12-5. Graphical Representation of Complex Numbers. In Chapter 1, real numbers were represented graphically as points on a line. In order to represent complex numbers, one line will not suffice, and it becomes necessary to use two dimensions. The complex number $x + jy$ may be represented on a rectangular coordinate system in the plane by a point whose abscissa is x and whose ordinate is y (see Fig. 12-1). One speaks

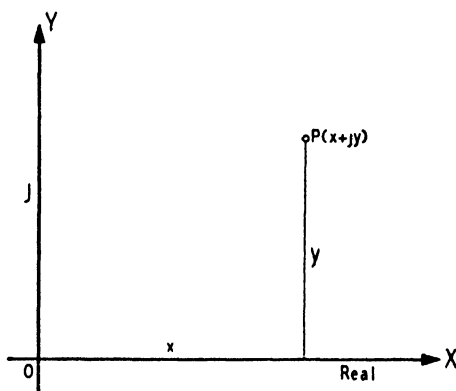


FIG. 12-1.

of P as the point $x + jy$. When complex numbers are so represented, the horizontal axis is the **axis of real numbers** or the **real axis**, and the vertical axis is called the **axis of imaginary numbers** or the **imaginary axis**. The entire plane when used for the representation of complex numbers is called the **complex plane**.

Thus in Fig. 12-2 the complex numbers $6 + j5$, $-5 + j3$, $-7 - j4$, and $7 - j3$ are represented by the points Q , R , S , and L respectively.

If in Fig. 12-1 a vector \mathbf{A}_P is drawn from the origin to point $P(x + jy)$, one can then think of $x + jy$ as the directed distance or vector from the origin to that point. (See Fig. 12-3.)

A complex quantity may thus be used to represent a vector, and it is in this way that complex numbers are used in electrical engineering.

Thus in Fig. 12-2, the complex quantity $6 + j5$ can represent a vector \mathbf{A}_Q , and the other complex numbers similarly, as may be seen in Fig. 12-4.

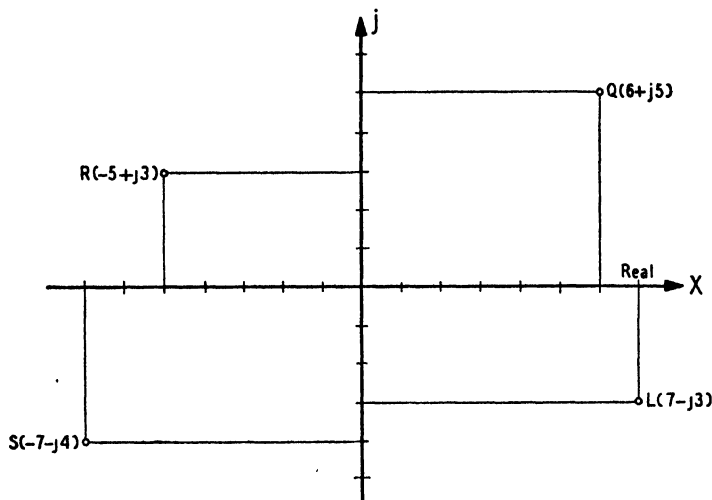


FIG. 12-2.

The addition and subtraction of complex numbers explained in the preceding section will now be performed graphically. Complex quantities are added, according to Sec. 12-4, by adding first the real parts and then the imaginary parts. This procedure corresponds to the law given

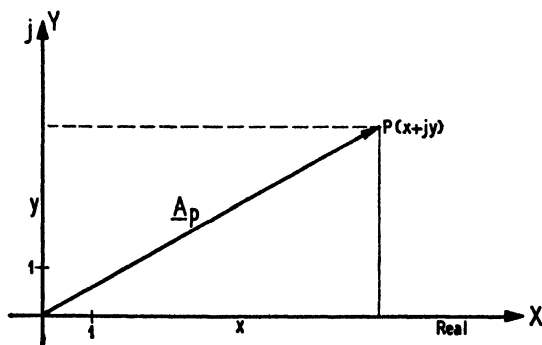


FIG. 12-3.

in Sec. 6-5 for the addition of vectors. This law stated that the sum of two vectors can be found by adding first the x -coordinates and then the y -coordinates. Thus the addition of two complex quantities $a + jb$ and $c + jd$ can be performed graphically by adding the corresponding vectors

as shown in Fig. 12-5. Hence the point P represents the complex number $(a + c) + j(b + d)$ which is the sum of the given complex numbers $a + jb$ and $c + jd$.

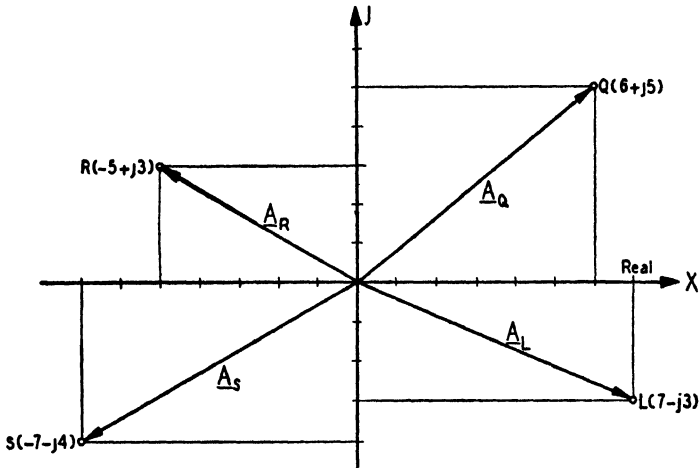


FIG. 12-4.

Thus to add two complex numbers graphically, complete the parallelogram which has as adjacent sides the lines drawn from the origin to the points representing the two complex numbers. The fourth vertex of the parallelogram will be the point representing the sum of the two complex numbers.

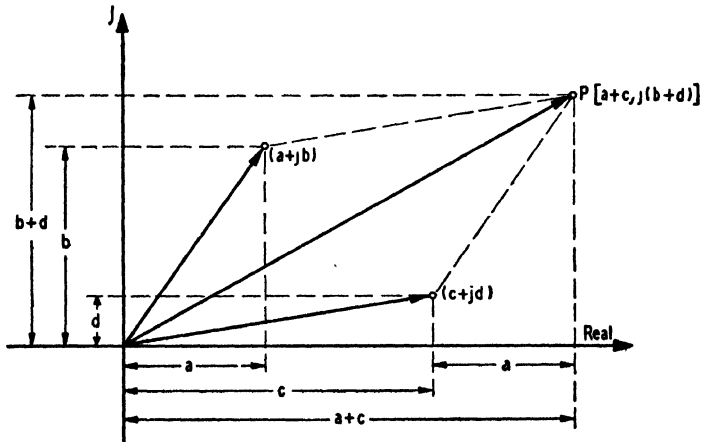


FIG. 12-5.

This method can be extended to find the sum of more than two complex numbers by first adding two of the numbers, then adding their sum to a third, and so on.

To subtract one complex number $c + jd$ from $a + jb$ graphically we merely add $a + jb$ and $-c - jd$ graphically.

Example 1. Add graphically $(7 + j5) + (9 - j2)$.

$$(7 + j5) + (9 - j2) = 16 + j3.$$

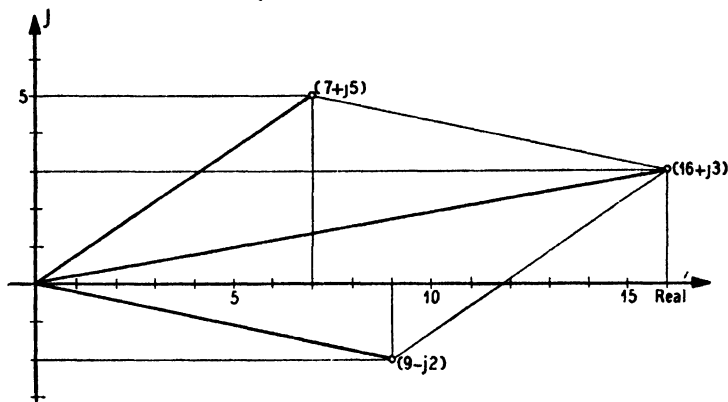


FIG. 12-6.

Example 2. Subtract graphically $(3 + j5) - (7 + j2)$.

This subtraction may be replaced by the addition $(3 + j5) + (-7 - j2)$.

$$(3 + j5) - (7 + j2) = -4 + j3.$$

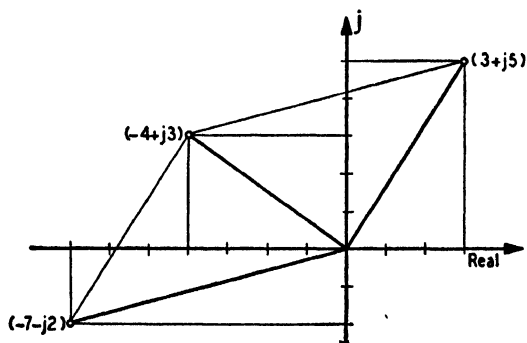


FIG. 12-7.

EXERCISES

Perform the indicated operations graphically. Use graph paper.

1. $(1 + j5) + (4 + j3)$.
2. $(2 + j7) + (5 - j3)$.
3. $(-3 + j) + (7 - j4)$.
4. $(-1 - j) + (5 - j2)$.
5. $(2 + j5) - (1 + j3)$.
6. $(8 + j6) - (4 + j5)$.
7. $(-4 + j2) + (9 - j2)$.
8. $(10 - j3) - (-7 + j5)$.
9. $(3 + j2) + (-4 + j5) + (7 - j2)$.
10. $(1 - j6) - (-2 + j3) + (4 - j5)$.

12-6. Trigonometric and Polar Forms of Complex Numbers. In Sec. 12-5, the vectorial representation of a complex number was given. In the present section a method of writing the complex number itself in terms of the length and direction angle of its vector will be given.

Let the complex number $a + jb$ be represented by the point P in Fig. 12-8. Join the point P to the origin. Denote the length of OP by r and an angle which OP makes with the positive direction of the real axis by θ . From the right triangle in Fig. 12-8 we obtain the following equations

$$(1) \quad a = r \cos \theta; \quad b = r \sin \theta.$$

$$(2) \quad r = \sqrt{a^2 + b^2}; \quad \tan \theta = \frac{b}{a}.$$

These equations hold irrespective of the quadrant in which θ terminates.

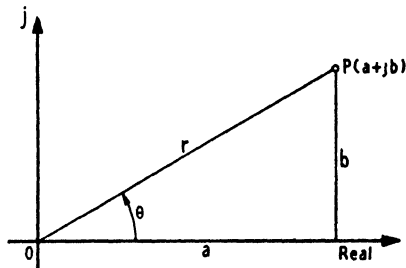


FIG. 12-8.

Using the equations in (1), the complex number can now be written in the form

$$(3) \quad a + jb = r(\cos \theta + j \sin \theta).$$

The form $r(\cos \theta + j \sin \theta)$ is called the **trigonometric form** of the complex number, while the form $a + jb$ is spoken of as the **rectangular form** of the complex number.

In the trigonometric form, r is sometimes called the **modulus** or the **absolute value** (magnitude) of the complex number, and the angle θ is called the **amplitude** or **argument** of the complex number. In our work we shall simply call r the **magnitude** of the complex number and θ the **angle** of the complex number.

A very convenient way of writing a complex number is

$$(4) \quad r/\theta.$$

This form is commonly called the **polar form** and is particularly useful when the complex number represents a vector. This polar form r/θ must be interpreted as a complex number having a magnitude r and making an angle of θ degrees with the axis of reals. In this notation the magnitude r and the angle θ do not form a product. The form r/θ is simply a shorthand notation for $r(\cos \theta + j \sin \theta)$.

We now summarize the various forms in which a complex number has so far been represented.

RECTANGULAR FORM	TRIGONOMETRIC FORM	POLAR FORM
$a + jb$	$r(\cos \theta + j \sin \theta)$	r/θ

It is important that the student be able to change one form of the complex number into another. This will be illustrated by the following examples.

Example 1. Express the complex number $3 + j4$ in trigonometric and polar forms. Using equations 2 with $a = 3$ and $b = 4$ and the trigonometric tables, we obtain:

$$r = \sqrt{3^2 + 4^2} = 5;$$

$$\tan \theta = \frac{4}{3} = 1.3333; \quad \theta = 53.13^\circ.$$

Hence

$$3 + j4 = 5(\cos 53.13^\circ + j \sin 53.13^\circ) = 5/\underline{53.13^\circ}$$

which gives, respectively, the rectangular, trigonometric, and polar forms. The graphical representation is given in Fig. 12-9.

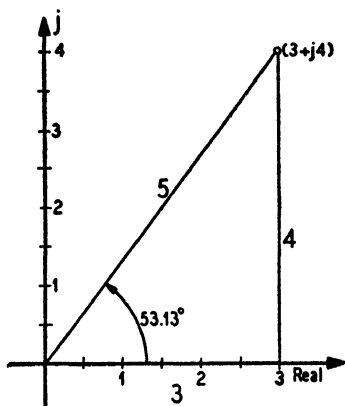


FIG. 12-9.

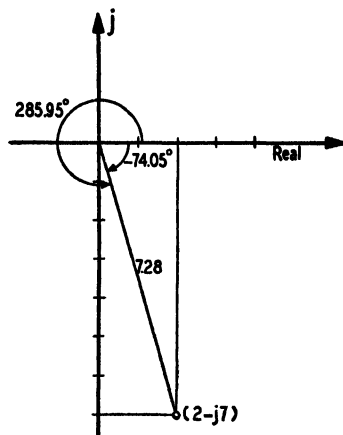


FIG. 12-10.

Example 2. Express the complex number $2 - j7$ in trigonometric and polar forms. Equations 2 with $a = 2$ and $b = -7$ yield

$$r = \sqrt{2^2 + (-7)^2} = \sqrt{53} = 7.28;$$

$$\tan \theta = \frac{-7}{2} = -3.50.$$

Since a is positive and b is negative, θ is in the fourth quadrant. From the trigonometric table we find

$$\tan 74.05^\circ = 3.50.$$

Therefore

$$\theta = 360^\circ - 74.05^\circ = 285.95^\circ \quad \text{or} \quad \theta = -74.05^\circ.$$

- Thus

$$2 - j7 = 7.28(\cos 285.95^\circ + j \sin 285.95^\circ) = 7.28/\underline{285.95^\circ},$$

or

$$2 - j7 = 7.28[\cos (-74.05^\circ) + j \sin (-74.05^\circ)] = 7.28/\underline{-74.05^\circ}.$$

These are respectively the rectangular, trigonometric, and polar forms of the complex number under consideration. The graphical representation is given in Fig. 12-10.

We summarize the method illustrated in the last two examples by stating that *in order to transform the rectangular form of a complex number into the trigonometric or polar form use expressions (2) of this section.*

By equation 3, any real number a can be regarded as a complex quantity of magnitude a and of angle 0° . Therefore

$$a = a (\cos 0^\circ + j \sin 0^\circ) = a/0^\circ$$

gives respectively the rectangular, trigonometric, and polar forms of any real number a .

Similarly, any imaginary number jb can be regarded as a complex quantity of magnitude b and of angle 90° . Therefore

$$jb = b (\cos 90^\circ + j \sin 90^\circ) = b/90^\circ$$

gives respectively the rectangular, trigonometric, and polar forms of any imaginary number jb .

Example 3. Express the complex number $4/132.7^\circ$ in trigonometric and rectangular forms.

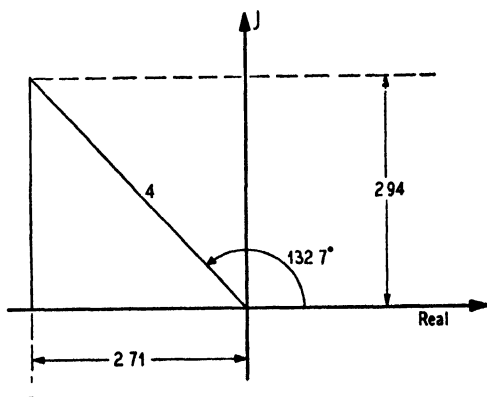


FIG. 12-11.

Changing into the trigonometric form we have at once

$$4/132.7^\circ = 4(\cos 132.7^\circ + j \sin 132.7^\circ).$$

From the trigonometric table, we find

$$\cos 132.7^\circ = -0.6782, \sin 132.7^\circ = 0.7349.$$

Therefore

$$\begin{aligned} 4(\cos 132.7^\circ + j \sin 132.7^\circ) &= 4(-0.6782 + j0.7349) \\ &= -2.7128 + j2.9396, \end{aligned}$$

which is the rectangular form of the given complex number. Thus $4/132.7^\circ = 4(\cos 132.7^\circ + j \sin 132.7^\circ) = -2.71 + j2.94$. The graphical representation is given in Fig. 12-11.

Thus, as illustrated by the above example, *to transform the polar or trigonometric form of a complex number into the rectangular form, evaluate the sine and cosine of the angle by trigonometric tables and simplify the result.*

EXERCISES

In each example write the complex number in polar form. Plot the corresponding point.

- | | | |
|------------------|---------------------|-----------------------|
| 1. $1 + j$. | 2. $\sqrt{3} + j$. | 3. $-1 + j\sqrt{3}$. |
| 4. 5. | 5. -8 . | 6. $j2$. |
| 7. $-j3$. | 8. $2 + j5$. | 9. $3 - j8$. |
| 10. $-12 + j7$. | 11. $-1 - j4$. | 12. $1.5 + j2.5$. |
| 13. $2 - j9$. | 14. $2.7 + j3.8$. | |

In each example find the rectangular form of the complex number. Plot the corresponding point.

- | | |
|--|---------------------------------------|
| 15. $7(\cos 90^\circ + j \sin 90^\circ)$. | 16. $5/\underline{180^\circ}$. |
| 17. $3(\cos 270^\circ + j \sin 270^\circ)$. | 18. $2/\underline{45^\circ}$. |
| 19. $4(\cos 37^\circ + j \sin 37^\circ)$. | 20. $6/\underline{81^\circ}$. |
| 21. $8/\underline{135^\circ}$. | 22. $9/\underline{-27^\circ}$. |
| 23. $1.5/\underline{53.7^\circ}$. | 24. $3.7/\underline{125.4^\circ}$. |
| 25. $3.1/\underline{213.8^\circ}$. | 26. $5.12/\underline{67.17^\circ}$. |
| 27. $6.08/\underline{-72.15^\circ}$. | 28. $1.17/\underline{137.08^\circ}$. |

12-7. The Exponential Form of Complex Numbers. In Sec. 8-4 the idea of an exponent was enlarged to include expressions of the form a^x where x was any rational number. In Sec. 8-14 we gave a meaning to a^x where x was any real value. Having introduced complex numbers, it would be very desirable to use them for exponents. One is free to give any meaning to the expression a^{x+jy} . In the light of the discussion given in Sec. 8-5, it is important to define a^{x+jy} in such a fashion that the laws of exponents remain true and so that it will be unnecessary to introduce new formulas.

In practical applications of mathematics such general expressions as a^{x+jy} are not needed. The expressions used in this connection are of the type $e^{j\theta}$ where e is the number discussed in Sec. 8-14, and θ is any real number. To give a meaning to this expression $e^{j\theta}$ we write

$$(1) \quad e^{j\theta} = \cos \theta + j \sin \theta$$

where the right-hand side defines the left-hand side. It should be stated that θ is measured in radians.

Using trigonometric identities, advanced mathematics books show that expressions of the form $e^{j\theta}$, as defined in (1), obey all the laws of exponents stated in Chapter 8.

The importance of (1) lies in the fact that we can use it now to give another form for the complex number. Comparing the definition of $e^{j\theta}$ with (3) in Sec. 12-6, one obtains

$$(2) \quad a + jb = r (\cos \theta + j \sin \theta) = re^{j\theta} = r \angle \theta.$$

The expression $re^{j\theta}$ is the **exponential form** of a complex number. The polar form $r \angle \theta$ is now a shorthand notation for the trigonometric form $r (\cos \theta + j \sin \theta)$ or for the exponential form $re^{j\theta}$.

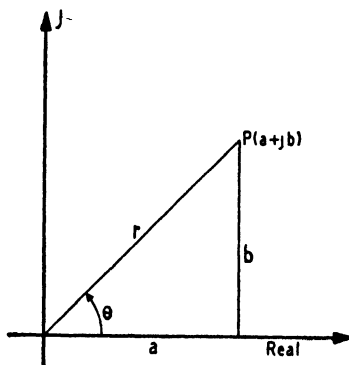


FIG. 12-12.

The following table with the graphical representation of Fig. 12-12 gives a summary of the various forms in which a complex number can be represented.

FORM	COMPLEX NUMBER
Rectangular	$a + jb$
Trigonometric	$r(\cos \theta + j \sin \theta)$
Exponential	$re^{j\theta}$
Polar	$r \angle \theta$

In the trigonometric and polar forms θ may be expressed in terms of radians or degrees (most of the electrical engineering books use the latter). In the exponential form, θ must be given in radians, although books on electrical engineering often express θ in degrees. However, when θ is expressed in degrees, it is to be understood that the corresponding angle in radians is intended. For example, since $\theta = 54.3^\circ = 0.948$ radian, engineering books write $e^{j54.3^\circ}$ for $e^{j0.948}$. Thus $e^{j54.3^\circ}$ and $\angle 54.3^\circ$ are simply notations for the correct $e^{j0.948}$.

It is important that the student be able to change one form of the complex number into another. This will be illustrated by the following examples.

Example 1. Express the complex number $3 + j4$ in the different forms. Using the results obtained in Example 1 of the preceding section, we obtain

$$3 + j4 = 5(\cos 53.13^\circ + j \sin 53.13^\circ) = 5e^{j0.927} = 5/\underline{53.13^\circ},$$

since $53.13^\circ = 0.927$ radians (see Table 6).

Example 2. Similarly from Example 2 of Sec. 12-6, one obtains

$$\begin{aligned} 2 - j7 &= 7.28[\cos(-74.05^\circ) + j \sin(-74.05^\circ)] = 7.28e^{-j1.292} \\ &= 7.28/\underline{-74.05^\circ}, \end{aligned}$$

since $74.05^\circ = 1.292$ radians.

Example 3. Express the complex number $10e^{j0.37}$ in rectangular form. Using Table 5,

$$\begin{aligned} 10e^{j0.37} &= 10(\cos 0.37 + j \sin 0.37) = 10(0.932 + j0.362) \\ &= 9.32 + j3.62. \end{aligned}$$

Example 4. Express the complex number $e^{j\pi}$ in rectangular form. We have at once $e^{j\pi} = \cos \pi + j \sin \pi = -1$.

EXERCISES

In each example write the complex number in exponential form.

- | | | |
|--|--|----------------------|
| 1. $2 + j2$. | 2. $\sqrt{3} - j$. | 3. $1 + j\sqrt{3}$. |
| 4. 9. | 5. -7 . | 6. $j3$. |
| 7. $-j5$. | 8. $3 + j7$. | 9. $5 - j8$. |
| 10. $1.5 + j3.5$. | 11. $2.2 - j7.5$. | 12. $-2.7 - j3.8$. |
| 13. $3 \left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2} \right)$. | 14. $5(\cos 270^\circ + j \sin 270^\circ)$. | |
| 15. $2(\cos 43^\circ + j \sin 43^\circ)$. | 16. $3.2/\underline{219.7^\circ}$. | |

In each example write the complex number in rectangular form.

- | | | |
|----------------------------|-----------------------------|-------------------|
| 17. $e^{j\frac{\pi}{2}}$. | 18. $e^{-j\frac{\pi}{2}}$. | 19. $e^{-j\pi}$. |
| 20. $e^{j2\pi}$. | 21. $e^{j1.7}$. | 22. $e^{j3.1}$. |
| 23. $e^{-j2.3}$. | 24. $e^{-j0.3}$. | |

12-8. Multiplication of Complex Numbers in Polar Form. In Sec. 12-4, the operations with complex numbers in the rectangular form $a + jb$ were discussed. Multiplication and division of complex numbers are much simpler when the numbers are given in a polar form.

Let r_1/θ_1 and r_2/θ_2 be two complex numbers in their polar form. By multiplication,

$$\begin{aligned} r_1/\theta_1 \cdot r_2/\theta_2 &= r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j\theta_1 + j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} \\ &= r_1 r_2 / \theta_1 + \theta_2. \end{aligned}$$

$$(1) \quad r_1/\theta_1 \cdot r_2/\theta_2 = r_1 r_2 / \theta_1 + \theta_2.$$

Therefore, the product of two complex numbers is a complex number whose magnitude is the product of the magnitudes of the numbers and whose angle is the sum of their angles.

It can readily be seen that this holds for the product of any number of complex quantities. Thus in case of three complex numbers we have

$$r_1/\theta_1 \cdot r_2/\theta_2 \cdot r_3/\theta_3 = r_1 r_2 r_3 / \theta_1 + \theta_2 + \theta_3.$$

Similarly, we have for the quotient of two complex numbers

$$\begin{aligned} \frac{r_1/\theta_1}{r_2/\theta_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j\theta_1 - j\theta_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \\ &= \frac{r_1}{r_2} / \theta_1 - \theta_2. \end{aligned}$$

$$(2) \quad \frac{r_1/\theta_1}{r_2/\theta_2} = \frac{r_1}{r_2} / \theta_1 - \theta_2.$$

Therefore, the quotient of two complex numbers r_1/θ_1 and r_2/θ_2 is a complex number whose magnitude is the quotient $\frac{r_1}{r_2}$ of the magnitudes of the numbers and whose angle is the difference $\theta_1 - \theta_2$ of their angles.

In order to use the methods of this section in multiplying and dividing complex numbers, they must first be expressed in polar form (see Sec. 12-6).

Example 1. Find the product $(3 + j4)(2 - j7)$.

Using the results obtained in Examples 1 and 2 of Sec. 12-6, we have

$$\begin{aligned} (3 + j4)(2 - j7) &= 5.00/53.13^\circ \cdot 7.28/285.95^\circ \\ &= 5.00 \cdot 7.28/53.13^\circ + 285.95^\circ \\ &= 36.40/339.08^\circ = 36.40/-20.92^\circ \\ &= 36.40(\cos 20.92^\circ - j \sin 20.92^\circ) \\ &= 36.40(0.9341 - j0.3570) \\ &= 34.00 - j12.99. \end{aligned}$$

The answer can be given in the form

$$36.40 / -20.92^\circ \quad \text{or} \quad 34.00 - j12.99.$$

Direct multiplication (method of Sec. 12-4) gives

$$(3 + j4)(2 - j7) = 34 - j13.$$

Example 2. Perform the multiplication and division in

$$\frac{(67.4 - j37.7)(35.0 + j75.0)}{(45.0 - j12.9)(14.6 + j8.86)}.$$

Converting from rectangular form to polar form, by methods of Sec. 12-6, we obtain:

$$(67.4 - j37.7) = 77.2 / -29.2^\circ.$$

$$(35.0 + j75.0) = 82.8 / 65.0^\circ.$$

$$(45.0 - j12.9) = 46.8 / -16.0^\circ.$$

$$(14.6 + j8.86) = 17.1 / 31.2^\circ.$$

The given expression is equal to

$$\begin{aligned} \frac{77.2 / -29.2^\circ \cdot 82.8 / 65.0^\circ}{46.8 / -16.0^\circ \cdot 17.1 / 31.2^\circ} &= \frac{77.2 \cdot 82.8 / -29.2^\circ + 65.0^\circ}{46.8 \cdot 17.1 / -16.0^\circ + 31.2^\circ} = 7.99 \frac{35.8^\circ}{15.2^\circ} \\ &= 7.99 / 35.8^\circ - 15.2^\circ = 7.99 / 20.6^\circ \\ &= 7.99(\cos 20.6^\circ + j \sin 20.6^\circ) \\ &= 7.99(0.9361 + j0.3518) = 7.47 + j2.81. \end{aligned}$$

Hence the answer is $7.99 / 20.6^\circ$ in polar form and $7.47 + j2.81$ in rectangular form.

12-9. Powers of Complex Numbers. The method of the previous section may now be applied to powers and roots of complex numbers.

Let r/θ be any complex number and n any positive integer, then

$$(r/\theta)^n = (re^{j\theta})^n = r^n e^{jn\theta} = r^n / n\theta.$$

$$(1) \quad (r/\theta)^n = r^n / n\theta.$$

Therefore when raising a complex number to the n th power, the result is a complex quantity whose magnitude is the n th power of the original magnitude and whose angle is n times the original angle.

The relation (1) holds for both integral and fractional values of n . Using fractional values we obtain a simple method for finding the roots of complex quantities. Thus one of the m th roots of a complex number is given by

$$(2) \quad \sqrt[m]{r/\theta} = (r/\theta)^{\frac{1}{m}} = \sqrt[m]{r} / \frac{\theta}{m}.$$

When the complex number is given in its trigonometric form, relations (1) and (2) can be written in the form

$$(3) \quad [r(\cos \theta + j \sin \theta)]^n = r^n(\cos n\theta + j \sin n\theta)$$

known as *DeMoivre's theorem*, and it holds true for all real values of n .

Example 1. Find $(4.60 + j2.82)^3$.

Converting into polar forms we have

$$(4.60 + j2.82)^3 = (5.40/\underline{31.5^\circ})^3 = 5.40^3/3 \cdot 31.5^\circ = 157.5/\underline{94.5^\circ}.$$

In polar form the answer is $157.5/\underline{94.5^\circ}$. Let the student express this answer also in rectangular form.

Example 2. Find $\sqrt{4.60 + j2.82}$.

Using the result of Example 1 and relation (2) we have

$$\sqrt{4.60 + j2.82} = (5.40/\underline{31.5^\circ})^{\frac{1}{2}} = \sqrt{5.40}/\underline{15.8^\circ} = 2.32/\underline{15.8^\circ}.$$

Since

$$\underline{31.5^\circ} = \underline{360^\circ + 31.5^\circ} = \underline{391.5^\circ},$$

we also have

$$\sqrt{4.60 + j2.82} = (5.40/\underline{391.5^\circ})^{\frac{1}{2}} = 2.32/\underline{195.8^\circ}.$$

Hence $2.32/\underline{15.8^\circ}$ and $2.32/\underline{195.8^\circ}$ are the two square roots of $4.60 + j2.82$.

Following the method of the last example, it can be shown that *any complex number r/θ has n distinct n th roots given by the formula*

$$\sqrt[n]{r} / \frac{\theta + k \cdot 360^\circ}{n}$$

where k takes the values $0, 1, 2, \dots, n - 1$.

12-10. Graphical Representation. In Sec. 12-5 the graphical representation of addition and subtraction of complex numbers (in rectangular form) was given. We shall now give the graphical representation of the operations discussed in Secs. 12-8 and 12-9.

Let P_1 and P_2 (Fig. 12-13) be the points representing the complex numbers r_1/θ_1 and r_2/θ_2 . Join point $A(1, 0)$ to P_1 . Construct $\angle P_2OP$ equal to θ_1 . Construct $\angle OP_2P$ equal to $\angle OAP_1$. From the fact that $\triangle OAP_1$ is similar to $\triangle OP_2P$, it can be shown that $OP = r_1r_2$. Hence point P represents the complex number $r_1r_2/\theta_1 + \theta_2$, which is the product of r_1/θ_1 and r_2/θ_2 .

Similar graphical representation can be given to division and also to raising to a power and extracting roots of complex numbers.

In Sec. 12-5 it was indicated that in engineering a complex quantity

is used to represent a vector. In particular, making use of the table for the values of the powers of j (Sec. 12-1), we can represent:

- 1 by a vector of length 1 and an angle 0°
- j by a vector of length 1 and an angle 90°
- j^2 by a vector of length 1 and an angle 180°
- j^3 by a vector of length 1 and an angle 270°
- j^4 by a vector of length 1 and an angle 360°

From Fig. 12-13 it follows that when a complex quantity is multiplied by j its vector is rotated through an angle of 90° in a counterclockwise

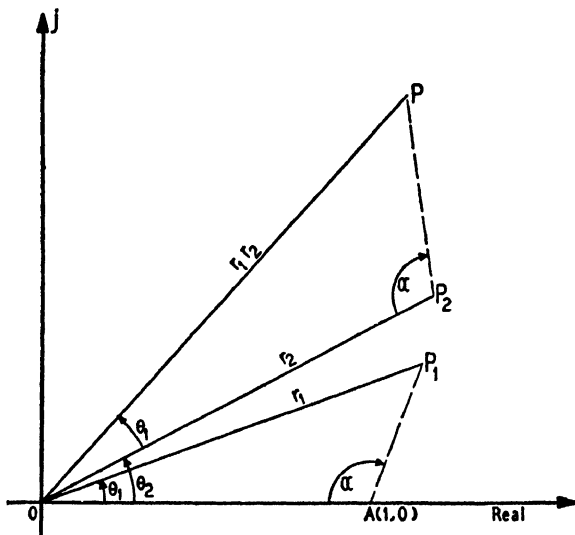


FIG. 12-13.

direction. Therefore j can be considered as an operator that rotates a vector in the complex plane through a positive angle of 90° .

Similar interpretations are given to the powers of j in what follows:

j^2 corresponds to a 180° counterclockwise rotation;

j^3 corresponds to a 270° counterclockwise rotation or to a 90° clockwise rotation.

EXERCISES

Perform the indicated operations, first reducing the numbers to polar form (if necessary). In the odd-numbered exercises express the results also in rectangular form.

1. $2/\underline{43^\circ} \cdot 3/\underline{15^\circ}$.

2. $5.1/\underline{53.7^\circ} \cdot 2.3/\underline{10.9^\circ}$.

3. $(3 + j2)(5 + j)$.

4. $(3.2 - j7.1)(8.5 + j3.4)$.

- | | |
|---|--|
| 5. $7/31^\circ \cdot 2/37^\circ \cdot 3/42^\circ$. | 6. $1.2/25.1^\circ \cdot 3.1/17.2^\circ \cdot 2.1/21.8^\circ$. |
| 7. $(3 + j)(7 - j2)(1 + j5)$. | 8. $(1.2 - j3.4)(7.3 + j3.7)(0.2 - j7.1)$. |
| 9. $77.2/43^\circ \div 28.7/21^\circ$. | 10. $1.87/123^\circ \div 0.27/46^\circ$. |
| 11. $(3 + j5) \div (2 + j)$. | 12. $(1.1 + j7.2) \div (7.5 - j3.4)$. |
| 13. $(1.2 - j3.7)^3$. | 14. $(5.3 + j1.9)^5$. |
| 15. $\frac{6.1}{(3.7 - j5.1)(9.3 + j7.8)}$. | 16. $\frac{(8.1 + j3.4)}{(2.4 + j)(7.7 - j2.1)}$. |
| 17. $\frac{(3.7 + j2.1)^3(7.3 - j3.1)^2}{(0.8 - j9.1)(2.8 + j3.1)}$. | 18. $\frac{3.2/7.1^\circ \cdot 2.1/31.5^\circ}{2.8/13.1^\circ \cdot 1.5/52.7^\circ}$. |

Perform the following multiplications graphically and check the results algebraically.

- | | | |
|-----------------------------|--------------------------------|-------------------------------|
| 19. $j \cdot (3 + j5)$. | 20. $j \cdot (4 - j9)$. | 21. $j \cdot 3/40^\circ$. |
| 22. $j \cdot 6/-90^\circ$. | 23. $j^2 \cdot (1 + j2)$. | 24. $j^2 \cdot 3/125^\circ$. |
| 25. $j^3 \cdot (5 - j)$. | 26. $j^3 \cdot 1/-270^\circ$. | |

Find the following roots. Express the results in rectangular form.

- | | | |
|---------------------------|-------------------------|---------------------|
| 27. $\sqrt{1 + j}$. | 28. \sqrt{j} . | 29. $\sqrt{-j}$. |
| 30. $\sqrt{3.9 + j4.5}$. | 31. $\sqrt[3]{2 + j}$. | 32. $\sqrt[3]{j}$. |

12-11. Summary of Operations with Complex Numbers. A complex quantity may be represented in a rectangular or polar form. One form may be used more advantageously than the other in various operations.

1. *Addition of complex numbers.* Write the numbers in the rectangular form. Add the real parts together and then add the j parts together.

2. *Subtraction of complex numbers.* Write the numbers in the rectangular form. Subtract the real parts and then subtract the j parts.

3. *Multiplication of complex numbers.* Write the numbers in the polar form. Multiply the magnitudes together and add the angles.

4. *Division of complex numbers.* Write the numbers in the polar form. Take the quotient of the magnitudes and subtract the angles.

5. *Powers of complex numbers.* Write the numbers in the polar form. Take the power of the magnitude and multiply the angle by the exponent of the power.

6. *Roots of complex numbers.* Write the numbers in the polar form. Extract the root of the magnitude and divide the angle by the index of the root.

The answer to a problem involving operations with complex numbers may be given either in the rectangular or polar form. In engineering applications, the answer is usually given in the polar form.

Example 1.

$$\begin{aligned} 20/\underline{23^\circ} + 11/\underline{-37^\circ} - 15/\underline{47^\circ} &= (18.4 + j7.81) + (8.78 - j6.63) - (10.2 + j11.0) \\ &= (18.4 + 8.78 - 10.2) + j(7.81 - 6.63 - 11.0) \\ &= 17.0 - j9.82. \end{aligned}$$

The student should convert the answer into the polar form.

Example 2. Using some of the results obtained in Example 2 of Sec. 12-8 in converting from the rectangular form to polar form, we have

$$\begin{aligned} \frac{(67.4 - j37.7)^2(14.6 + j8.86)}{(45.0 - j12.9)^3} &= \frac{(77.2/\underline{-29.2^\circ})^2(17.1/\underline{31.2^\circ})}{(46.8/\underline{-16.0^\circ})^3} \\ &= \frac{5960/\underline{-58.4^\circ} \cdot 17.1/\underline{31.2^\circ}}{103,000/\underline{-48.0^\circ}} \\ &= \frac{5960 \cdot 17.1/\underline{-27.2^\circ}}{103,000/\underline{-48.0^\circ}} = 0.989/\underline{20.8^\circ}. \end{aligned}$$

The student should convert this answer into the rectangular form.

EXERCISES

Perform the indicated operations. Express the results in rectangular and polar forms.

- $10.8/\underline{42^\circ} + 23.5/\underline{31^\circ} - 1.92/\underline{-27^\circ}.$
- $(3.1 + j4.5)(7.2 - j1.1) + (4.1 - j1.3).$
- $7/\underline{-81^\circ} - 2/\underline{110^\circ} + 9/\underline{-34^\circ}.$
- $8.21/\underline{-5.7^\circ} + 12.1/\underline{17^\circ} - 15.4/\underline{95^\circ}.$
- $(23.1 - j7.52) + (12.9 + j4.09) - (19.4 - j1.70).$
- $1.8/\underline{112^\circ} - 1/\underline{-51^\circ} - 4.2/\underline{21^\circ}.$
- $\frac{(5 + j2)^3(1 - j2)^2}{(2 - j4)}.$
- $\frac{j2.6}{(7.1 - j2.3)^3}.$
- $\frac{5}{(3 - j5)^2(9 + j2)^3}.$
- $\frac{\sqrt{j}}{(5 - j2)(7 + j)}.$
- $\frac{(13.1 + j2.72)^3(3.10 - j42.7)^2}{(71.5 + j39.1)^2}.$
- $\frac{5.1/\underline{17^\circ} \cdot 1.2/\underline{-25^\circ}}{3.5/\underline{45^\circ} \cdot 9.3/\underline{-81^\circ}}.$

12-12. The Impedance Triangle. The vector representation of an alternating voltage across a resistor, inductor, and condenser were given in Sec. 6-8. It has been stated that the voltage drop across a resistance R is in phase with the current I and may conveniently be represented by a vector \mathbf{E}_R which has the same direction as the vector representing the current. From Ohm's law $E_R = RI$, where E_R is expressed in volts, R in ohms, and I in amperes.

The voltage drop across an inductance L leads the current by 90° and may conveniently be represented by a vector \mathbf{E}_L which makes an angle of 90° with the current vector. The magnitude in volts of \mathbf{E}_L is $E_L = X_L I$, where X_L is the inductive reactance expressed in ohms.

The voltage drop across a condenser of capacitive reactance X_C (measured in ohms) lags the current by 90° and may conveniently be represented by a vector \mathbf{E}_C which makes an angle of -90° with the current vector. The magnitude in volts of \mathbf{E}_C is $E_C = X_C I$. Kirchhoff's law states that the impressed voltage \mathbf{E} is given by the vector sum

$$(1) \quad \mathbf{E} = \mathbf{E}_R + \mathbf{E}_L + \mathbf{E}_C.$$

If now we choose the current vector \mathbf{I} along the positive x -axis, then the terminal points of the vectors \mathbf{E}_R , \mathbf{E}_L , and \mathbf{E}_C are $(E_R, 0)$, $(0, E_L)$, and $(0, -E_C)$, respectively. If we superimpose a complex plane upon the vector plane so that the horizontal and vertical axes coincide and have the same directions, then these terminal points correspond to the complex numbers E_R , jE_L , and $-jE_C$, as we saw in Sec. 12-5. Then the vector sum (1) corresponds, as we have seen, to the sum of the complex numbers E_R , jE_L , and $-jE_C$. Thus it is common in practice to identify the complex numbers and the corresponding vectors by writing

$$\mathbf{E}_R = E_R, \quad \mathbf{E}_L = jE_L, \quad \mathbf{E}_C = -jE_C,$$

whence the sum (1) can be written as

$$(2) \quad \mathbf{E} = E_R + jE_L - jE_C = IR + jI(X_L - X_C).$$

The total **impedance** \mathbf{Z} (Z being in ohms) offered to the flow of current is represented by a complex number \mathbf{Z} defined by the equation

$$(3) \quad \mathbf{E} = \mathbf{Z}\mathbf{I}.$$

Of course, the term vector cannot be used in the sense of Chapter 6 in connection with (3), for no vector product was defined in that chapter. Furthermore the vector product defined in advanced works do not behave like the complex numbers in the product (3). Therefore, in order to have meaning, (3) must be thought of as expressing a relation between

quantities given in the form of complex numbers. However, in texts on electrical engineering, the term vector is quite frequently used in this connection as synonymous with complex number, rather than in the sense of Chapter 6.

Considering only resistance and inductance in series in an alternating-current circuit (Fig. 12-14), the impressed voltage across this circuit is by (1)

$$\mathbf{E} = \mathbf{E}_R + \mathbf{E}_L,$$

where \mathbf{E}_R is in phase with the current and \mathbf{E}_L leads the current by 90° . Using the current as the horizontal axis (axis of reference), we may

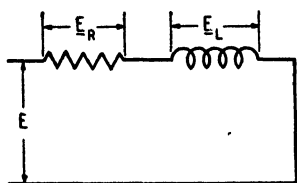


FIG. 12-14.

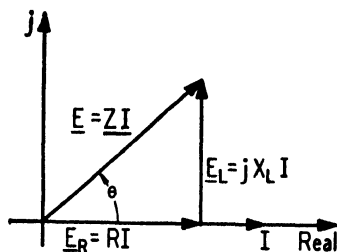


FIG. 12-15.

draw the vector diagram (Fig. 12-15). The magnitude of \mathbf{E}_R is $E_R = RI$, the magnitude of \mathbf{E}_L is $E_L = X_L I$. Introducing complex numbers, we can state that the vector \mathbf{E} corresponds to the complex number $\mathbf{E} = E_R + jE_L = RI + jX_L I$. From (3)

$$\mathbf{Z}\mathbf{I} = \mathbf{E} = E_R + jE_L = RI + jX_L I.$$

Since \mathbf{I} lies along the positive x -axis, the complex number corresponding to \mathbf{I} is I , whence this relation can be written

$$\mathbf{I}\mathbf{Z} = RI + jX_L I,$$

or, dividing by I ,

$$\mathbf{Z} = R + jX_L$$

which gives rise to the **impedance triangle** of Fig. 12-16. The impedance may be stated in polar form as

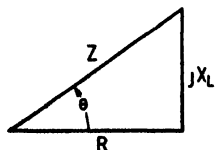


FIG. 12-16.

$$\mathbf{Z}/\theta = \sqrt{R^2 + X_L^2} / \text{Arctan} \frac{X_L}{R}.$$

The student will note that this impedance triangle is similar to the voltage vector diagram and that the impedance is equal to a vector sum of the resistance and inductive reactance.

Example 1. A 5-ohm resistance is connected in series with a 12-ohm inductive reactance across a 60-cycle 110-volt alternating-current supply line. Calculate the impedance, resistance voltage drop, the reactance voltage drop, and the current.

The impedance of the circuit is given by the expression $Z = 5 + j12$. Changing to polar form $Z/\theta = \sqrt{(5)^2 + (12)^2} \angle \text{Arctan } \frac{12}{5} = 13/67.4^\circ$ ohms.

From the impedance triangle and the calculated value of impedance we note that the voltage leads the current by 67.4° in the similar voltage vector diagram. Using the current vector as reference, the voltage vector is then given as $110/67.4^\circ$ volts.

Converting this to the rectangular form

$$\begin{aligned} E &= 110/67.4^\circ = 110(\cos 67.4^\circ + j \sin 67.4^\circ) \\ &= 110(0.3843 + j0.9232) \\ &= 42.27 + j101.55. \end{aligned}$$

From this we note that the component in phase with the current is 42.27 volts, therefore

$$RI = 42.27 \text{ volts.}$$

The component voltage at right angles to the current vector or the j term is 101.55 volts, therefore

$$X_L I = 101.55 \text{ volts.}$$

The current is calculated from the relationship $I = \frac{E}{Z}$. Substituting the values for E and Z obtained above

$$I = \frac{42.27 + j101.55}{5 + j12} = \frac{(42.27 + j101.55)(5 - j12)}{(5 + j12)(5 - j12)} = \frac{1430}{169} = 8.46 \text{ amperes,}$$

which in this case may be written in the polar form

$$I = 8.46/0^\circ \text{ amperes.}$$

In a series alternating-current circuit composed of a resistor and a condenser (Fig. 12-17), the impressed voltage E is equal to the vector

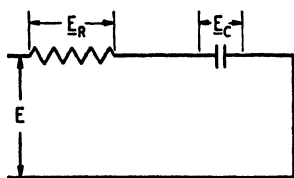


FIG. 12-17.

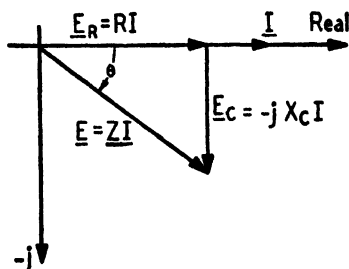


FIG. 12-18.

sum of the components E_R and E_C where E_R is in phase with the current, and E_C lags the current by 90° . Again using the current as the reference line, the voltage vector diagram is drawn as shown in Fig. 12-18;

considering that the magnitude of \mathbf{E}_R is $E_R = RI$, the magnitude of \mathbf{E}_C is $E_C = X_C I$.

The voltage drop $X_C I$ lags the current by 90° and therefore the complex quantity corresponding to the vector \mathbf{E} is $RI - jX_C I$. We note that since the voltage drop across the condenser is directed -90° with respect to the current, the minus sign is used. The impressed voltage may then be written in rectangular form as

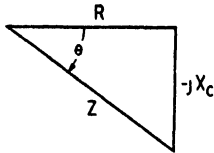


FIG. 12-19.

$$\mathbf{E} = \mathbf{ZI} = RI - jX_C I;$$

hence, dividing by I ,

$$\mathbf{Z} = R - jX_C$$

which gives rise to the impedance triangle (Fig. 12-19). The impedance may be stated in polar form as

$$\underline{Z/\theta} = \sqrt{R^2 + C^2} \angle \text{Arctan} \frac{-X_C}{R}.$$

As in the previous case, the impedance triangle is similar to the voltage vector diagram and the impedance is equal to a vector sum of the resistance and capacitive reactance.

Example 2. A resistance of 30 ohms is connected in series with a capacitive reactance of 40 ohms across a 1000-cycle 20-volt alternating-current supply line. Calculate the impedance, the resistance voltage drop, the reactance voltage drop, and the current.

The impedance of the circuit is $\mathbf{Z} = 30 - j40$. Changing to polar form

$$\underline{Z/\theta} = \sqrt{(30)^2 + (40)^2} \angle \text{Arctan} \frac{-40}{30} = 50 \angle -53.1^\circ \text{ ohms.}$$

The current leads the voltage by 53.1° . Using the current as reference, the voltage vector may then be written in polar and rectangular forms

$$\begin{aligned} \mathbf{E} &= 20 \angle -53.1^\circ = 20(\cos 53.1^\circ - j \sin 53.1^\circ) \\ &= 20(0.6000 - j0.8000) \\ &= 12.0 - j16.0. \end{aligned}$$

From the rectangular expression of the voltage vector \mathbf{E} we note that the component in phase with the current is 12.0 volts and hence

$$RI = 12.0 \text{ volts.}$$

The capacitive reactance voltage drop at right angles to the current vector, or the $-j$ term, is 16.0 volts and hence

$$X_C I = 16 \text{ volts.}$$

The current may be obtained by the equation

$$I = \frac{E}{Z} = \frac{12.0 - j16.0}{30 - j40} = \frac{20 / -53.1^\circ}{50 / -53.1^\circ} = 0.4 / 0^\circ.$$

EXERCISES

1. An alternating-current relay operating on 110 volts has a resistance of 16 ohms and an inductive reactance of 42 ohms. Calculate the impedance and the current.

2. A spark-quenching circuit consists of a 10-ohm resistor and a condenser having a capacitive reactance of 24 ohms connected in series. If the voltage applied to the combination is 80 volts, what are the impedance, resistance voltage drop, reactance voltage drop, and the current in the circuit?

3. An alternating voltage of 220 volts is applied to a solenoid having a resistance of 45 ohms and an inductive reactance of 180 ohms. What is the current flowing through the solenoid?

4. An inductive reactance of 6 ohms is connected in series with a resistance of 18 ohms across a 110-volt 60-cycle alternating-current circuit. What is the voltage drop across the resistance?

5. Two resistors having values of 40 and 10 ohms respectively are connected in series, and they are connected in series with an inductive reactance of 72 ohms across a 240-volt alternating-current circuit. What is the current that flows in the circuit and what is the voltage drop across the 40-ohm resistor?

6. A capacitive reactance of 3200 ohms is connected in series with a resistance of 2400 ohms across a 1000-cycle 1.2-volt circuit. What are the voltage drop in millivolts across the condenser and the current in milliamperes through the condenser?

PROGRESS REPORT

In this chapter, the system of real numbers has been extended to the system of complex numbers so that all operations, including the computation of square roots of negative numbers, can be performed. The geometric representation of complex numbers revealed a close connection between these numbers and vectors so that each vector can be represented by a complex number and conversely. This connection proved to be very useful in the discussion of alternating currents. Alternating voltages, currents, and impedances can be represented by complex numbers; and so the whole technique of algebra can be applied to the examination of electrical phenomena.

CHAPTER 13

LINEAR EQUATIONS AND DETERMINANTS

Students of engineering, physics, and other sciences deal with equations at every stage of their work. In particular, systems of linear equations are used frequently in mechanics, electricity, statistics, and in other applied sciences. This chapter gives some of the graphical and algebraic methods for solving such equations. It also contains a discussion of the theory of determinants and their application to systems of linear equations.

13-1. Systems of Linear Equations with Two and Three Unknowns. Linear equations with one and two unknown quantities were discussed in Chapter 2. The system of two linear equations with two unknowns was solved by deriving from the given system of two equations a single equation containing only one of the two unknowns. This procedure is called elimination of one of the unknowns, and this unknown is said to be eliminated. Two elimination methods were discussed.

A. Elimination by substitution. This method consists in solving one equation for one of the unknowns and substituting the result in the second equation.

Example 1. Find x and y from the system of the two equations

$$(1) \quad x + y = 2,$$

$$(2) \quad x + 2y = 3.$$

Equation 1 solved for x gives

$$(3) \quad x = 2 - y,$$

and this value when substituted in (2) gives an equation with only one unknown quantity:

$$2 - y + 2y = 3,$$

whence

$$y = 1.$$

The substitution of this value in (3) gives $x = 2 - 1 = 1$. Thus $x = 1$, $y = 1$ is the solution of the given system of two simultaneous equations.

B. Elimination by addition or subtraction.

Example 2. We shall study the same system as before.

$$(1) \quad x + y = 2,$$

$$(2) \quad x + 2y = 3.$$

If equation 1 is subtracted from equation 2, x is eliminated and the result is $y = 1$. The substitution of this value in (1) gives $x + 1 = 2$, whence $x = 1$ as before.

The procedure in this case was extremely simple because the unknown x had the same coefficient in both equations. Otherwise, one or both equations must be multiplied by numerical factors which are chosen so that x or y has the same numerical coefficient in both equations.

Example 3. Solve the system

$$(4) \quad 5x - 2y = 4,$$

$$(5) \quad 2x + 3y = 10.$$

If equation 4 is multiplied by 3 and equation 5 by 2, the following equivalent system is obtained.

$$15x - 6y = 12,$$

$$4x + 6y = 20.$$

The unknown y can be eliminated by adding these two equations, thus obtaining

$$19x = 32$$

or

$$x = \frac{32}{19}.$$

In order to find y , the variable x can be eliminated in a similar way. Multiplying the first equation by 2, the second by 5, and subtracting the first from the second, we obtain

$$10x - 4y = 8$$

$$10x + 15y = 50$$

$$19y = 72$$

$$y = \frac{72}{19}.$$

This method of elimination can easily be extended to systems of more than two unknowns. If, for example, a system of three equations with three unknowns x , y , and z is given, the system can be solved in the following way.

1. Eliminate one of the unknowns, say x , by combining two of the given equations by addition or subtraction after having multiplied them by factors such that x has the same numerical coefficient in both equations. The result is an equation containing y and z .

2. Eliminate the same unknown x , using another combination of two of the given equations. The result is again a linear equation in y and z .

3. The two linear equations obtained in the first two steps form a system of two equations with two unknowns which can be solved by the methods described before.

4. The values of y and z obtained in the third step are substituted in one of the given equations in order to find the unknown quantity x .

Example 4. Solve the following system of three simultaneous equations with three unknowns.

$$(A) \quad x + 2y + z = 3,$$

$$(B) \quad 2x + y + 3z = 7,$$

$$(C) \quad 3x + 3y - 4z = 2.$$

The equations are denoted by letters (A) , (B) , (C) so that it is possible to indicate schematically the operations performed with these equations.

In order to eliminate x from the first two equations, multiply equation A by -2 and add the result to equation B

$$(-2A) \quad -2x - 4y - 2z = -6$$

$$(B) \quad 2x + y + 3z = 7$$

$$(-2A + B = B') \quad -3y + z = 1.$$

For brevity, the equation $-2A + B$ is denoted by B' . In order to have a second equation for y and z , the variable x is eliminated from (A) and (C) as follows.

$$(-3A) \quad -3x - 6y - 3z = -9$$

$$(C) \quad 3x + 3y - 4z = 2$$

$$(-3A + C = C') \quad -3y - 7z = -7.$$

Having eliminated x we have now the following system of equations:

$$(B') \quad -3y + z = 1,$$

$$(C') \quad -3y - 7z = -7.$$

The unknown y is eliminated by subtraction and we have

$$(B' - C') \quad 8z = 8$$

or

$$z = 1.$$

The substitution of this value in (B') gives

$$-3y + 1 = 1,$$

and

$$-3y = 0,$$

$$y = 0.$$

The result of the substitution of the values $y = 0$, $z = 1$ in equation A is

$$x + 0 + 1 = 3$$

or

$$x = 2.$$

The given system of equations has, therefore, the solution

$$x = 2, \quad y = 0, \quad z = 1.$$

A check for the computation is obtained by substituting these values in (B) and (C) :

$$2 \cdot 2 + 0 + 3 \cdot 1 = 7,$$

$$3 \cdot 2 + 3 \cdot 0 - 4 \cdot 1 = 2.$$

EXERCISES

Solve the following systems of equations.

1. $R_1 + R_2 + R_3 = 6$,
 $2R_1 + R_2 - R_3 = 3$,
 $R_1 + 2R_2 + 3R_3 = 13$.
2. $3V_1 + V_2 - V_3 = 15$,
 $V_1 + 3V_2 - V_3 = 17$,
 $V_1 + V_2 - 3V_3 = -7$.
3. $x + y + 2z = 7$,
 $2x + 2y + 2z = 10$,
 $3x - 2y + z = -9$.
4. $2Z_1 - 3Z_2 + 3Z_3 = 36$,
 $Z_1 + 3Z_2 + 2Z_3 = 13$,
 $3Z_1 + 4Z_2 = 1$.
5. $V_1 - V_2 + V_3 = -9$,
 $-V_1 + 3V_2 - V_3 = 51$,
 $2V_1 - V_2 + 7V_3 = 63$.
6. $I_1 + 2I_2 + 3I_3 = 6$,
 $2I_1 + 3I_2 + 4I_3 = 7$,
 $3I_1 + 4I_2 + 5I_3 = 8$.
7. $2I_1 + 3I_2 + I_3 = 3$,
 $3I_1 + 5I_2 + 3I_3 = -6$,
 $4I_1 + I_2 + 2I_3 = 3$.
8. $P_1 + P_3 = 4$,
 $3P_1 - 2P_2 - 5P_3 = 6$,
 $4P_1 + P_2 + P_3 = 5$.
9. $2x - 3y + z = 3$,
 $x + 4y - 3z = -5$,
 $3x + y + 4z = 4$.
10. $R_1 + 6R_2 - 2R_3 = 4$,
 $3R_1 + 2R_2 + 4R_3 = -2$,
 $6R_1 - 7R_2 + 5R_3 = 3$.
11. $3I_1 + 6I_2 + 5I_3 - 7 = 0$,
 $I_1 + 2I_2 + 3I_3 - 6 = 0$,
 $-2I_1 + 4I_2 + 7I_3 + 5 = 0$.
12. $3E_1 + 2E_2 + 2E_3 = 4$,
 $5E_1 + 3E_2 + 7E_3 = -3$,
 $6E_1 + 6E_2 - E_3 = 2$.
13. $2c_0 = 8$,
 $3c_1 + 5c_0 = 11$,
 $4c_2 + 7c_1 + 9c_0 = 23$.
14. $2p - 2q + r = 8$,
 $5q - 2r = 7$,
 $3r = 12$.
15. $x_1 + x_2 + x_3 = 17$,
 $x_2 + x_3 = 8$,
 $x_1 + x_3 = 7$.
16. $-A + B + C = 30$,
 $A - B + C = 24$,
 $A + B - C = 20$.
17. $2E_1 - 3E_2 + 5E_3 = 7$,
 $E_1 + 6E_2 + 4E_3 = 8$,
 $7E_1 + 2E_2 + 20E_3 = 9$.
18. $X + Z + R = 9$,
 $X + 2Z + 4R = 15$,
 $X + 3Z + 9R = 23$.
19. $r + s = 27$,
 $r + t = 25$,
 $s + t = 22$.
20. $I_1 + I_2 - I_3 = 0$,
 $I_1 - I_2 = 6$,
 $I_1 + I_3 = 4$.

13-2. Solution of Systems of Linear Equations by the Doolittle Method. A great amount of numerical work is involved in the solution of a system of linear equations if the coefficients are numbers with two or more significant digits. The amount of work is immensely increased if systems of four or more equations have to be solved. Since such systems arise in many practical applications, it is important to have a method which meets the following requirements.

1. The work should be arranged so that what has already been done is obvious at every step of the computation, and so that there is no doubt as to the next step to be performed.

2. The method should be such that the number of necessary multiplications and divisions is as small as possible.

3. The method should be suited to the use of the slide rule or a computing machine.

4. A check for each step of the work should be available.

The mathematician Gauss showed how the method of elimination by adding and subtracting can be arranged so that the conditions enumerated above are satisfied. A modification of his method, which saves a certain amount of work, is due to the American mathematician Doolittle. This method, known as the Doolittle method, will be discussed in this section. It will be explained by means of examples.

An important feature of the Doolittle method is the way each step of the computation is checked. In this check, the sum of all the coefficients and the right-hand member of each equation is computed. This sum is called the **check sum** of the equation.

Example 1. The check sum of the equation $3x - 4y + 5z = 10$ is $3 - 4 + 5 + 10 = 14$.

The check sums are always written on the same line as the equation itself, separated by a vertical line.

Example 2. A system of three equations with its check sums is written in this way.

$$\begin{array}{r|l} x + 2y + z = 3 & 7 \\ 2x + y + z = 7 & 11 \\ 3x + 3y - 4z = 2 & 4 \end{array}$$

The check sums are treated exactly as the coefficients of the corresponding equations. Thus, if the equation is multiplied by a factor, the check sum is to be multiplied by the same factor. Or, if two equations are added or subtracted, the corresponding check sums are to be added or subtracted. *The result of the operations carried out with the check sums of the original equations has to be equal to the check sum of the final equation. This fact is used as the check on the computation.*

Example 3. If the first equation of Example 2 and its check sum are multiplied by 3, the result is

$$3x + 6y + 3z = 9 \quad | \quad 21$$

The check sum of the new equation is $3 + 6 + 3 + 9 = 21$ and is equal to the number which was obtained by multiplying the original check number 7 by 3.

Example 4. If the first two equations of Example 2 and their check sums are added, the equation

$$3x + 3y + 2z = 10 \quad | \quad 18$$

is obtained. Its check sum is $3 + 3 + 2 + 10 = 18$ and is equal to the sum of the check sums 7 and 11.

How this check is used throughout the computation for each single step will be seen from the following example in which Doolittle's method for solving a system of linear equations is explained.

Solve the system of three simultaneous equations.

$$\begin{array}{l|l} (A) & 3.42x - 5.93y + 1.44z = -2.20 & -3.27 \\ (B) & 1.95x + 4.68y - 2.73z = 3.00 & +6.90 \\ (C) & 2.14x + 3.64y + 2.92z = 13.58 & +22.28 \end{array}$$

The check sum of each equation is written in the right-most column, to the right of the vertical line. In the left-most column, each equation is denoted by a letter in order that we shall be able to indicate the operations performed with the equations.

All the computations in this example are made with the slide rule.

Step 1. The first equation is divided by the coefficient 3.42 of x in order to obtain an equivalent equation in which the coefficient of x is equal to 1. The divided equation is given by

$$\left(A' = \frac{A}{3.42} \right) \quad \left| \quad x - 1.734y + 0.421z = -0.643 \quad \right| \quad -0.956$$

The check sum -0.956 is found by dividing the check sum -3.27 of the original equation by 3.42. The check consists in investigating whether the number -0.956 obtained in this way is equal to the check sum of the new equation. This check sum is

$$1 - 1.734 + 0.421 - 0.643 = -0.956$$

which is equal to the number -0.956 found before. It is, therefore, not likely that a numerical error has been made so far.

Step 2. In order to eliminate x , equation A' is multiplied by -1.95 , and the result is added to equation B . This computation is performed below.

(B)	$1.95x + 4.68y - 2.73z = 3.00$	+6.90
(-1.95A')	$-1.95x + 3.38y - 0.82z = 1.25$	+1.86
(B - 1.95A')	$8.06y - 3.55z = 4.25$	8.76

After writing down a line, the check sum test has to be made immediately. Thus, for the last equation, the number 8.76 was obtained by adding the check sums $6.90 + 1.86$, and this has to be compared with the check sum of the new equation

$$8.06 - 3.55 + 4.25 = 8.76.$$

If this check sum had another value, then the computation of the last line would have to be investigated for errors. A small difference of one or two units in the right-most place of the check sum may result from the fact that all the coefficients are incomplete numbers, computed only with a certain number of digits. It is obvious that in adding a few incomplete numbers the result may differ from the correct result by a few units in the right-most digit.

The last equation contains only y and z . Dividing by 8.06, the coefficient of y , we have the following equivalent equation in which the first coefficient is one.

$$\left(B' = \frac{B - 1.95A'}{8.06} \right) \quad \left| \quad y - 0.440z = 0.528 \quad \right| \quad 1.087.$$

Step 3. The two equations A' and B' are now used in order to eliminate x and y from the equation C . This is done in the following way.

First write equation C

$$(C) \quad \left| \quad 2.14x + 3.64y + 2.92z = 13.58 \quad \right| \quad +22.28$$

and write below the product of equation A' and the number -2.14 , the negative coefficient of x in (C) . This product is

$$(-2.14A') \quad \left| \quad -2.14x + 3.71y - 0.90z = 1.38 \quad \right| \quad +2.05.$$

If the equations C and $(-2.14A')$ are added, the unknown x is eliminated, and the unknown y has the coefficient $3.64 + 3.71 = 7.35$. In

order to eliminate y simultaneously, equation B' is multiplied by -7.35 , giving

$$(-7.35B') \quad | \quad -7.35y + 3.24z = -3.88 \quad | \quad -7.99.$$

If now the three last equations are added, x and y are eliminated, and the following equation results.

$$(C - 2.14A' - 7.35B') \quad | \quad 5.26z = 11.08 \quad | \quad 16.34.$$

Dividing this equation by 5.26 gives

$$(C') \quad | \quad z = 2.11 \quad | \quad 3.11.$$

The check sum test indicates no error, because $1 + 2.11 = 3.11$. The value of z obtained in the last step is now substituted in equation B' in order to find y . The result is

$$y = 0.528 + 0.440z = 0.528 + 0.440 \times 2.11 = 1.456.$$

Finally, the values of z and y are substituted in equation A' , from which

$$\begin{aligned} x &= -0.643 - 0.421z + 1.734y \\ &= -0.643 - 0.421 \times 2.11 + 1.734 \times 1.456 \\ &= 0.99. \end{aligned}$$

Writing all the results with the same number of decimal places, the solution is:

$$x = 0.99, \quad y = 1.46, \quad z = 2.11.$$

A check for the whole computation can be made by substituting these values in equation C . The substitution in equation A or B would not furnish a complete test because these equations were used for the computation for y and x .

The whole computation, as described, can be carried out on much less space than has been used here, omitting all explanations and writing only the coefficients instead of the whole equations. In this way it is possible to have a layout for the whole procedure which can be filled in systematically. The following table gives the computation carried out before, but omits everything which is not required for the computation itself or the check.

Name of the Equation	Coefficient of			Right Member	Check Sum
	x	y	z		
A	+3.42	-5.93	+1.44	-2.20	-3.27
$A' = \frac{A}{3.42}$	+1	-1.734	+0.421	-0.643	-0.956
B	+1.95	+4.68	-2.73	+3.00	+6.90
$-1.95A'$	-1.95	+3.38	-0.82	+1.25	+1.86
$B - 1.95A'$	0	+8.06	-3.55	+4.25	+8.76
$B' = \frac{1}{8.06} (B - 1.95A')$		+1	-0.440	+0.528	+1.087
C	+2.14	+3.64	+2.92	+13.58	+22.28
$-2.14A'$	-2.14	+3.71	-0.90	+1.38	+2.05
$-7.35B'$		-7.35	+3.24	-3.88	-7.99
$C - 2.14A' - 7.35B'$	0	0	+5.26	+11.08	+16.34
$C' = \frac{1}{5.26} (C - 2.14A' - 7.35B')$			+1	+2.11	+3.11

$$z = 2.11,$$

$$y = 0.528 + 0.440z = 1.46,$$

$$x = -0.643 - 0.421z + 1.734y = 0.99.$$

The answers can be checked by substituting the values of x , y , z in equation C' :

$$2.14x + 3.64y + 2.92z = 2.12 + 5.31 + 6.17 = 13.60.$$

This result is close enough to the correct value 13.58.

The Doolittle method was explained here with a system of three simultaneous equations with three unknown quantities. It can be used for any number of equations. If the system consists of only two equations with two unknowns, the third step in the computation is not required. If more than three equations are given, the procedure can be continued to the point where only one equation with one unknown is obtained.

EXERCISES

Solve the following systems of simultaneous equations (slide rule precision).

1. $1.67x + 4.83y = 7,$

$8.90x + 11.92y = 14.$

2. $0.077x + 0.66y = -2.15,$

$0.08x - 0.053y = -0.0842.$

3. $60x + 77y = 1209,$

$24x - 35y = 152.3.$

5. $3.4I_1 - 1.2I_2 = -8.16,$

$5.6I_1 + 1.2I_3 = -13.44,$

$5.6I_2 - 3.4I_3 = 38.08.$

7. $2.14a - 0.91b + 0.56c = 1.27,$

$-0.91a + 3.72b - 0.38c = 1.98,$

$0.56a - 0.38b + 2.87c = 2.67.$

4. $1.5x - 1.33y = 4.42,$

$4.5x - 0.33y = -21.58.$

6. $216x + 322y - 508z = 3.19,$

$121x - 97y - 195z = 488,$

$395x - 192y - 213z = -205.$

8. $x + 0.237y - 0.189z = 1.067,$

$0.237x + y + 0.345z = 1.617,$

$-0.189x + 0.345y + z = 1.203.$

13-3. Graphical Representation and Discussion of a System of Two Linear Equations. The problem of finding the solution of a system of simultaneous linear equations may sometimes present difficulties. For a system with two unknowns these difficulties will be discussed in this section. The graphical method explained in Chapter 3 is very helpful in obtaining a clear understanding of the different possibilities.

The graph corresponding to the linear equation $ax + by = c$ consists of all points whose coordinates x and y satisfy this equation. It was stated in Chapter 3 that the graph of a linear equation is always a straight line. This line, as described in Sec. 3-12, can be constructed after finding two of its points.

In order to find the coordinates of a point on the line, an arbitrary value can be assumed for one of the two variables and the corresponding value of the other variable can then be computed from the given equation. In particular, the x -intercept, that is, the abscissa of the point where the straight line meets the x -axis, is found by substituting $y = 0$ in the given equation; the y -intercept is found by substituting $x = 0$.

Example 1. Construct the straight line L corresponding to the equation

$$3x + 4y = 6.$$

Points on this line can be found by substituting particular values for one of the variables. Thus, for $x = 3$, the corresponding value of y is

$$y = \frac{6 - 3x}{4} = \frac{6 - 9}{4} = -\frac{3}{4},$$

and the point $(3, -\frac{3}{4})$ is on the line L . In this way any number of points on the line can be found.

The x -intercept of the line is the point on the line whose ordinate has the value $y = 0$ and is found, therefore, from the equation

$$3x + 4 \cdot 0 = 6,$$

$$3x = 6,$$

$$x = 2.$$

The y -intercept is found by substituting $x = 0$, which gives

$$3 \cdot 0 + 4y = 6,$$

$$y = \frac{3}{2}.$$

The graph of this straight line is plotted in Fig. 13-1.

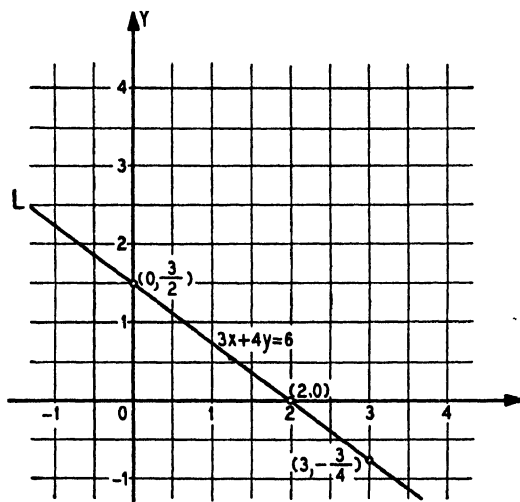


FIG. 13-1.

This graphical representation can be used to solve a system of two simultaneous equations with two unknowns. Each of the two equations can be regarded as an equation between two variables, and the two corresponding straight lines can be constructed. The coordinates of a point where the two lines intersect one another satisfy both equations and are, therefore, a solution of the given system of equations.

Example 2. Solve the following system of equations:

$$4x - 3y = 3,$$

$$3x + 4y = 6.$$

The corresponding straight lines are plotted in Fig. 13-2. They intersect at the point with coordinates $x = 1.2$, $y = 0.6$. This pair of values satisfies, therefore, both equations and is the solution of the given system.

Two straight lines have no point in common if they are parallel. Thus a system of two simultaneous linear equations has no solution if the corresponding straight lines are parallel. Consider, for example, the system of equations

$$3x + 4y = 6,$$

(1)

$$6x + 8y = 3.$$

The graphs of the two equations are plotted in Fig. 13-3. Since the two straight lines are parallel and therefore have no point of intersection, it can be concluded that the given system of equations has no solution.

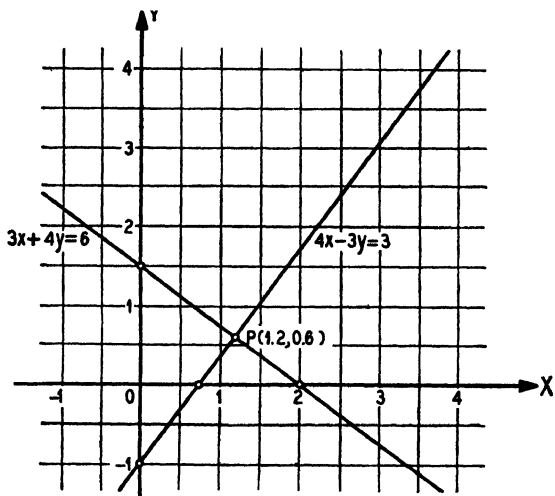


FIG. 13-2.

The fact that there is no solution in this case can be shown without using a graphical representation in the following way. We start with the

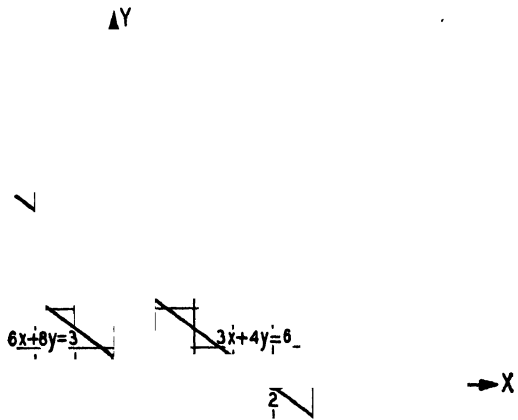


FIG. 13-3.

hypothesis that the given system of equations has a solution and then show that this hypothesis leads to an impossible conclusion and, there-

fore, must be discarded. Thus, if it is assumed that the system has solutions, two numbers x and y could be found such that

$$3x + 4y = 6,$$

$$6x + 8y = 3$$

are satisfied simultaneously. Multiplying the first equation by 2 and subtracting it from the second equation we have $0 = -9$, which is obviously impossible. The hypothesis, therefore, that the given equations have a solution is wrong because it leads to a false statement.

A system of simultaneous equations which have no common solutions is called **inconsistent**. Correspondingly, the expression **consistent** is used for any system of simultaneous equations which have common solutions. Thus, the system of Example 2 is consistent, the system (1) is inconsistent.

In addition to the two cases discussed above, there is a third possibility, namely, that the two lines may be identical. Since in this case the same straight line corresponds to both equations, the coordinates of every point on this line satisfy both equations, and therefore infinitely many solutions can be found.

Example 3. Solve the system

$$3x + 4y = 6,$$

$$5.19x + 6.92y = 10.38.$$

Both equations correspond to the same straight line, which is plotted in Fig. 13-1, and the coordinates of each point of this line satisfy both equations. Multiplying the first equation by 1.73, we obtain the second equation, whence the given equations are equivalent.

The discussion of a system of two simultaneous linear equations with two unknowns given in this section can be now summarized in the following three cases.

1. *The two straight lines corresponding to the equations of the system are not parallel.* In this case the system is **consistent** and has just one solution, and the equations are said to be **independent**.

2. *The two straight lines corresponding to the equations of the system are parallel and different from each other.* In this case the system is **inconsistent** and has no solution.

3. *The two straight lines corresponding to the two equations of the system coincide.* This system is consistent and has infinitely many solutions. In this case the two equations are always equivalent. They are therefore called **dependent** on one another, and the system itself is called **dependent**. Every solution of one equation is also a solution of the other.

EXERCISES

Construct the graphs corresponding to the following systems of equations and indicate the number of solutions.

1. $2x + 7y = 20,$
 $5x - 8y = 10.$

2. $3x - 5y = 15,$
 $2x - y = 2.$

3. $15x - 21y = 20,$
 $10x - 14y = 10.$

4. $0.51A + 0.68B = 0.85,$
 $0.63A + 0.84B = 1.26.$

5. $x + y = 5,$
 $x + y = 12.$

6. $7.4u - 11.1v = 18.5,$
 $22u - 33v = 55.$

7. $35u + 30v = 15,$
 $42u + 36v = 18.$

8. $3m - 2n = 0,$
 $3m + 2n = 6.$

9. $4x + 5y = 0,$
 $x - y = 0.$

10. $4x - 12y = 0,$
 $-5x + 15y = 0.$

11. $A + B = 0,$
 $A - B = 0.$

12. $6u = 8v,$
 $12v = 9u.$

13. $3u + v = 7,$
 $u + 2v = -6.$

14. $p - q = 2,$
 $3p - 3q = 7.$

15. $1.2x - 2.4y = 3.6,$
 $1.8x - 3.6y = 5.4.$

13-4. Solutions of Systems of Two Linear Equations by Determinants.

The graphical method of Sec. 13-3 is often inconvenient and inaccurate, and it cannot easily be extended to systems with three or more unknowns. It is therefore desirable to have a method which permits us to examine a system of linear equations by computations.

We shall start with a system of two equations with two unknowns. In order to make statements which will be true for all such systems, the equations will be written with letters as coefficients,

$$(1) \quad \begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2. \end{aligned}$$

In order to eliminate y , the first equation is multiplied by b_2 , the second by b_1 , giving

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2, \\ a_2b_1x + b_2b_1y &= c_2b_1. \end{aligned}$$

Subtracting the second equation from the first it is found that

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

or

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

Similarly, it can be shown that

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Thus, two formulas have been found by which the solutions of the system 1 can be computed. They look complicated, but a closer examination will show that it is very easy to write them down. The two formulas have identical denominators, given by the expression

$$a_1b_2 - a_2b_1.$$

Expressions of this kind occur frequently in different branches of mathematics. It is, therefore, useful to have a name for them and to examine their properties.

The expression $a_1b_2 - a_2b_1$ is called a **determinant of the second order** and it is denoted by the symbol:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The four numbers a_1, b_1, a_2, b_2 are arranged in two horizontal lines called **rows** and two vertical lines called **columns**. Each number is called an **element** of the determinant. Thus, the first row consists of the elements a_1, b_1 , the second of the elements a_2, b_2 . The first column consists of the elements a_1, a_2 , the second of the elements b_1, b_2 . The four elements form a square with two diagonals, one from the upper left to the lower right corner, the other from the lower left to the upper right corner, as indicated in Fig. 13-4.

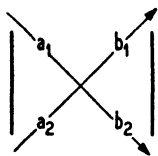


FIG. 13-4.

A determinant of the second order is computed by taking the product a_1b_2 of the elements in the first diagonal and subtracting the product a_2b_1 of the elements in the other diagonal.

The definition of a determinant of the second order and the rule of computing remain the same when the letters a_1, b_1, a_2, b_2 are replaced by particular numbers.

Example 1. Compute the determinant

$$\begin{vmatrix} 7 & 2 \\ 4 & 3 \end{vmatrix}.$$

According to the rule given above the product $7 \cdot 3$ of the elements in the diagonal going from the left down has to be formed, and the product $4 \cdot 2$ of the elements in the diagonal from the left up has to be subtracted, so that

$$\begin{vmatrix} 7 & 2 \\ 4 & 3 \end{vmatrix} = 7 \cdot 3 - 4 \cdot 2 = 21 - 8 = 13.$$

Example 2. Similarly, it is found that

$$\begin{vmatrix} -2 & -5 \\ 4 & 10 \end{vmatrix} = (-2) \cdot 10 - 4 \cdot (-5) = -20 + 20 = 0.$$

Example 3. The computation is simplified if there is an element of the determinant which is zero,

$$\begin{vmatrix} 3 & 9 \\ 0 & -4 \end{vmatrix} = 3 \cdot (-4) - 0 \cdot 9 = -12.$$

The formulas for the solution of a system of two linear equations

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}, \quad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

can be written using the determinant notation:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

This result can be stated in the following rule.

The values of x and y which satisfy the system

$$\begin{aligned} (2) \quad & a_1 x + b_1 y = c_1, \\ & a_2 x + b_2 y = c_2 \end{aligned}$$

are two fractions, each one having the denominator

$$(3) \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is the determinant formed by the coefficients of x and y in the given equations 2. The numerator in the expressions for x is

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

which is the determinant obtained by replacing the elements of the first column of (3) by the right members c_1 and c_2 of (2). The numerator in the expression for y is obtained similarly by replacing the elements of the second column of (3) by the right members of the given equations 2.

Example 4. Using determinants find the solution of

$$\begin{aligned} 7x - 5y &= 9, \\ 3x + 4y &= 10. \end{aligned}$$

The denominator of x and y is

$$\begin{vmatrix} 7 & -5 \\ 3 & 4 \end{vmatrix} = 28 - 3 \cdot (-5) = 28 + 15 = 43.$$

The numerator of x is the determinant obtained when the elements 7 and 3 of the first column of the above determinant are replaced by 9 and 10. This numerator has the value

$$\begin{vmatrix} 9 & -5 \\ 10 & 4 \end{vmatrix} = 36 - 10(-5) = 36 + 50 = 86.$$

Similarly, the numerator of y is equal to

$$\begin{vmatrix} 7 & 9 \\ 3 & 10 \end{vmatrix} = 7 \cdot 10 - 3 \cdot 9 = 70 - 27 = 43,$$

and therefore

$$x = \frac{86}{43} = 2, \quad y = \frac{43}{43} = 1.$$

EXERCISES

Solve the following systems of equations by determinants.

1. $7x + 2y = 10,$
 $3x + 9y = 7.$
2. $2.3u - 5.6v = 9.5,$
 $1.7u - 1.3v = 7.8.$
3. $2E_1 + 3E_2 = 6,$
 $7E_1 + E_2 = 2.$
4. $I_1 - I_2 = 4,$
 $2I_1 + 6I_2 = 16.$
5. $3v_1 - 2v_2 = 1,$
 $3v_1 + 2v_2 = 5.$
6. $2E + 6v = -20,$
 $3E + v = 2.$
7. $2z_1 + z_2 = 3,$
 $8z_1 - 7z_2 = 1.$
8. $I_1 + I_2 = 4,$
 $4I_1 - 3I_2 = 5.$
9. $2R - 5x = 0,$
 $R - x = 3.$
10. $P_1 - 2P_2 = 10,$
 $2P_1 - P_2 = 10.$
11. $6500A + 5300B = 120,000,$
 $8200A - 2100B = 96,000.$
12. $0.5m + 0.7n = 28,$
 $0.2m + 0.8n = 17.$
13. $10r - 5s + 19 = 0,$
 $3r + 2s - 12 = 0.$
14. $15I_1 - 13I_2 = 0,$
 $9I_1 + 7I_2 = 26.$
15. $982x - 587y = 0,$
 $375x + 234y = 0.$
16. $y = 2x - 3,$
 $x + 2y = 14.$
17. $m - \frac{11-n}{3} = 18,$
 $2m = 29 - \frac{n-13}{4}.$
18. $7 - 2L = 1.5(5 - 3R),$
 $R - L = 4.$
19. $\frac{u+1}{10} = \frac{3v-5}{2},$
 $\frac{u-v}{8} = \frac{u+1}{10}.$
20. $2x + y = 7,$
 $\frac{x+2}{2} + \frac{y+3}{3} = 4.$

13-5. Discussion of a System of Two Linear Equations by Means of Determinants. The solutions of a system of two linear equations are, according to the rule of the preceding section, fractions with the denominator $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. These fractions can always be computed if this determinant is not zero. Therefore, the following theorem is true:

Two simultaneous linear equations

$$(1) \quad \begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \end{aligned}$$

are independent, which means that they may be solved and that there is only one pair of values in the solution, if the determinant of the system

$$(2) \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

If the determinant (2) is equal to zero, then the equations in (1) are either inconsistent or dependent. We shall state here without proof that *when the three determinants*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

all have the value zero, the system is dependent. If the first is zero but the other two are not zero, then the system is inconsistent. A few examples will show how to apply these criteria.

Example 1. Consider the system of equations

$$\begin{aligned} 3x - 7y &= 10, \\ 5x + 2y &= 20. \end{aligned}$$

Find the value of the determinant:

$$\begin{vmatrix} 3 & -7 \\ 5 & 2 \end{vmatrix} = 6 - 5(-7) = 6 + 35 = 41.$$

Since this value is different from zero, we conclude, without solving the equations, that the system is independent and has only one solution.

There are many cases in which it is sufficient to know that the system is independent and has only one solution and in which it is not required to compute the values of the unknowns.

Example 2. The system of equations

$$\begin{aligned} 6x + 8y &= 10, \\ 9x + 12y &= 12 \end{aligned}$$

is not independent, for

$$D = \begin{vmatrix} 6 & 8 \\ 9 & 12 \end{vmatrix} = 72 - 72 = 0.$$

In order to decide whether this system is inconsistent or dependent, two other determinants must be computed. These determinants are obtained by replacing, respectively, the first and the second column of D by the right members of the given equations. We have then

$$D_1 = \begin{vmatrix} 10 & 8 \\ 12 & 12 \end{vmatrix} = 120 - 96 = 24,$$

$$D_2 = \begin{vmatrix} 6 & 10 \\ 9 & 12 \end{vmatrix} = 72 - 90 = -18.$$

Since D_1 and D_2 are different from zero, the given system of equations is inconsistent and has, therefore, no solution. The graphs of the two equations are, in this case, two parallel lines.

Example 3. Consider the system of equations

$$6x + 8y = 10,$$

$$9x + 12y = 15.$$

The three determinants whose values we need are,

$$D = \begin{vmatrix} 6 & 8 \\ 9 & 12 \end{vmatrix} = 72 - 72 = 0,$$

$$D_1 = \begin{vmatrix} 10 & 8 \\ 15 & 12 \end{vmatrix} = 120 - 120 = 0,$$

$$D_2 = \begin{vmatrix} 6 & 10 \\ 9 & 15 \end{vmatrix} = 90 - 90 = 0.$$

Hence it follows that the given system is dependent and that every solution of the first equation is also a solution of the second. The two equations are equivalent, for the second equation is obtained by multiplying the first one by $\frac{3}{2}$.

Example 4. Given the following system of two simultaneous equations with the two unknowns x and y :

$$kx + 8y = 10,$$

$$9x + 2ky = 15.$$

Without solving the system, find the values of the quantity k for which the equations are independent, inconsistent, or dependent.

The first step is to compute the determinant of the coefficients of x and y . This gives

$$D = \begin{vmatrix} k & 8 \\ 9 & 2k \end{vmatrix} = 2k^2 - 72.$$

The system is independent for all values of k for which $2k^2 - 72 \neq 0$. For example, if $k = 4$, then

$$D = 2k^2 - 72 = 2 \cdot 16 - 72 = -40 \neq 0,$$

and the system obtained by substituting $k = 4$,

$$4x + 8y = 10,$$

$$9x + 8y = 15,$$

has a solution which can be easily computed.

The system is not independent if the determinant

$$D = 2k^2 - 72 = 0$$

or when

$$k = \pm 6.$$

In order to decide whether in this case the system is inconsistent or dependent the two other determinants

$$D_1 = \begin{vmatrix} 10 & 8 \\ 15 & 2k \end{vmatrix} = 20k - 120,$$

$$D_2 = \begin{vmatrix} k & 10 \\ 9 & 15 \end{vmatrix} = 15k - 90$$

must be computed for $k = \pm 6$. The result is the following.

Case 1. For $k = +6$: $D_1 = 120 - 120 = 0$, $D_2 = 90 - 90 = 0$. The given system is in this case *dependent*.

Case 2. For $k = -6$: $D_1 = -120 - 120 = -240 \neq 0$, $D_2 = -90 - 90 = -180 \neq 0$. The given system is in this case *inconsistent*.

EXERCISES

Decide whether the following systems are independent, inconsistent, or dependent, and compute the values of the unknowns when the equations are independent.

1. $21x - 15y = 12$,
 $3x - 2y = 17$.
2. $24x + 15y = 17$,
 $16x + 10y = 12$.
3. $0.08A - 0.06B = 0.1$,
 $12A - 9B = 15$.
4. $7m = 3 - 5n$,
 $5m = 7n - 6$.
5. $12u + 9 = 4v$,
 $10v + 9 = 30u$.
6. $9.2A = 16.1B + 11.5$,
 $3.5B = 2.0A - 2.5$.
7. $3.8x - 1.4y = 3.6$,
 $5.7x - 2.1y = 5.4$.
8. $6M = 9N + 7$,
 $6N = 4M - 5$.
9. $20p - 19q = 22$,
 $21p - 20q = 23$.
10. $0.65I_1 + 0.85I_2 = 2.35$,
 $0.39I_1 + 0.51I_2 = 1.41$.
11. $3x + 9 = 5y$,
 $3y + 9 = 5x$.
12. $4x - 15 = 8y$,
 $6y - 20 = 4x$.
13. $E_1 + 2E_2 = 3$,
 $2E_1 - E_2 + \frac{1}{2} = 0$.
14. $p + 2q + 11 = 0$,
 $3p + q + 13 = 0$.

Find the values of k for which the following systems are inconsistent or dependent.

$$15. \begin{cases} (2+k)x + (5+k)y = 3, \\ (3+k)x + (4+k)y = 1. \end{cases}$$

$$17. \begin{cases} kx - y = k, \\ x + ky = 1. \end{cases}$$

$$19. (7-k)a + 3b = 12,$$

$$4a + (3-k)b = 8.$$

$$21. \frac{A}{4-k} + \frac{B}{6-k} = 1,$$

$$\frac{A}{5-k} - \frac{B}{7-k} = 1.$$

$$23. \begin{cases} 10m + kn = 2k, \\ 5m + 8n = k. \end{cases}$$

$$16. \begin{cases} (1-k)u + 5v = 7, \\ 5v + (25-k) = 0. \end{cases}$$

$$18. \begin{cases} kx - y = m, \\ kx + y = -m. \end{cases}$$

$$20. \frac{A}{4-k} + \frac{B}{6-k} = 1,$$

$$\frac{x}{5-k} + \frac{y}{7-k} = 1.$$

$$22. \frac{x}{k} = \frac{y}{k+2} + 1,$$

$$\frac{y}{k} = \frac{x}{k+2} + 1.$$

13-6. Systems of Two Linear Homogeneous Equations. A linear equation is called **homogeneous** if its right member is zero. Thus, $3x + 4y = 0$ and $a_1x + b_1y = 0$ are homogeneous equations. There are many problems leading to the solution of a system of two homogeneous equations, and, therefore, such systems will be studied in this section.

The graph of a linear equation without the absolute term as, for example, $3x + 4y = 0$ is a straight line passing through the origin of the coordinate system. This is so because the equation is satisfied when $x = 0$ and $y = 0$ are substituted. A second point of this line can be easily found by choosing an arbitrary value for x and computing the corresponding value of y . Thus, when $x = 4$, $y = -3$. The graph of $3x + 4y = 0$ is plotted in Fig. 13-5.

Suppose a second homogeneous equation be given by $7x - 5y = 0$. The straight line corresponding to this equation also passes through the origin. To obtain a second point on it, the value 5 is substituted for x , giving $y = 7$. From Fig. 13-5, it is obvious that the two straight lines have only the origin as a common point; therefore, the coordinates of this point, $x = 0$, $y = 0$, are the only solution of the system

$$3x + 4y = 0,$$

$$7x - 5y = 0.$$

A similar discussion would show that any two linear homogeneous equations are represented by two straight lines which intersect at the origin, and that, therefore, the coordinates $x = 0$, $y = 0$ are the only solution of the system formed by the two equations. But there is one

exception. It may happen that *the two equations are represented by the same straight line*. The coordinates of every point on this line satisfy both equations, so that there are infinitely many solutions besides $x = 0, y = 0$. This exceptional case is very important for the application of the theory, because we are often required to investigate whether

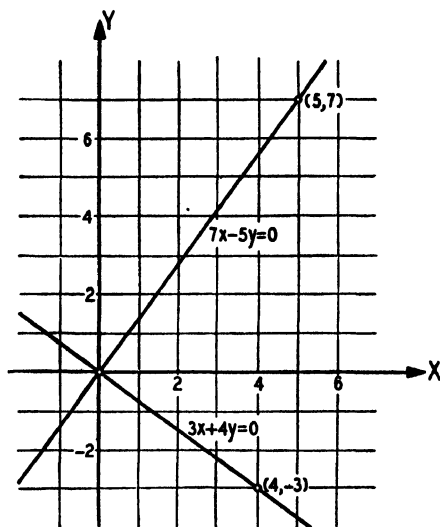


FIG. 13-5.

or not there are solutions different from zero. A test by which the two cases may be distinguished will be found by the rules of Sec. 13-5.

In general, consider the system

$$(1) \quad \begin{aligned} a_1x + b_1y &= 0, \\ a_2x + b_2y &= 0. \end{aligned}$$

The determinant of the coefficients of x and y is given by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In Sec. 13-5 it was stated that the system has only one solution if this determinant is different from zero. This solution is, in this case, $x = 0, y = 0$. This gives the theorem:

When the determinant of the system (1) of two homogeneous linear equations is not zero, then the only solution is given by $x = 0, y = 0$.

Consider now the case when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

and hence

$$a_1b_2 - a_2b_1 = 0,$$

or

$$a_1b_2 = a_2b_1,$$

so that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

If this quotient is denoted by a letter k , we have then,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$$

and

$$a_1 = ka_2, \quad b_1 = kb_2,$$

or

$$a_1x + b_1y = k(a_2x + b_2y).$$

The first equation $a_2x + b_2y = 0$ is, therefore, equivalent to the second equation $a_1x + b_1y = 0$, from which it is derived by multiplication by the factor k . A solution of one of the two equations satisfies, therefore, the second equation, and the lines representing the two equations are identical. Thus, the second part of the theorem can be stated as follows.

A system (1) of two homogeneous linear equations has infinitely many solutions if its determinant is zero.

A few examples will show how to apply the rules of this section.

Example 1. The system

$$3.7x - 4.9y = 0,$$

$$1.3x + 8.5y = 0$$

has the only solution $x = 0, y = 0$, because its determinant

$$\begin{vmatrix} 3.7 & -4.9 \\ 1.3 & 8.5 \end{vmatrix} = 3.7 \cdot 8.5 + 1.3 \cdot 4.9 \neq 0.$$

Example 2. For what values of k has the system

$$(7 - k)x + 2y = 0,$$

$$2x + (4 - k)y = 0$$

only the solution $x = 0, y = 0$?

The determinant of the system is

$$\begin{vmatrix} 7-k & 2 \\ 2 & 4-k \end{vmatrix} = (7-k)(4-k) - 2^2 \\ = 28 - 11k + k^2 - 4 \\ = k^2 - 11k + 24.$$

The system has solutions different from $x = 0, y = 0$ only if this determinant is zero, or only if

$$k^2 - 11k + 24 = 0.$$

The solution of this quadratic equation gives the two values

$$k_1 = 8, \quad k_2 = 3.$$

Hence, it can be stated that the given system has only the solution $x = 0, y = 0$, if a value is chosen for k different from 8 or 3.

If $k = 8$ is substituted, then the given system becomes

$$-x + 2y = 0,$$

$$2x - 4y = 0.$$

These two equations are equivalent, the second being the product of the first equation and -2 . Each pair of numbers satisfying the first equation is a solution of the system. For example,

$$x = 2, \quad y = 1, \quad \text{or} \quad x = 10, \quad y = 5, \quad \text{or} \quad x = 400, \quad y = 200,$$

satisfy both equations.

The substitution $k = 3$ into the given system yields

$$4x + 2y = 0,$$

$$2x + y = 0.$$

Again, both equations are equivalent, the first being twice the second.

EXERCISES

Investigate whether the following systems have solutions different from $x = 0, y = 0$.

$$1. \quad 17x - 15y = 0, \\ 15x - 17y = 0.$$

$$2. \quad 12A + 27B = 0, \\ 8A + 18B = 0.$$

$$3. \quad x + y = 0, \\ x - y = 0.$$

$$4. \quad 6x + 8y = 0, \\ 9x + 12y = 0.$$

$$5. \quad 0.357m - 0.437n = 0, \\ 0.391m - 0.398n = 0.$$

$$6. \quad 3.1P_1 = 5.3P_2, \\ 3.41P_1 = 5.83P_2.$$

$$7. \quad 12E_1 + 56E_2 = 0, \\ -3E_1 - 14E_2 = 0.$$

$$8. \quad 7.1r = 8.3s, \\ 2.3r = 1.5s.$$

$$9. \quad 1.5x + 35y = 0, \\ 0.6x + 14y = 0.$$

Find values for k such that the following systems have solutions different from $x = 0, y = 0$.

$$\begin{array}{lll} 10. \quad 2A + 5B = 0, & 11. \quad (7 - k)x - 5y = 0, & 12. \quad x = ky, \\ & 7A + (4 + k)B = 0. & 8x - 9y = 0. & y = kx. \end{array}$$

$$\begin{array}{ll} 13. \quad kF_1 + 2.7F_2 = 0, & 14. \quad 4A - kB = 0. \\ & 1.5F_1 + kF_2 = 0. & kA - 9B = 0. \end{array}$$

$$15. \quad \frac{x}{7 - k} + \frac{y}{9 - k} = 0, \quad 16. \quad 11x + 3y = kx,$$

$$\frac{x}{11 - k} + \frac{y}{13 - k} = 0. \quad 11y + 3x = ky.$$

$$\begin{array}{ll} 17. \quad (11 - k)A + 7B = 0, & 18. \quad (13 - 2k)m - (15 + 4k)n = 0, \\ & 2A + (6 - k)B = 0. & 11m + 6n = 0. \end{array}$$

13-7. Determinants of the Third Order. In the previous sections determinants of the second order were defined and used for the discussion of systems of two linear equations with two unknowns. Determinants of higher order must be introduced if systems of linear equations with more than two unknowns are to be examined.

A determinant of the third order is related to a system of nine numbers which are arranged in three rows and three columns. It is denoted by enclosing the system of nine numbers by two vertical lines, like

$$\begin{vmatrix} 3 & 5 & 11 \\ 9 & -3 & 8 \\ 1 & 5 & 7 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Each of the nine numbers is called an **element** of the determinant. Every element is located in a certain row and in a certain column. For example, in the last determinant the element c_2 is in the second row and in the third column; this can be also expressed by saying that the row number of c_2 is 2 and its column number is 3.

An element is called **even** if the sum of its row and column numbers is even; it is called an **odd** element if this sum is odd. Thus the element c_2 is odd because $2 + 3 = 5$. The element a_1 is even because it is in the first row and first column and the sum $1 + 1 = 2$ is an even number. *Throughout this section the terms even and odd will be used exclusively in the sense just described.*

The computation of a determinant of the third order can be reduced to the computation of determinants of the second order. How this can be done will be shown in what follows.

If in a determinant of the third order one row and one column are omitted, four elements remain from which a determinant of the second order can be formed. Thus, if in

$$\begin{vmatrix} 3 & 5 & 11 \\ 9 & -3 & 8 \\ 1 & 5 & 7 \end{vmatrix}$$

the second row and the third column are crossed out,

$$\begin{vmatrix} 3 & 5 & 11 \\ \hline 9 & -3 & 8 \\ 1 & 5 & 7 \end{vmatrix},$$

the remaining four elements

$$\begin{vmatrix} 3 & 5 \\ 1 & 5 \end{vmatrix}$$

form a determinant of the second order.

To each element of the third-order determinant, there corresponds a second-order determinant consisting of the elements which remain if the row and the column of the given element are crossed out. This procedure is used in the following definition of a **cofactor**.

The cofactor of a given element in a determinant of the third order is defined in the following way.

- (a) *The row and the column of the given element are crossed out.*
- (b) *The second-order determinant of the four remaining elements is formed.*
- (c) *This determinant is the cofactor of the given element if this element is even. For an odd element, the negative value of this determinant is the cofactor of the given element.*

Example 1. Find the cofactors of the elements of the first row of the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In order to find the cofactor of a_1 , the first line and the first column have to be omitted, which gives

$$\begin{vmatrix} \hline a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The four remaining elements form a determinant of the second order

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

Since element a_1 is even, its cofactor is equal to this determinant.

The element b_1 is odd. Its cofactor, therefore, is the negative value of the determinant obtained after crossing out the first row and second column in D . Thus the cofactor of b_1 is

$$-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

Similarly, the cofactor of c_1 is

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Example 2. Compute the cofactor of the element -3 in the determinant

$$\begin{vmatrix} 3 & 5 & 11 \\ 9 & -3 & 8 \\ 1 & 5 & 7 \end{vmatrix}.$$

This element, being in the second row and second column, is even. Its cofactor, therefore, is

$$\begin{vmatrix} 3 & 11 \\ 1 & 7 \end{vmatrix} = 21 - 11 = 10.$$

The definition of a determinant of the third order can now be stated as follows:

In order to compute a determinant of the third order, each element of the first row is multiplied by its cofactor, and the three products are added.

When this definition is applied in order to find the value of

$$(1) \quad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

we obtain

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

where the values of the cofactors are taken from Example 1.

If the three second-order determinants of the last expression are computed, we have then,

$$\begin{aligned} D &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ &= a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1a_2b_3 - c_1a_3b_2. \end{aligned}$$

Rearranging the last sum by writing first the positive terms, we have the following expression for the determinant of the third order given in (1):

$$D = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1.$$

This expression, which looks complicated, can be found by the following simple rule, which applies *only* to determinants of third order.

To the right of the given determinant, the first two columns are repeated so that the following is obtained:

$$\left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right|$$

Then multiply the elements in the diagonal of the original square which runs from the upper left corner downward and the elements in the two lines parallel to this diagonal indicated in Fig. 13-6. These three products are the positive terms of D . In order to obtain the negative terms, multiply the elements in the diagonal of the original square, running from the lower left corner upward, and the elements in the two lines parallel to this diagonal as shown in Fig. 13-6.

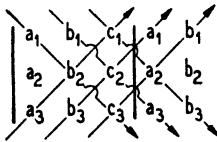


FIG. 13-6.

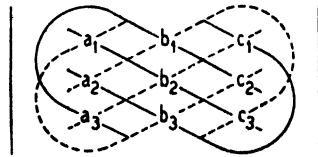


FIG. 13-7.

An alternative rule which gives the same result is indicated in Fig. 13-7.

Example 3. Compute the value of the determinant

$$D = \left| \begin{array}{ccc} 3 & 5 & 11 \\ 9 & -3 & 8 \\ 1 & 5 & 7 \end{array} \right|.$$

By the rule given above we write

$$\left| \begin{array}{ccc|cc} 3 & 5 & 11 & 3 & 5 \\ 9 & -3 & 8 & 9 & -3 \\ 1 & 5 & 7 & 1 & 5 \end{array} \right|.$$

Hence we have for our determinant:

$$D = 3 \cdot (-3) \cdot 7 + 5 \cdot 8 \cdot 1 + 11 \cdot 9 \cdot 5 - 1 \cdot (-3) \cdot 11 - 5 \cdot 8 \cdot 3 - 7 \cdot 9 \cdot 5 \\ = -63 + 40 + 495 + 33 - 120 - 315 = 568 - 498 = 70.$$

Example 4. Compute the value of the determinant

$$D = \begin{vmatrix} 10 & 0 & -5 \\ 1 & 6 & 0 \\ 0 & 9 & 2 \end{vmatrix}.$$

As in the previous example we write

$$\begin{vmatrix} 10 & 0 & -5 \\ 1 & 6 & 0 \\ 0 & 9 & 2 \end{vmatrix} \begin{vmatrix} 10 & 0 \\ 1 & 6 \\ 0 & 9 \end{vmatrix}.$$

which gives

$$D = 10 \cdot 6 \cdot 2 + 0 + (-5) \cdot 1 \cdot 9 - 0 - 0 - 0 = 120 - 45 = 75.$$

When the elements are complicated numbers, the computation of determinants may be very troublesome. Such computations can be greatly simplified by several rules relating to determinants which are given below without proof. The same rules hold also for determinants of the second order and for determinants of higher orders.

The value of a determinant of the third order was originally defined as the sum of the products of the elements of the first row by their cofactors. The same value is obtained if the elements of the first row are replaced by the elements of another row or of a column. Thus, the first of the theorems to be stated is:

1. *The value of a determinant is obtained if the elements of any row or any column are multiplied by their respective cofactors and these products are added.*

2. *If only one element in a row or in a column is different from zero, the determinant is the product of this element and its cofactor.*

This statement is an immediate consequence of the first theorem.

Example 5. Using the last result we have

$$\begin{vmatrix} 3 & 4 & 2 \\ 0 & 0 & 4 \\ 2 & 1 & 3 \end{vmatrix} = -4 \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -4(3 - 8) = 20.$$

Other consequences of the first theorem are the following.

3. *A determinant is zero if all elements of one row or all elements of one column are zero.*

4. *The value of the determinant changes its sign if two rows or two columns are interchanged.*

This last theorem will be illustrated in the next two examples.

Example 6. Consider the determinant

$$\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 5 - 8 = -3.$$

Interchanging the two rows we obtain

$$\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8 - 5 = 3.$$

Example 7. Consider the determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 + 2 \cdot 4 \cdot 2 + (-1) \cdot 3 \cdot 3 - 2 \cdot 1 \cdot (-1) - 3 \cdot 4 \cdot 1 - 1 \cdot 3 \cdot 2$$

$$= 1 + 16 - 9 + 2 - 12 - 6 = -8.$$

The determinant obtained by interchanging the first and third columns is given by

$$\begin{vmatrix} -1 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (-1) \cdot 1 \cdot 2 + 2 \cdot 3 \cdot 1 + 1 \cdot 4 \cdot 3 - 1 \cdot 1 \cdot 1 - 3 \cdot 3 \cdot (-1) - 2 \cdot 4 \cdot 2$$

$$= -2 + 6 + 12 - 1 + 9 - 16 = +8.$$

5. *A determinant is zero if the elements of one row or of one column are equal to the corresponding elements of another row or column, respectively.*

Example 8. To illustrate the last theorem, consider the following determinant having the elements of the first column equal to the corresponding elements of the third column:

$$\begin{vmatrix} 3 & 1 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 3 \cdot 3 \cdot 1 + 1 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 2 - 1 \cdot 3 \cdot 3 - 2 \cdot 2 \cdot 3 - 1 \cdot 2 \cdot 1$$

$$= 9 + 2 + 12 - 9 - 12 - 2 = 0.$$

6. *A determinant is multiplied or divided by a number k if all elements of one row or of one column are multiplied or divided respectively by this number k .*

It follows from this theorem that any factor common to all the elements of a row or column may be removed and written before the determinant.

Example 9.

$$\begin{vmatrix} 24 & 32 & 16 \\ 5 & 1 & 3 \\ 3 & 4 & 7 \end{vmatrix} = \begin{vmatrix} 8 \cdot 3 & 8 \cdot 4 & 8 \cdot 2 \\ 5 & 1 & 3 \\ 3 & 4 & 7 \end{vmatrix} = 8 \begin{vmatrix} 3 & 4 & 2 \\ 5 & 1 & 3 \\ 3 & 4 & 7 \end{vmatrix}.$$

7. If to the elements of any column we add or subtract k times the corresponding elements of any other column, the value of the determinant remains unchanged. The same rule holds for rows.

Example 10. If the elements of the second column of the determinant

$$\begin{vmatrix} 3 & 4 & 10 \\ 1 & 2 & 6 \\ 2 & 6 & 15 \end{vmatrix}$$

are multiplied by 3 and then subtracted from the third column, the result is

$$\begin{vmatrix} 3 & 4 & -2 \\ 1 & 2 & 0 \\ 2 & 6 & -3 \end{vmatrix}.$$

In order to check the given theorem, both determinants can be computed:

$$\begin{vmatrix} 3 & 4 & 10 \\ 1 & 2 & 6 \\ 2 & 6 & 15 \end{vmatrix} = 3 \cdot 2 \cdot 15 + 4 \cdot 6 \cdot 2 + 10 \cdot 1 \cdot 6 - 2 \cdot 2 \cdot 10 - 6 \cdot 6 \cdot 3 - 15 \cdot 1 \cdot 4$$

$$= 90 + 48 + 60 - 40 - 108 - 60 = -10.$$

$$\begin{vmatrix} 3 & 4 & -2 \\ 1 & 2 & 0 \\ 2 & 6 & -3 \end{vmatrix} = -3 \cdot 2 \cdot 3 + 4 \cdot 0 \cdot 2 + (-2) \cdot 1 \cdot 6 - 2 \cdot 2 \cdot (-2) - 6 \cdot 0 \cdot 3 - (-3) \cdot 1 \cdot 4$$

$$= -18 - 12 + 8 + 12 = -10.$$

The computation of a determinant can often be greatly simplified by the application of these rules. Theorem 7 is especially useful in decreasing the numerical value of elements of the determinant or, as in Example 10, in reducing an element to zero.

A few examples will show how these theorems are used for the computation of determinants.

Example 11. Find the value of the determinant

$$D = \begin{vmatrix} 32 & 31 & 15 \\ 40 & 44 & 23 \\ 35 & 34 & 18 \end{vmatrix}.$$

By Theorem 7 the value of D is not changed if the second column is subtracted from the first. This operation diminishes the terms of the first column and makes

$$D = \begin{vmatrix} 1 & 31 & 15 \\ -4 & 44 & 23 \\ 1 & 34 & 18 \end{vmatrix}.$$

In order to continue with the simplification of D , the first row is subtracted from the third, making one element zero. The result is

$$D = \begin{vmatrix} 1 & 31 & 15 \\ -4 & 44 & 23 \\ 0 & 3 & 3 \end{vmatrix}.$$

All the elements of the third row are divisible by 3. Hence the application of Theorem 6 gives

$$D = 3 \begin{vmatrix} 1 & 31 & 15 \\ -4 & 44 & 23 \\ 0 & 1 & 1 \end{vmatrix}.$$

Subtraction of the third column from the second gives

$$D = 3 \begin{vmatrix} 1 & 16 & 15 \\ -4 & 21 & 23 \\ 0 & 0 & 1 \end{vmatrix}.$$

The last determinant can be easily computed because of the zeros in the third row. We have finally

$$D = 3[1 \cdot 21 - (-4) \cdot 16] = 3 \cdot 85 = 255.$$

Example 12. Compute the determinant

$$D = \begin{vmatrix} 13 & 19 & 11 \\ 7 & 9 & 5 \\ 10 & 14 & 8 \end{vmatrix}.$$

The elements of the third row are divisible by 2, therefore

$$D = 2 \begin{vmatrix} 13 & 19 & 11 \\ 7 & 9 & 5 \\ 5 & 7 & 4 \end{vmatrix}.$$

The terms of D can be diminished in different ways. For example, in the last determinant the elements of the third row are multiplied by 2 and subtracted from the

first, and then the elements of the third row are subtracted from the second. Both operations, when performed simultaneously, give

$$D = 2 \begin{vmatrix} 3 & 5 & 3 \\ 2 & 2 & 1 \\ 5 & 7 & 4 \end{vmatrix}.$$

Subtraction of the second row from the third in the last determinant gives

$$D = 2 \begin{vmatrix} 3 & 5 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 3 \end{vmatrix}.$$

In this determinant, the elements of the first and third row are respectively equal; therefore, according to Theorem 5,

$$D = 0.$$

Example 13. Compute the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

If the first column is subtracted from the second and from the third column, we obtain

$$\begin{aligned} D &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}. \end{aligned}$$

All the terms of the second column are divisible by $b-a$ and all the terms of the third column by $c-a$. Applying Theorem 6 we get,

$$D = (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}.$$

To simplify the last determinant, the second column is subtracted from the third, yielding

$$D = (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & b+a & c-b \end{vmatrix},$$

and therefore

$$D = (b-a)(c-a)(c-b).$$

EXERCISES

Compute the values of the following determinants.

$$1. \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 5 \\ 6 & 2 & 3 \end{vmatrix}.$$

$$2. \begin{vmatrix} 3 & 0 & -4 \\ -2 & 4 & 5 \\ 0 & 3 & -1 \end{vmatrix}.$$

$$3. \begin{vmatrix} 1 & -3 & 2 \\ 4 & 6 & -3 \\ 3 & -2 & 2 \end{vmatrix}.$$

$$4. \begin{vmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \\ 5 & 7 & 3 \end{vmatrix}.$$

$$5. \begin{vmatrix} -3 & 5 & 7 \\ 2 & 4 & 6 \\ 7 & 3 & 1 \end{vmatrix}.$$

$$6. \begin{vmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{vmatrix}.$$

$$7. \begin{vmatrix} -5 & 2 & 3 \\ 2 & -3 & 1 \\ 1 & 4 & 6 \end{vmatrix}.$$

$$8. \begin{vmatrix} -8 & 2 & 3 \\ 5 & 1 & 2 \\ -3 & 6 & 3 \end{vmatrix}.$$

$$9. \begin{vmatrix} 9 & 3 & 7 \\ -2 & 1 & -3 \\ 3 & 2 & 4 \end{vmatrix}.$$

$$10. \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

$$11. \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

$$12. \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

$$13. \begin{vmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix}.$$

$$14. \begin{vmatrix} 1 & 5 & 6 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{vmatrix}.$$

$$15. \begin{vmatrix} 0 & 9 & 3 \\ 0 & 0 & 4 \\ 2 & 8 & 6 \end{vmatrix}.$$

$$16. \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix}.$$

$$17. \begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}.$$

$$18. \begin{vmatrix} 0 & -m & -n \\ m & 0 & -p \\ n & p & 0 \end{vmatrix}.$$

$$19. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

$$20. \begin{vmatrix} 10 & 9 & 8 \\ 9 & 8 & 7 \\ 8 & 7 & 6 \end{vmatrix}.$$

$$21. \begin{vmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \\ 6 & 10 & 14 \end{vmatrix}.$$

$$22. \begin{vmatrix} 17 & 36 & 28 \\ 19 & 21 & 12 \\ 11 & 16 & 19 \end{vmatrix}.$$

$$23. \begin{vmatrix} 57 & 63 & 48 \\ 31 & 26 & 19 \\ 64 & 71 & 80 \end{vmatrix}.$$

$$24. \begin{vmatrix} 3.3 & -2.2 & 4.5 \\ -2.9 & 1.7 & -3.2 \\ 4.2 & 3.8 & 4.7 \end{vmatrix}.$$

$$25. \begin{vmatrix} 7 & 9 & 16 \\ 0 & 0 & 0 \\ 1 & 9 & 5 \end{vmatrix}.$$

$$26. \begin{vmatrix} 1 & 0 & 8 \\ 3 & 0 & 14 \\ 7 & 0 & 9 \end{vmatrix}.$$

$$27. \begin{vmatrix} 27 & 93 & 28 \\ 324 & 213 & 629 \\ 0 & 0 & 0 \end{vmatrix}.$$

$$28. \begin{vmatrix} 8 & 11 & -17 \\ -9 & 20 & 3 \\ -9 & 20 & 3 \end{vmatrix}.$$

$$29. \begin{vmatrix} 17 & 5 & 17 \\ 1 & 6 & 1 \\ 2 & 9 & 2 \end{vmatrix}.$$

$$30. \begin{vmatrix} 3 & 0 & 2 \\ 6 & 4 & 7 \\ 3 & 0 & 2 \end{vmatrix}.$$

$$31. \begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 9 & 81 \end{vmatrix}.$$

$$32. \begin{vmatrix} 1 & 1 & 1 \\ 3 & 10 & 4 \\ 9 & 100 & 16 \end{vmatrix}.$$

$$33. \begin{vmatrix} 64 & 8 & 1 \\ 36 & 6 & 1 \\ 9 & 3 & 1 \end{vmatrix}.$$

$$34. \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}.$$

$$35. \begin{vmatrix} 1 & 1 & 1 \\ A & B & C \\ A^2 & B^2 & C^2 \end{vmatrix}.$$

$$36. \begin{vmatrix} 1 & 2u & 4u^2 \\ 1 & 3v & 9v^2 \\ 1 & 4w & 16w^2 \end{vmatrix}.$$

$$37. \begin{vmatrix} k-1 & 2 & 3 \\ 1 & k-1 & 2 \\ -1 & -2 & k-1 \end{vmatrix}.$$

$$38. \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$

$$39. \begin{vmatrix} 0 & r & s \\ -r & 0 & -p \\ -s & p & 0 \end{vmatrix}.$$

13-8. Systems of Linear Equations with Three Unknowns. Determinants of the third order can be used to obtain the solution of a system of three simultaneous linear equations just as determinants of the second order were used for solving a system of two simultaneous equations. Since space does not permit the development in this book of the general theory of systems of three or more equations, only three of the more important problems concerning such systems will be discussed. These problems are:

1. Is the given system independent?
2. When does a system of homogeneous equations have a solution in which not all the unknowns are equal to zero?
3. What is the solution of a system of two homogeneous equations in three unknowns?

Each one of these three problems is solved by a theorem. The resulting three theorems will be stated in this section without proof.

A system of three linear equations with three unknown quantities is called **independent** if it can be solved and also if there is only one system of values of the unknowns satisfying the given equations. This is exactly the same definition of the idea of independence which was given in the discussion of systems of two equations (Sec. 13-3).

The theorem solving the first of the above stated problems is given in the following.

1. *The system of equations*

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3,$$

is independent if the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is formed from the coefficients of the unknowns is not zero.

If this determinant is zero, the system is either inconsistent or dependent. In the first case there is no solution, and in the second case there are infinitely many solutions.

Example 1. Consider the system of equations:

$$x + y + z = 3,$$

$$2x - y + 3z = 1,$$

$$4x + y + 5z = 5.$$

This system is not independent, because the determinant of the coefficients

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \end{vmatrix} = -5 + 12 + 2 + 4 - 3 - 10 = 0.$$

By theorems which are not given here it could be shown that this system of equations is inconsistent. This fact, however, can be shown directly by multiplying the first equation by 2, the second by 1, the third by -1 , and adding the resulting equations.

$$2x + 2y + 2z = 6$$

$$2x - y + 3z = 1$$

$$-4x - y - 5z = -5$$

$$0 = 2$$

This, obviously, is impossible.

The method for solving an independent system of three linear equations with three unknowns was given in Sec. 13-1.

The second problem deals with a system of equations of the form

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0.$$

In these equations the constant term is missing, and they are called homogeneous. It is obvious that these equations are satisfied if

$$x = 0, \quad y = 0, \quad z = 0.$$

In many problems it is important to know whether or not there are other solutions such that not all the unknowns are zero. The answer to this problem is given by the following theorem.

2. *The system of three homogeneous equations with three unknowns*

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0$$

has only one solution,

$$x = 0, \quad y = 0, \quad z = 0,$$

if the determinant of the system

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is not zero. It has infinitely many solutions if this determinant D is equal to zero. In this case all the solutions can be found by multiplying the cofactors of the elements of the first (or any other) row of D by an arbitrary number, so that these solutions are given by

$$\begin{aligned} x &= t \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = t(b_2c_3 - b_3c_2), \\ y &= -t \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = t(a_3c_2 - a_2c_3), \\ z &= t \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = t(a_3b_3 - a_3b_2), \end{aligned}$$

where t is arbitrary.

Example 2. Solve the system of equations

$$x + y + z = 0,$$

$$2x - y + 3z = 0,$$

$$4x + y + 5z = 0.$$

The determinant of this system as computed in Example 1 is

$$(1) \quad \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \end{vmatrix} = 0.$$

According to Theorem 2, the given system has infinitely many solutions. In order to find them, we compute the cofactors of the elements of the first row of the above determinant:

$$\begin{vmatrix} -1 & 3 \\ 1 & 5 \end{vmatrix} = -5 - 3 = -8, \quad - \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -(10 - 12) = 2,$$

$$\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2 + 4 = 6.$$

All the solutions of the system are now given by

$$(2) \quad x = -8t, \quad y = 2t, \quad z = 6t,$$

where t is an arbitrary number. This solution can be checked by substituting the expressions for x , y , and z in the given system of equations. Thus we obtain

$$x + y + z = -8t + 2t + 6t = 0,$$

$$2x - y + 3z = -16t - 2t + 18t = 0,$$

$$4x + y + 5z = -32t + 2t + 30t = 0.$$

A simple solution is obtained by choosing, for example, $t = \frac{1}{2}$, so that

$$x = -4, \quad y = 1, \quad z = 3.$$

The same solution is obtained if the cofactors of another row of the determinant in (1) are used. Computing the cofactors of the third row we obtain

$$\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3 + 1 = 4, \quad - \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3 - 2) = -1,$$

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3.$$

Hence, if u is an arbitrary number, we have as the solution

$$(3) \quad x = 4u, \quad y = -u, \quad z = -3u.$$

These expressions give exactly the same values for x , y , z as those obtained in (2), for when $u = -2t$ we see that (3) is the same as (2).

From Theorem 2 it follows that, if the determinant of a system of three homogeneous linear equations is zero, the equations are dependent in such a way that each set of solutions satisfying two of the equations is also a solution of the third equation. This statement can be used to solve a system of two homogeneous equations with three unknowns

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0.$$

For if these two equations are regarded as the first two equations of a system of three equations, the determinant of the coefficients is zero, so that the solution of this system can be expressed by the cofactors of the missing third row, which have the values

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

We have now the following theorem.

3. *The solutions of the system of two simultaneous linear equations with three unknowns*

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0$$

are given by the expressions

$$x = t \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad y = t \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad z = t \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

where t is an arbitrary number.

Example 3. Solve the system of equations

$$3x - 4y + 5z = 0,$$

$$2x + y + z = 0.$$

The three determinants, used in Theorem 3, are in this case

$$\begin{vmatrix} -4 & 5 \\ 1 & 1 \end{vmatrix} = -4 - 5 = -9, \quad \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7, \\ \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 + 8 = 11.$$

Hence all the solutions of the given system are given by

$$x = -9t, \quad y = 7t, \quad z = 11t,$$

where t is arbitrary. If, for example, the value $t = 1$ is chosen, then

$$x = -9, \quad y = 7, \quad z = 11.$$

These values are easily checked to be solutions by substitution in the given equations. The result is

$$3(-9) - 4 \cdot 7 + 5 \cdot 11 = -27 - 28 + 55 = 0,$$

$$2(-9) + 7 + 11 = -18 + 7 + 11 = 0.$$

The definition of determinants of the third order by means of the cofactors of the elements of a row or column can be extended to define determinants of any order. It can be shown that the theorems of Sec. 13-7 hold for determinants of any order. Likewise, the theorems about systems of linear equations given above hold for systems with any number of unknowns.

EXERCISES

Decide whether or not the following systems are independent.

1. $3x - 4y + 5z = 1,$
 $-4x + 2y + 3z = 2,$
 $5x + 3y + z = 3.$

3. $y + z = 1,$
 $x + z = 2,$
 $x + y = 3.$

5. $2A + 3B + 4C = 0,$
 $2A + 3B - 4C = 0,$
 $2A - 3B + 4C = 1.$

7. $x + y = 10,$
 $x + z = 19,$
 $y + z = 23.$

2. $3x - 4y + 5z = 2,$
 $7x - 2y - z = 8,$
 $2x + y - 3z = 3.$

4. $q - r = 4,$
 $-q + r = -5,$
 $p - q = 1.$

6. $7u + 8v + 9w = 10,$
 $8u + 9v + 10w = 11,$
 $9v + 10v + 11w = 12.$

8. $x + 3y + 5z = 1,$
 $2x + 6y + 10z = 3,$
 $8x + 9y + 11z = 0.$

Solve the following systems of homogeneous equations.

9. $3x - 4y + 5z = 0,$
 $7x - 2y - z = 0,$
 $2x + y - 3z = 0.$

11. $2x + 5y - 8z = 0,$
 $3x + 6y - 9z = 0,$
 $4x + 7y - 10z = 0.$

13. $3q - 5r = 0,$
 $5r + 7p = 0,$
 $3q - 7p = 0.$

15. $2x + 5y - 6z = 0,$
 $x - 8y + 3z = 0.$

17. $3A_x + 7A_y - 5A_z = 0,$
 $A_x - 12A_y + 8A_z = 0$

19. $e_1 - 2e_2 - 4e_3 = 0,$
 $4e_1 + e_2 - e_3 = 0.$

10. $x + y + z = 0,$
 $x - y + z = 0,$
 $x + y - z = 0.$

12. $3M - 5N + P = 0,$
 $7M + 3N - 5P = 0,$
 $5M - N - 2P = 0.$

14. $2.7u - 3.6v + 5.2z = 0,$
 $1.2u + 2.8v - 1.9z = 0,$
 $5.1u + 2.0v + 1.4z = 0.$

16. $7A + 5B + 3C = 0,$
 $2A - 8B + 9C = 0.$

18. $3u - 2v - w = 0,$
 $4u - 2v - 4w = 0.$

20. $2i_1 + i_2 - 3i_3 = 0,$
 $i_1 - 4i_2 + i_3 = 0.$

Solve the following equations for A , B , C and give the particular solution for which the arbitrary constant t , used in the third theorem of this section, is equal to 1.

21. $2A + 3B + C = 0,$
 $5A - 7B + C = 0.$

23. $k^2A + kB + C = 0,$
 $A + B + C = 0.$

22. $A + 2B - 3C = 0,$
 $3A - B + C = 0.$

24. $Ax_1 + By_1 + C = 0,$
 $Ax_2 + By_2 + C = 0.$

Find, if possible, values of k such that the following systems of equations have other solutions besides $x = 0, y = 0, z = 0$.

$$\begin{aligned} 25. \quad & x + y + z = 0, \\ & x + 2y + kz = 0, \\ & x + 4y + 2kz = 0. \end{aligned}$$

$$\begin{aligned} 26. \quad & (2 - k)x + (3 + 2k)y - (1 + k)z = 0, \\ & 3x - 4y + 2z = 0, \\ & 4x + 3y - 6 = 0. \end{aligned}$$

$$\begin{aligned} 27. \quad & (1 - k)x - 2y + z = 0, \\ & -2x + (2 - k)y + 6z = 0, \\ & x + 6y - 31z = 0. \end{aligned}$$

$$\begin{aligned} 28. \quad & kx + 2y - z = 0, \\ & x - 3y + kz = 0, \\ & 3x + 2y + z = 0. \end{aligned}$$

13-9. Engineering Applications of Simultaneous Equations. Simultaneous linear equations arise in many engineering problems, especially in electrical networks in electrical engineering and in the theory of structures in civil engineering. We shall consider in this section a system of simultaneous linear equations arising in connection with an electrical network.

In an automobile, airplane, tank, or tractor, an electric generator charges a storage battery, and both of these in turn supply the current for the ignition system and lights.

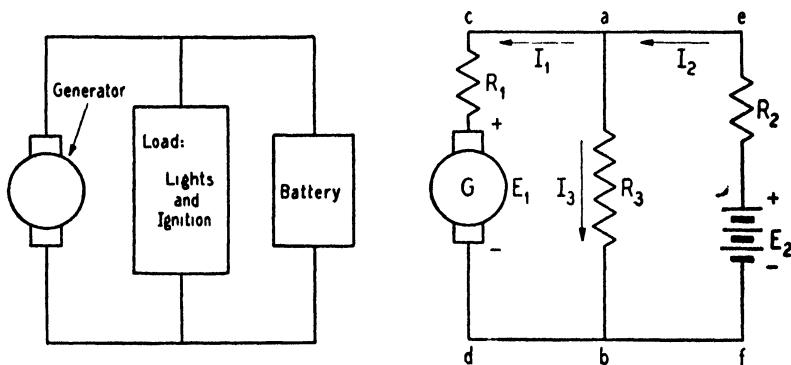


FIG. 13-8.

diagram and a schematic diagram of such a circuit. The voltage of the generator is E_1 and the voltage of the battery is E_2 . The internal resistance of the generator is R_1 and the internal resistance of the battery is R_2 . The load will be considered simply as a resistance R_3 . The currents flowing in the branch circuits are I_1 , I_2 , and I_3 as shown. Their direction of flow as indicated by the arrows is chosen arbitrarily. If we should later find that in the solution of a particular problem a current comes out with a negative value, then the current flows in a

direction opposite to that made in the original assumption. The current junction points are marked a and b .

Kirchhoff's laws state that

1. The algebraic sum of the currents at a junction is equal to zero.
2. In an electric circuit the algebraic sum of the voltage rises and voltage drops is zero.

The algebraic sign given to any voltage rise is positive if the direction of travel is from negative to positive. The sign associated with a voltage drop across a resistance is positive if the direction of travel is in the assumed direction of current flow.

From the circuit given above, by these laws and Ohm's law (see Sec. 2-18) we can write the circuit equations. If we regard the voltage and resistance values as known, then there are three unknown currents, the current through the generator, the current through the battery, and the current through the load. By Kirchhoff's first law, we see that

$$(1) \quad I_3 = I_2 - I_1.$$

Adding the voltage rises and drops in the circuit $acdb$ by going around the circuit in a counterclockwise direction

$$(2) \quad -I_1 R_1 - E_1 + I_3 R_3 = 0.$$

Similarly, for the circuit $abfe$ we obtain

$$(3) \quad -I_3 R_3 + E_2 - I_2 R_2 = 0.$$

We have then the system of equations:

$$(1) \quad I_1 - I_2 + I_3 = 0,$$

$$(2) \quad -R_1 I_1 + R_3 I_3 = E_1,$$

$$(3) \quad R_2 I_2 + R_3 I_3 = E_2,$$

which can be solved by the methods of this chapter for I_1 , I_2 , and I_3 .

EXERCISES

1. In the circuit given in Fig. 13-8 let the generator voltage be 8.5 volts and that of the battery 6.8 volts. The internal resistance of the generator is 0.6 ohm and that of the battery 0.2 ohm. The load has a resistance of 0.3 ohm. What is the current delivered by the generator and the battery?

2. In a farm lighting plant arranged as shown in Fig. 13-8, the generator voltage is 36 volts and that of the battery is 32 volts. The internal resistance of the generator is 0.2 ohm and that of the battery 0.08 ohm. The load resistance is 0.6 ohm. What are the values of the generator, battery, and load currents?

3. In the above circuit write the three simultaneous equations assuming that (a) the polarity of the storage battery is reversed and (b) the polarity of both the battery and the generator is reversed.

4. Rewrite the simultaneous equations of the circuit in Sec. 13-8, assuming that the internal resistances of the battery and generator are zero.

PROGRESS REPORT

In this chapter graphical and algebraic methods were given for solving systems of linear equations in two and three unknowns. Several algebraic methods of elimination were discussed. When the coefficients in a system of linear equations have several digits, it is convenient to use the Doolittle method to solve the system. This method is adapted to the use of the slide rule or the computing machine.

In this chapter determinants were also discussed. It was shown how determinants can be used to find out whether the linear equations of a system are independent, dependent, or inconsistent, and also how they can be used to find out whether the linear equations of a homogeneous system are independent or dependent. In the last section it was shown how a system of linear equations arises in the solution of an electrical network.

CHAPTER 14

QUADRATIC EQUATIONS AND EQUATIONS OF HIGHER DEGREE

In many applications it is necessary to know how to operate with polynomials and how to find the roots of polynomial equations in one unknown. In this chapter we shall learn to handle operations with polynomials and how to find the real roots, both rational and irrational, of polynomials in one unknown.

14-1. Polynomials. An expression of the form

$$(1) \quad 7x^3 + 2x^2 - 5x^4 + 9 - x$$

is called a **polynomial** or more precisely, a **polynomial in x** . The exponent of the highest power of x (in this case 4) is called the **degree** of the polynomial. It is usually convenient to arrange the terms of the polynomial according to descending or ascending exponents of x . The polynomial (1), if arranged according to descending exponents of x , can be written as

$$-5x^4 + 7x^3 + 2x^2 - x + 9.$$

It is a polynomial of the fourth degree. The numbers $-5, 7, 2, -1, 9$ are called the **coefficients** of the polynomial, -5 being the coefficient of the highest power of x . The number 9 is called the **absolute term** of the polynomial. As examples: $10 + 30t - 16t^2$ is a polynomial of second degree in t ; the absolute term is 10 . $Q^3 - 2Q$ is a polynomial of third degree in Q ; the absolute term is zero.

When speaking about polynomials in general, we denote the coefficients by letters. Thus, an arbitrary polynomial of the fourth degree can be written in the form

$$Ax^4 + Bx^3 + Cx^2 + Dx + E.$$

It is sometimes useful to denote all the coefficients by the same letter and to distinguish them by subscripts. Thus, a polynomial of the n th degree can be written as

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n.$$

Polynomials of the first degree, namely,

$$A_0x + A_1,$$

are also called **linear**. These were studied in Sec. 2-12. Polynomials of the second degree, namely,

$$A_0x^2 + A_1x + A_2$$

are called **quadratic**. The next few paragraphs are devoted to the study of quadratic polynomials and equations connected with them.

14-2. Quadratic Polynomials and Functions. An arbitrary quadratic polynomial in x will be written, in the following paragraphs, in the form

$$Ax^2 + Bx + C,$$

A being the coefficient of the highest or quadratic term, B the coefficient of the linear term, and C the absolute term.

If x is regarded as variable, the function

$$y = Ax^2 + Bx + C$$

is called a **quadratic function of x** . For convenience we shall sometimes use the functional notation discussed in Chapter 3 and write

$$y = f(x) = Ax^2 + Bx + C.$$

In order to obtain a good idea of the behavior of the quadratic function, we shall construct tables of values for several examples of quadratic functions and plot them.

Example 1. Constructing a table of values and a graph for the quadratic function

$$y = \frac{1}{2}x^2 - x + 2,$$

we obtain

x	-3	-2	-1	0	1	2	3	4
y	9.5	6	3.5	2	1.5	2	3.5	6

Example 2. Similarly we obtain for the quadratic function

$$y = 5x^2 + 2x - 4,$$

x	-3	-2	-1	0	1	2	3
y	35	12	-1	-4	3	20	47

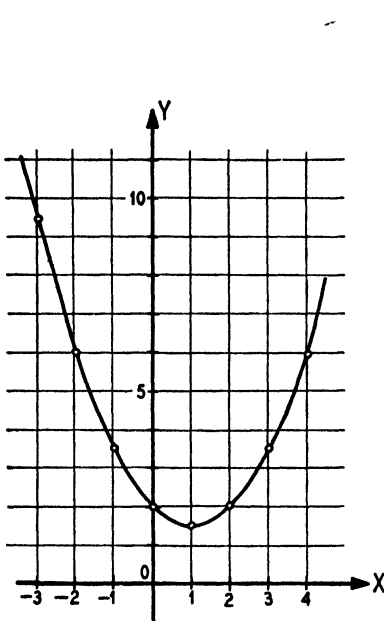


FIG. 14-1.

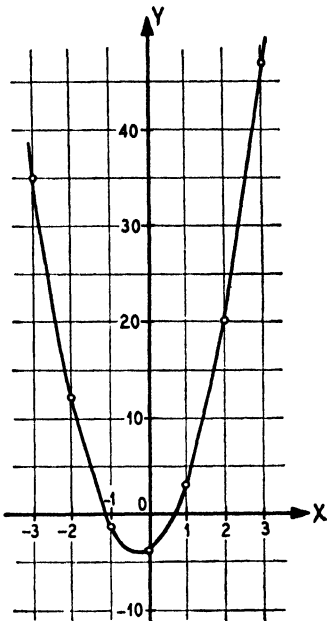


FIG. 14-2.

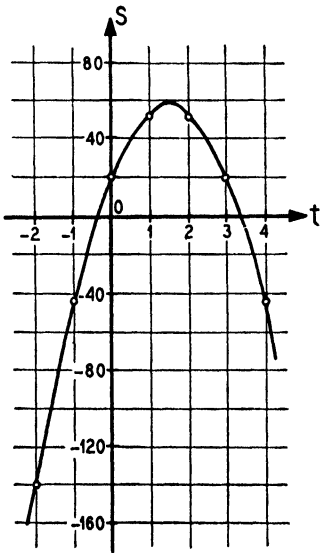


FIG. 14-3.

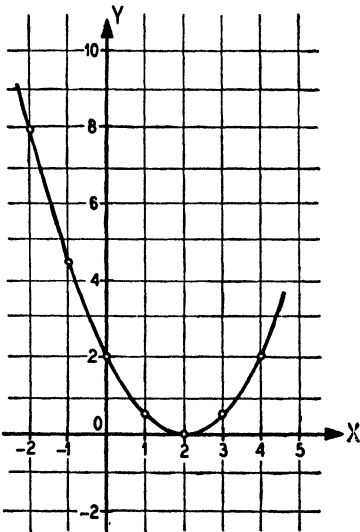


FIG. 14-4.

Example 3. In a like manner we obtain for the quadratic function

$$s = 20 + 48t - 16t^2$$

the following table and graph.

t	-2	-1	0	1	2	3	4
s	-140	-44	20	52	52	20	-44

Example 4. For the quadratic function

$$y = \frac{1}{2}x^2 - 2x + 2$$

we have the following values.

x	-2	-1	0	1	2	3	4
y	8	4.5	2	0.5	0	0.5	2

The curves obtained in Fig. 14-1, 14-2, 14-3, and 14-4 by plotting the graphs of quadratic functions are called **parabolas**. In general, *the graph of the quadratic function*

$$y = Ax^2 + Bx + C$$

is called a parabola. Inspecting the graphs plotted above it can be seen that a parabola has the following characteristics.

1. If the coefficient A of the second degree term is positive, the graph has a lowest point, called the **vertex** of the parabola.
2. If the coefficient A of the second degree term is negative, the graph has a highest point, also called the **vertex** of the parabola.
3. It is sometimes convenient to know that the coordinates of the

vertex are $x = -\frac{B}{2A}$, $y = -\frac{B^2 - 4AC}{4A}$.

14-3. Graphical Solution of Quadratic Equations. An equation whose left-hand side is a polynomial and whose right-hand side is zero is said to be an equation in the **polynomial form**. An equation which can be **reduced** to the polynomial form is called an **algebraic equation**. For example, equations involving radicals of the unknown are algebraic equations since, as shown in Sec. 8-10, they can be reduced to the polynomial form. Throughout this chapter, the term algebraic equation will refer to equations in the polynomial form.

In particular, an equation which can be reduced to $Ax^2 + Bx + C = 0$ is called a **quadratic equation**. The problem of finding its roots can be put in this way: *Find the values of x such that the function*

$$y = Ax^2 + Bx + C$$

assumes the value zero. This can be done by plotting the graph of the function and estimating the abscissas of the points where the graph crossed the x -axis. To illustrate this we use the examples of the preceding section.

Example 1. Find the roots of $\frac{1}{2}x^2 - x + 2 = 0$.

From Fig. 14-1 the student will see immediately that the graph of the function $y = \frac{1}{2}x^2 - x + 2$ has no intersection with the x -axis. The given equation, therefore, has no real roots.

Example 2. Find the roots of $5x^2 + 2x - 4 = 0$.

Figure 14-2 shows that the graph has two x -intercepts. The given equation, therefore, has two roots; measuring the x -intercepts, the roots are found to be approximately -1.1 and 0.7 .

Example 3. Find the roots of $20 + 48t - 16t^2 = 0$.

From Fig. 14-3 it is seen that the graph of the corresponding quadratic has two intercepts with the t -axis. Hence the corresponding equation has two roots; they are approximately -0.4 and 3.4 .

Example 4. Find the roots of $\frac{1}{2}x^2 - 2x + 2 = 0$.

Inspection of the corresponding graph (Fig. 14-4) shows that there is only one point where the curve meets the x -axis. The corresponding abscissa is 2; the given equation has therefore only one root $x = 2$.

If the graph of the curve in Fig. 14-4 were slightly lowered (see Fig. 14-5), one would obtain two roots which are very near to one another. The single root $x = 2$ of this example can therefore be thought of as replacing two roots of a more general case and is called, therefore, a **double root**.

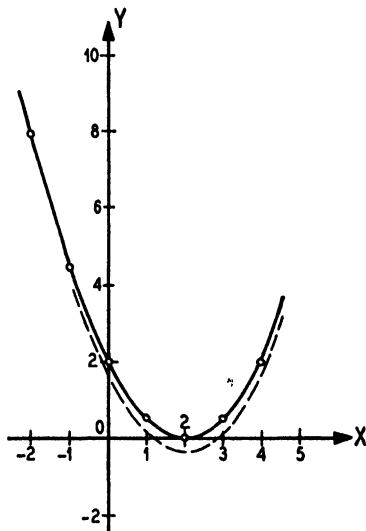


FIG. 14-5.

EXERCISES

1. On the same coordinate system plot the graphs of

(a) x^2 , (b) $\frac{1}{2}x^2$, (c) $2x^2$, (d) $-x^2$, (e) $-\frac{1}{3}x^2$.

2. On the same coordinate system plot the graphs of

(a) $x^2 - 2x$, (b) $x^2 - 2x - 1$, (c) $x^2 - 2x + 1$,
(d) $x^2 - 2x + 2$, (e) $-x^2 + 2x + 3$, (f) $-2x^2 + 5x + 6$.

3. On the same coordinate system plot the graphs of

$$\begin{array}{lll} (a) x^2 - 1, & (b) (x - 1)^2 - 1, & (c) (x + 2)^2 - 1, \\ (d) -x^2 + 4, & (e) -(x + 3)^2 + 8, & (f) -(x + 1)^2 - 1. \end{array}$$

Estimate graphically the roots, if any exist, of the following equations.

$$\begin{array}{lll} 4. 3x^2 - 6x + 2 = 0. & 5. 16t^2 - 35t + 15 = 0. & 6. u^2 + 3u + 4 = 0. \\ 7. y^2 + 3y - 5 = 0. & 8. 4x^2 + 12x + 9 = 0. & 9. -2y^2 + y + 7 = 0. \end{array}$$

10. Given $u = 3w^2 - 6w + 5$, find the values of w such that

$$\begin{array}{lll} (a) u = 1, & (b) u = 2, & (c) u = 3, \\ (d) u = 4, & (e) u = -2, & (f) u = 0. \end{array}$$

14-4. Solution by Factoring. The graphical method described in Sec. 14-3 gives only approximate answers. If more accurate values of the roots are required, other methods must be used.

One method, which can be used only occasionally, uses factoring of the quadratic polynomial $Ax^2 + Bx + C$. This method has been described in Sec. 2-13. The student should review this section before proceeding.

Example 1. Consider the equation $x^2 + x - 6 = 0$. The polynomial on the left side of this equation can easily be factored by writing

$$x^2 + x - 6 = (x + 3)(x - 2).$$

Thus the given equation can be written as

$$x^2 + x - 6 = (x + 3)(x - 2) = 0.$$

As pointed out in Sec. 2-13, the product of two factors can be zero only if at least one of the factors is zero. It follows then that this equation can be satisfied if either

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0.$$

The first possibility gives $x = -3$, the second $x = 2$. The equation has, therefore, the two roots -3 and 2 .

The method of factoring can always be applied, if there is no absolute term in the given equation.

Example 2. To solve $40t - 16t^2 = 0$, we factor the left-hand member and obtain

$$t(40 - 16t) = 0.$$

This equation is satisfied when

$$t = 0 \quad \text{or} \quad t = 2.5.$$

14-5. Pure Quadratic Equations. A quadratic equation whose linear term is missing can be solved very easily. If, in the general quadratic

$Ax^2 + Bx + C = 0$, we have $B = 0$, the equation takes the form sometimes referred to as a **pure quadratic equation**:

$$(1) \quad Ax^2 + C = 0.$$

To solve this equation we have at once

$$x^2 = -\frac{C}{A} \quad \text{and} \quad x = \pm \sqrt{-\frac{C}{A}}.$$

Thus, equation 1 has two real roots when A and C have opposite signs, and two imaginary roots when A and C have like signs.

Example 1. To solve the equation

$$3x^2 - 75 = 0,$$

we have at once

$$x^2 = 25,$$

$$x = \pm 5.$$

Example 2. For the equation $3x^2 + 10 = 0$ we have

$$x^2 = -\frac{10}{3},$$

$$x = \pm \sqrt{-\frac{10}{3}} = \pm j \frac{1}{3} \sqrt{30} = \pm j1.826.$$

Example 3. Compute to three significant digits a positive value of ω such that

$$3 \times 10^{-3}\omega - \frac{1}{2 \times 10^{-6}\omega} = 0.$$

Multiplication by the denominator, which is permitted since $\omega \neq 0$, yields

$$6 \times 10^{-9}\omega^2 = 1$$

or

$$\omega^2 = \frac{1}{6} \times 10^9 = \frac{10}{6} \times 10^8 = 1.67 \times 10^8,$$

and

$$\omega = 1.29 \times 10^4.$$

EXERCISES

Solve the following equations using the slide rule for the numerical computations.

- | | | |
|---|--------------------------|---------------------------------|
| 1. $2.7u^2 = 19.3.$ | 2. $R^2\pi = 12.47.$ | 3. $300 = 17l^2.$ |
| 4. $4\pi^2f^2 = 3 \times 10^8.$ | 5. $13Q^2 + 15 = 16Q^2.$ | 6. $5 \cdot 10^{12} = 10^4u^2.$ |
| 7. $13i^2 - 47 = 0.$ | 8. $3.4E^2 - 8.1E = 0.$ | 9. $23R^2 + 11 = -7R^2.$ |
| 10. $\pi L^2 - 3.85 = L^2.$ | | 11. $0.17E_1 = 0.91E_1^2.$ |
| 12. $1.9R^2 - 3.2R = 4.5R^2.$ | | 13. $9.23Z^2 = 1.72Z.$ |
| 14. $91.7\omega^2 + 1.79 = 1.03\omega^2.$ | | |

15. The impedance of an alternating-current circuit is given by the formula $Z = \sqrt{R^2 + X^2}$ where R is the resistance in ohms and X the reactance in ohms. Compute the value of the reactance if $Z = 60$ ohms and $R = 40$ ohms.

16. If L is the inductance in henries and C the capacitance in farads of an alternating-current circuit, the resonance frequency f in cycles per second satisfies the relation

$$2\pi fL - \frac{1}{2\pi fC} = 0.$$

Solve this equation for f , and compute f if $L = 2 \times 10^{-3}$ henry and $C = 300 \mu\text{f}$. (1 micromicrofarad = 10^{-12} farad).

17. The pull which an electromagnet exerts on its armature is given by the equation $F = \frac{B^2 A}{72,130,000}$ where F is in pounds, B is the magnetic flux density in lines per square inch and A is the area in square inches of the pole face of the magnet.

(a) Solve this equation for B .

(b) In a given electromagnet the flux density is 60,000 lines per square inch and the pole face diameter is 1.0 in. What is the pull in pounds exerted on the armature?

18. The energy stored in a magnetic field is given by the expression $W = L \frac{I^2}{2}$

where W is the energy in watts, L the inductance in henries, and I the current in amperes. Compute the current when $W = 20$ watts and $L = 4.5$ henries.

19. The alternating-current power P in watts in the load resistance of a vacuum tube circuit is given as $P = \frac{\mu^2 E_g^2}{8R_p}$ where μ is the amplification factor of the tube, E_g the peak input voltage to the tube, and R_p the load resistance in ohms. In a certain tube having an amplification factor of 6.2 and a load resistance of 10,500 ohms, what peak input voltage must be supplied in order that the power in the load resistance be 0.6 watts?

20. The kinetic energy of a moving electron is given by the formula $\text{K.E.} = \frac{1}{2}mv^2$ where the kinetic energy is given in ergs, the mass m in grams, and the velocity v in centimeters per second. If the mass of the electron is 9.03×10^{-28} gram and the kinetic energy in a given case is 3.5×10^{-10} erg, what is the velocity?

21. The volume of a cylinder is $0.785d^2h$ where d is the diameter and h the height. What is the diameter of a steel rod 4 in. long which contains a volume of 3.5 cu. in.?

22. The centrifugal force acting on a revolving body is given by the expression $F = 0.00034Wrn^2$ where F is the force in pounds, W is weight in pounds of the body, r is the radius in feet, and n is the number of revolutions per minute the body makes. A 30-lb. weight revolves at a radius of 4.5 ft. At what speed will the centrifugal force on a radial member be equal to 2000 lb.?

23. A general formula for the brake horsepower of a single cylinder gasoline engine is $P = \frac{d^2ln}{18,000}$ where d is the diameter of the cylinder in inches, l the stroke in inches, and n the number of revolutions per minute. Calculate the diameter of the cylinder of an engine which has a stroke of 4 in. and is to deliver 20 horsepower at a speed of 1200 r.p.m.

24. The energy W in joules stored in a condenser of capacity C farads connected to a supply circuit of E volts is given by the equation $W = \frac{CE^2}{2}$. What voltage is required to store 0.061 joule in a condenser of $10 \mu\text{f}$?

25. Fairbain's modified equation for the collapse of a short tube under external pressure is given as $W = 9,675,000 \frac{t^2}{ld}$. Given an external pressure W of 200 lb. per sq. in., a length l of 6 in., and an outside diameter d of 4 in., what will be the thickness t in inches of the tube which will just collapse at that pressure?

26. The heat, measured in calories, developed in a direct-current circuit during t seconds is $H = 0.24I^2Rt$, where R is the resistance in ohms of the circuit and I the current in amperes.

(a) Solve this equation for I .

(b) What current is required to develop 50,000 calories of heat in 10 minutes if the resistance is 20 ohms?

14-6. Quadratic Formula. In practical problems the methods of the two preceding paragraphs can be used only very occasionally. We shall, therefore, develop in the present section a method which can always be applied.

Consider the general quadratic equation

$$(1) \quad Ax^2 + Bx + C = 0,$$

where we may suppose that $A \neq 0$, since otherwise we obtain a linear and not a quadratic equation. When this equation is divided by A , the following equivalent equation is obtained.

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0,$$

or

$$(2) \quad x^2 + \frac{B}{A}x = -\frac{C}{A}.$$

If $\frac{B^2}{4A^2}$ is added to both sides of equation 2, the left member becomes the square of a linear expression, since

$$\left(x + \frac{B}{2A}\right)^2 = x^2 + 2x \frac{B}{2A} + \left(\frac{B}{2A}\right)^2 = x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}.$$

From (2) we obtain, therefore,

$$\left(x + \frac{B}{2A}\right)^2 = x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} = -\frac{C}{A} + \frac{B^2}{4A^2},$$

or

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}.$$

Hence by computing the square root

$$x + \frac{B}{2A} = \pm \sqrt{\frac{B^2 - 4AC}{4A^2}} = \pm \frac{\sqrt{B^2 - 4AC}}{2A},$$

and therefore

$$(3) \quad x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

gives us the formula for the two roots of the general quadratic equation 1. The expression in (3) is called the **quadratic formula**.

For practical applications when using the quadratic formula it is convenient to apply it by the following steps.

1. Find the values of A , B , C .
2. Compute the quantity $D = B^2 - 4AC$. This is called the **discriminant** of the quadratic equation.
3. Compute the roots by the quadratic formula

$$x = \frac{-B \pm \sqrt{D}}{2A}.$$

There are three essentially different cases of the quadratic equation according to the value of D , which are enumerated in the following table.

Case 1. $D > 0$. \sqrt{D} is real.	There are <i>two real and unequal roots</i> .
Case 2. $D = 0$. $\sqrt{D} = 0$.	There is only <i>one real root</i> called a <i>double root</i> .
Case 3. $D < 0$. \sqrt{D} is an imaginary number.	There are <i>two unequal complex roots</i> , which are <i>conjugates</i> .

In order to illustrate the various cases, we apply the quadratic formula to the equations plotted in Sec. 14-2.

Example 1. To solve the quadratic equation

$$\frac{1}{2}x^2 - x + 2 = 0$$

we follow the steps outlined above.

Step 1. $A = \frac{1}{2}$, $B = -1$; $C = 2$.

Step 2. $D = B^2 - 4AC = (-1)^2 - 4 \cdot \frac{1}{2} \cdot 2 = -3$.

Step 3. $x = \frac{-B \pm \sqrt{D}}{2A} = \frac{1 \pm \sqrt{-3}}{2 \cdot \frac{1}{2}} = 1 \pm j\sqrt{3}$.

Since $D = -3 < 0$, this quadratic equation belongs to Case 3. The roots are conjugate complex.

The student should compare Example 1 of Sec. 14-2 and Example 1 of Sec. 14-3.

Example 2. For the equation

$$5x^2 + 2x - 4 = 0$$

we have:

$$A = 5, \quad B = 2, \quad C = -4,$$

$$D = 2^2 - 4 \cdot 5 \cdot (-4) = 4 + 80 = 84.$$

$$\sqrt{D} = \sqrt{84} = 9.17.$$

$$x = \frac{-2 \pm 9.17}{10}.$$

$$x_1 = \frac{-2 + 9.17}{10} = \frac{7.17}{10} = 0.717.$$

$$x_2 = \frac{-2 - 9.17}{10} = \frac{-11.17}{10} = -1.117.$$

Since $D = 84 > 0$, the equation belongs to Case 1, and has two real roots 0.717 and -1.117. This should be compared with the answers obtained in Example 2 of Sec. 14-3.

Example 3. Solve the equation discussed in Example 3 of Sec. 14-3:

$$20 + 48t - 16t^2 = 0.$$

It is always useful to investigate whether a given equation can be simplified. In this case, the whole equation may be divided by 4. If we simplify the equation and rearrange it, we have

$$-4t^2 + 12t + 5 = 0.$$

To solve this equation we write,

$$A = -4, \quad B = 12, \quad C = 5,$$

$$D = 12^2 - 4 \cdot (-4) \cdot 5 = 144 + 80 = 224 > 0, \text{ Case 1.}$$

$$\sqrt{D} = \sqrt{224} = 14.97.$$

$$t_1 = \frac{-12 + 14.97}{-8} = -\frac{2.97}{8} = -0.37.$$

$$t_2 = \frac{-12 - 14.97}{-8} = \frac{26.97}{8} = 3.37.$$

Example 4. To solve the equation

$$\frac{1}{2}x^2 - 2x + 2 = 0$$

we write:

$$A = \frac{1}{2}, \quad B = -2, \quad C = 2,$$

$$D = 4 - 4 \cdot \frac{1}{2} \cdot 2 = 0.$$

$$x = \frac{2 \pm 0}{2 \cdot \frac{1}{2}} = 2.$$

Since $D = 0$, by Case 2, the equation has a double root $x = 2$. Again the student should compare this result with Example 4 of Sec. 14-3.

Example 5. Solve the equation

$$x + \frac{1}{x} = k,$$

and discuss the character of the roots.

Clearing fractions, the given equation may be written in the form

$$x^2 - kx + 1 = 0.$$

To solve this equation we write

$$A = 1, \quad B = -k, \quad C = 1,$$

$$D = k^2 - 4.$$

Hence the values of the roots of the above equation found by the quadratic formula are

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2}.$$

To discuss the character of the roots for different values of k , we must examine carefully the discriminant $D = k^2 - 4$.

Case 1. When

$$D = k^2 - 4 > 0,$$

the roots are real and different. In this case we can write $k^2 > 4$ or $k > +2$ and $k < -2$.

Case 2. When

$$D = k^2 - 4 = 0 \quad \text{or} \quad k = \pm 2,$$

the roots are equal and real.

Case 3. When

$$D = k^2 - 4 < 0 \quad \text{or} \quad k^2 < 4,$$

the roots are conjugate complex.

When the notation $|k|$ for the absolute value of k is used, the result can be stated in the following way.

The equation $x + \frac{1}{x} = k$ has:

- (a) Two different real roots if $|k| > 2$,
- (b) A double root if $|k| = 2$,
- (c) Two complex roots if $|k| < 2$.

14-7. Properties of the Roots of a Quadratic Equation. The roots of the quadratic equation

$$(1) \quad Ax^2 + Bx + C = 0$$

were found by the previous section to be given by the formulas

$$x_1 = \frac{-B + \sqrt{D}}{2A}, \quad x_2 = \frac{-B - \sqrt{D}}{2A}$$

where

$$D = B^2 - 4AC.$$

From these formulas we have at once

$$\begin{aligned}x_1 + x_2 &= \frac{-B + \sqrt{D}}{2A} + \frac{-B - \sqrt{D}}{2A} = \frac{-2B}{2A} = -\frac{B}{A}, \\x_1x_2 &= \frac{-B + \sqrt{D}}{2A} \cdot \frac{-B - \sqrt{D}}{2A} = \frac{B^2 - (\sqrt{D})^2}{4A^2} = \frac{B^2 - D}{4A^2} \\&= \frac{B^2 - (B^2 - 4AC)}{4A^2} = \frac{4AC}{4A^2} = \frac{C}{A}.\end{aligned}$$

Hence it follows that when x_1 and x_2 are the roots of equation 1

$$(2) \quad x_1 + x_2 = -\frac{B}{A}, \quad x_1x_2 = \frac{C}{A}.$$

In the special case when $A = 1$, we may state that the sum of the roots is equal to the negative value of the coefficient of x and the product of the roots equal to the absolute term, i.e., for the equation $x^2 + Bx + C = 0$ we have $x_1 + x_2 = -B$, $x_1x_2 = C$.

The sum and the product of the roots of a quadratic equation can be found without solving the equation and are given by formulas (2). This will be illustrated by the following examples.

Example 1. Consider the equation

$$3x^2 - 7x - 20 = 0$$

where $A = 3$, $B = -7$, $C = -20$. Using formulas (2) we have

$$x_1 + x_2 = \frac{7}{3} \quad \text{and} \quad x_1x_2 = -\frac{20}{3}.$$

Thus without solving the above equation we know that the sum of its roots is $\frac{7}{3}$ and their product $-\frac{20}{3}$. This can be verified by solving the equation, for $D = 49 - 4 \cdot 3(-20) = 289$ or $\sqrt{D} = 17$, and hence

$$x_1 = \frac{7 + 17}{6} = 4 \quad \text{and} \quad x_2 = \frac{7 - 17}{6} = -\frac{5}{3},$$

so that

$$x_1 + x_2 = 4 - \frac{5}{3} = \frac{7}{3} \quad \text{and} \quad x_1x_2 = 4(-\frac{5}{3}) = -\frac{20}{3}.$$

The formulas for the sum and the product of the roots can be used to show that every quadratic polynomial can be factored. This can be stated as follows.

Let $Ax^2 + Bx + C$ be a given polynomial and let x_1, x_2 be the roots of $Ax^2 + Bx + C = 0$. Then $Ax^2 + Bx + C = A(x - x_1)(x - x_2)$.

This statement is very easily proved if the multiplication of the right member of the last equation is performed as follows.

$$\begin{aligned} A(x - x_1)(x - x_2) &= A[x^2 - (x_1 + x_2)x + x_1x_2] \\ &= A\left[x^2 - \left(-\frac{B}{A}\right)x + \frac{C}{A}\right] \\ &= Ax^2 + Bx + C. \end{aligned}$$

Example 2. The roots of the equation

$$3x^2 - 7x - 20 = 0$$

have been found to be

$$x_1 = 4, \quad x_2 = -\frac{5}{3}.$$

We can now factor the corresponding quadratic polynomial by writing

$$3x^2 - 7x - 20 = 3(x - 4)(x + \frac{5}{3}) = (x - 4)(3x + 5).$$

Example 3. Factor the polynomial $x^2 - x - 1$. The roots of the corresponding equation $x^2 - x - 1 = 0$ are found to be

$$x = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2},$$

whence

$$x_1 = \frac{3.236}{2} = 1.618, \quad x_2 = \frac{-1.236}{2} = -0.618,$$

and therefore

$$x^2 - x - 1 = (x - 1.618)(x + 0.618).$$

Example 4. Factor the expression $3x^2 - 5xy + 2y^2$.

First solve the equation

$$3x^2 - 5xy + 2y^2 = 0$$

for x . Using the quadratic formula we have in this case:

$$A = 3, \quad B = -5y, \quad C = 2y^2,$$

$$D = B^2 - 4AC = 25y^2 - 24y^2 = y^2,$$

$$x = \frac{5y \pm y}{6},$$

and the roots are given by

$$x_1 = y, \quad x_2 = \frac{2}{3}y.$$

Therefore

$$\begin{aligned} 3x^2 - 5xy + 2y^2 &= 3(x - x_1)(x - x_2) = 3(x - y)(x - \frac{2}{3}y) \\ &= (x - y)(3x - 2y). \end{aligned}$$

EXERCISES

Solve the following quadratic equations by any of the methods discussed in this chapter. Give all real roots in decimal form correct to three significant digits.

1. $x^2 - 9x + 14 = 0$.
2. $p^2 + 6p - 16 = 0$.
3. $y^2 - 6y + 34 = 0$.
4. $M^2 + 19M + 48 = 0$.
5. $x^2 + 9 = 6x$.
6. $n^2 = 10 - n$.
7. $3E + 9 = E^2 - 9$.
8. $\frac{1}{R^2} + \frac{2}{R} = 24$.
9. $2s^2 + 10s + 15 = 0$.
10. $4t^2 - 2t + 3 = 0$.
11. $5N^2 - 13N - 6 = 0$.
12. $21k^2 + k - 2 = 0$.
13. $12 = 7t + 5t^2$.
14. $6x = 5x^2 + 9$.
15. $k^2 + 3k + 5 = 0$.
16. $5w + 3 = 7w^2$.
17. $15 + 4z - 32z^2 = 0$.
18. $7u^2 + 3 = 4u$.
19. $4Q^2 - 7 = 12Q$.
20. $16 - 24m = 9m^2$.
21. $d^2 + d = 1$.
22. $57y^2 + 43y = 31$.
23. $10t^2 + 11t - 6 = 0$.
24. $3x^2 - 2x = 8$.
25. $7y^2 - 8y = 9$.
26. $x^2 + \sqrt{2}x + 1 = 0$.
27. $1.73x^2 - 4.18x + 1.93 = 0$.
28. $7.8N = 5.07 + 3N^2$.
29. $257k^2 + 362k - 174 = 0$.
30. $2000 = 650t - 16.2t^2$.
31. $0.035u^2 - 3.83 = 0.78u$.
32. $8.7 \times 10^{12}n^2 + 2.3 \times 10^6n = 3.6$.
33. $2.57 \times 10^{-4}t^2 = 3.18 \times 10^{-2} \times 4.64$.
34. $x + \frac{1}{x} = 5$.
35. $\frac{1}{t-2} + \frac{1}{t-3} = \frac{1}{2}$.
36. $3 \times 10^{-3}\omega - \frac{1}{5 \times 10^{-10}\omega} = 4.10^{-6}$.
37. $\frac{1}{R} + \frac{1}{R+1} = \frac{4}{R+3}$.
38. $\frac{1}{E-1} + \frac{1}{E+1} = \frac{1}{E+3}$.
39. $5x + 7 = \frac{6}{2x-3}$.
40. $\frac{4}{3l+2} = \frac{5}{2l^2}$.

Discuss the character of the roots of the following equations without computing the roots.

41. $11x^2 - 20x + 10 = 0$.
42. $7t - 3t^2 = 2$.
43. $2E^2 + 3E + 2 = 0$.
44. $14s^2 + 13s + 12 = 0$.
45. $0 = 20.365t + 28.963t^2 - 51.477$.
46. $3 \times 10^{-6}p^2 - 8 \times 10^{-5}p + 4 \times 10^{-4} = 0$.
47. $3.2m^2 + 0.45 = 2.4m$.
48. $5.1Z^2 + 5.1Z + 1.2 = 0$.

Solve the following equations for x and discuss the character of the roots.

49. $Rx^2 + 3x - 5 = 0$.
50. $x^2 + 7x + 5 = k$.
51. $(x-2)(x-3) = p$.
52. $2x^2 - xy + 3y^2 = 18$.
53. $x^2 - 3x + 1 = p$.
54. $(x+1)(x+3) = q$.

Determine k (real) so that the following equations have double roots.

55. $(k+1)x^2 - 2kx + k - 1 = 0.$

56. $2x^2 - 12x + 7 - k(x^2 + 1) = 0.$

57. $3ky^2 - 12y + k + 1 = 0.$

58. $kx^2 - 5kx + 25 = 0.$

59. $9x^2 + 6kx + 8k = 0.$

60. $(k+1)x^2 = 20x - 9k + 2.$

Factor the following expressions by solving quadratic equations. Express irrationals in decimal form with three significant digits.

61. $x^2 + 2x - 5.$

62. $6x^2 + 5xy - 11y^2.$

63. $3.1p^2 - 4.7pq + 1.6q^2.$

64. $x^2 - 7x + 8.$

65. $2s^2 - 3s - 1.$

66. $36m^2 - 24m + 13.$

67. $35L^2 - 13L - 66.$

68. $2.7t^2 - 6.1t + 4.6.$

69. $8x^2 - 2xy - 15y^2.$

70. $4x^2 - 4xy + 46y^2.$

Solve the equations in Exercises 71-84 and check the results. These equations involve radicals and before solving them the students should review Sec. 8-10, where such equations were discussed.

71. $\sqrt{9x - 7} = 5.$

72. $\sqrt{12.92 - 3p^2} = 3.81.$

73. $\sqrt[3]{y^2 + 10} = 7.$

74. $\sqrt[3]{13.61 + 2.58q^2} = 4.79.$

75. $(4A + 5)^{\frac{1}{2}} = 5.$

76. $(10 + Q^2)^{\frac{1}{3}} = 4.$

77. $\sqrt{4 + 2x} - \sqrt{15 - x} = 1.$

78. $\sqrt{P^2 + 1} + \sqrt{P^2 - 4} = 5.$

79. $\sqrt{2t + 10} + \sqrt{t + 15} = 3.$

80. $2\sqrt{u} + 5 = 3\sqrt{u} + 6.$

81. $\sqrt{2\omega + 1} + 3\sqrt{\omega} = 11.$

82. $\sqrt{2\omega + 1} + \sqrt{3\omega} = 11.$

83. $\sqrt{x + \sqrt{2x + 6}} = 3.$

84. $\sqrt{\sqrt{7y} - 2\sqrt{y - 9}} = 3.$

85. Find two consecutive integers whose product is 342.

86. The length and breadth of a rectangular lot are 80 ft. and 60 ft. Both have to be increased by the same amount so that the area is doubled. What are the dimensions of the new rectangle?

87. An open box containing 480 cu. in. is made by cutting out a 5-in. square from each corner of a square piece of tin and turning up the sides. Find the dimensions of the original piece of tin.

88. A motorboat takes 3 hours to travel 12 miles downstream and back on a river which flows at a rate of 3 miles per hour. Find the speed at which the boat would travel in still water.

89. After traveling 60 miles at a certain speed, a motorist increases his speed by 5 miles per hour and travels 48 miles farther. If he took 4 hours to cover the whole distance of 108 miles, find his speed during the first 60 miles.

90. An object shot upwards with a speed of 400 ft. per second will be, after t seconds, at a height of approximately

$$s = 400t - 16t^2$$

if air resistance is neglected. When will the object be 2000 ft. above the starting point?

91. The e.m.f. of a storage battery is 6.2 volts and its internal resistance is 0.02 ohm. What current will the battery deliver to an electromagnet that requires 200 watts? Use the formulas $I(R_i + R_e) = E$ and $I^2 R_e = P$, where I is the current in amperes, R_i the internal resistance of the battery in ohms, R_e the resistance of the electromagnet in ohms, and P the power in watts.

92. In a vacuum tube with cylindrical electrodes, the amplification constant μ depends on the radius r_g of the grid, the radius r_p of the plate, and on a constant K according to the equation $\mu = K \left(r_g - \frac{r_g^2}{r_p} \right)$. Solve this equation for r_g .

93. The alternating component i_p of a vacuum tube plate current of a variable-mu tube can be expressed approximately in terms of grid voltage e_g by the equation $i_p = c_1 e_g + c_2 e_g^2$ where c_1 and c_2 are constants. Solve this equation for e_g .

94. In an electric circuit comprising a given heater in series with a resistance, the following relationship exists $I^2 R + IE = 600$ where R is the resistance in ohms and E is the applied voltage. If the resistance has a value of 2 ohms and the applied voltage is 110, what is the current I in amperes which flows in the circuit?

14-8. Equations Reducible to Quadratics. There are many complicated equations which can be reduced by a simple substitution to quadratic equations. A few such examples are given below.

Example 1. Solve the fourth-degree equation $x^4 - x^2 - 12 = 0$.

When x^2 is replaced by y this equation becomes

$$y^2 - y - 12 = 0$$

which is a quadratic. This equation has the two roots:

$$y_1 = 4 \quad \text{and} \quad y_2 = -3,$$

from which it follows that

$$x^2 = 4 \quad \text{or} \quad x^2 = -3.$$

The first possibility yields

$$x = \pm\sqrt{4},$$

and the second

$$x = \pm j\sqrt{3}.$$

The given equation has the four roots 2, -2, $j\sqrt{3}$, and $-j\sqrt{3}$. On substitution it can be verified that these numbers satisfy the given equation.

Example 2. Solve the equation $12 \frac{x^2 + 1}{x^2 - 1} + 25 \frac{x^2 - 1}{x^2 + 1} = 35$.

Let

$$\frac{x^2 + 1}{x^2 - 1} = y, \quad \text{then} \quad \frac{x^2 - 1}{x^2 + 1} = \frac{1}{y},$$

and the given equation is reduced to

$$12y + 25 \frac{1}{y} = 35,$$

or the quadratic form:

$$12y^2 - 35y + 25 = 0.$$

The solutions of this equation are

$$y = \frac{35 \pm \sqrt{35^2 - 4 \cdot 12 \cdot 25}}{24} = \frac{35 \pm 5}{24};$$

$$y_1 = \frac{5}{3}, \quad y_2 = \frac{5}{4}.$$

There are, therefore, two possibilities.

$$\frac{x^2 + 1}{x^2 - 1} = \frac{5}{3},$$

$$\frac{x^2 + 1}{x^2 - 1} = \frac{5}{4},$$

$$3x^2 + 3 = 5x^2 - 5,$$

$$4x^2 + 4 = 5x^2 - 5,$$

$$x^2 = 4,$$

$$x^2 = 9,$$

$$x_1 = +2,$$

$$x_3 = +3,$$

$$x_2 = -2.$$

$$x_4 = -3.$$

The numbers 2, -2, 3, and -3 are the roots of the given equation since, on substitution, they are seen to satisfy the given equation.

Example 3. Find all the angles between 0° and 360° so that $25 \sin^2 x + 30 \sin x - 7 = 0$.

The substitution $y = \sin x$ reduces the given equation to

$$25y^2 + 30y - 7 = 0,$$

the solution of which is given by

$$y = \frac{-30 \pm \sqrt{900 + 700}}{50} = \frac{-30 \pm 40}{50},$$

$$y_1 = 0.2, \quad y_2 = -1.4.$$

We have therefore two possibilities.

$$\sin x = 0.2,$$

$$\sin x = -1.4,$$

$$x_1 = 11.5^\circ,$$

which gives no solution, since the absolute value of $\sin x$ cannot be greater than 1.

$$x_2 = 180^\circ - 11.5^\circ$$

$$= 168.5^\circ.$$

The two angles 11.5° and 168.5° are the required roots of the given equation.

Example 4. Find all the angles between 0° and 360° so that $\tan x + \cot x = 3$.

Let $\tan x = y$, then $\cot x = \frac{1}{y}$, and the given equation is reduced to quadratic form:

$$y + \frac{1}{y} = 3 \quad \text{or} \quad y^2 - 3y + 1 = 0.$$

The solutions of this equation are

$$y = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2},$$

$$y_1 = \frac{3 + 2.236}{2} = \frac{5.236}{2} = 2.618,$$

$$y_2 = \frac{3 - 2.236}{2} = \frac{0.764}{2} = 0.382.$$

We have therefore two possibilities.

$$\tan x = 2.618,$$

$$x_1 = 69.1^\circ,$$

or

$$\begin{aligned} x_2 &= 180^\circ + 69.1^\circ \\ &= 249.1^\circ. \end{aligned}$$

$$\tan x = 0.382,$$

$$x_3 = 20.9^\circ,$$

or

$$\begin{aligned} x_4 &= 180^\circ + 20.9^\circ \\ &= 200.9^\circ. \end{aligned}$$

The four angles 20.9° , 69.1° , 200.9° , and 249.1° are the required roots of the given equation.

EXERCISES

Solve the following equations. Give all irrational answers in decimal form, correct to three significant digits.

1. $x^4 - 7x^2 + 4 = 0.$

3. $u^4 + u^2 = 8.$

5. $y^4 - 8y^2 - 9 = 0.$

7. $s^4 - 7s^2 + 1 = 0.$

9. $1 + 0.27y^3 = 0.015y^6.$

11. $\left(x + \frac{6}{x}\right)^2 + 7\left(x + \frac{6}{x}\right) = 60.$

13. $6z^{-4} - z^{-2} - 12 = 0.$

15. $\left(R + \frac{1}{R}\right)^2 + 2\left(R + \frac{1}{R}\right) - 35 = 0.$

16. $(E^2 - E)^2 - 26(E^2 - E) + 120 = 0.$

17. $\frac{1}{x^2 + 3x} + \frac{6}{(x^2 + 3x)^2} = \frac{1}{8}.$

19. $3 \tan^2 \theta - 7 \tan \theta = 5.$

21. $5 \tan A = 2 \cot A + 6.$

23. $5 - 2 \cot u = \frac{3}{\tan^2 u}.$

25. $3^x + 3^{-x} = 10.$

27. $\cos \alpha + \sqrt{1 - \cos \alpha} = 1.$

29. $\sqrt{1 + \tan^2 \theta} = 2 \tan \theta - 3.$

31. $4 \sin x + 3 \cos x = 6.$

2. $3.1t^4 - 6.3 = 12.7t^2.$

4. $M^6 - 5M^3 - 24 = 0.$

6. $6R^4 + 7R^2 - 5 = 0.$

8. $E^{-2} - E^{-1} - 20 = 0.$

10. $15x^{-4} - 26x^{-2} - 371 = 0.$

12. $2(2u^2 - 5)^4 - 27 = 15(2u^2 - 5)^2.$

14. $(L + 3) + \sqrt{L + 3} - 12 = 0.$

18. $(x^2 - 1)^2 + (x^2 - 1)^{-2} = 4.25.$

20. $6.7 \sin^2 x = 2.3 - 5.2 \sin x.$

22. $3 \cos^2 \alpha - \sin^2 \alpha = 4 - 8 \cos \alpha.$

24. $7 + 2^x = 2 + 2^{2x} - 15.$

26. $3^{2x} + 3^x - 5 = 0.$

28. $\sqrt{2 \sin \theta - 1} + 3 = 5 \sin \theta.$

30. $\sin x + \cos x = 1.5.$

32. $1.2 \sin x - 1.7 \cos x = 0.8.$

Solve for x .

33. $\frac{e^x - e^{-x}}{2} = u.$

34. $\frac{e^x - e^{-x}}{e^x + e^{-x}} = v.$

14-9. Polynomials and Algebraic Equations with Two Unknowns. A function of two variables x and y is called a polynomial in x and y if it is a sum of terms of the form

$$ax^p y^q$$

where p and q are positive integers or zero. The sum $p + q$ is called the **degree of the term**, and the highest degree of any term of the sum is called the **degree of the polynomial**.

Example 1. $3x^2 + 2xy - y^2 + 5x - 4y + 6$ is a polynomial of the second degree, or a quadratic polynomial in x and y .

$P^4 + Q^4$ is a polynomial of the fourth degree in P and Q .

$P + 2P^2Q + Q^2 - P^3$ is a polynomial of the third degree in P and Q .

$uv - 2u + 3v$ is a polynomial of second degree in u and v .

$st^2 - 1$ is a polynomial of the third degree in s and t .

An equation between two quantities x and y is called an **algebraic equation of n th degree** if it can be reduced to $P(x, y) = 0$ where $P(x, y)$ is a polynomial of n th degree.

A pair of particular values for x and y , satisfying the equation $P(x, y) = 0$, is called a **pair of solutions** or briefly a **solution** of the given equation. There are infinitely many solutions of such an equation. They can be found by solving the equation for one of the variables; for instance, we can solve for y and then compute the values of y which correspond to arbitrary values of x . This is illustrated in the following example.

Example 2. Let the given equation be $P(x, y) = x^3 - 3x^2y - 10 = 0$.

Solving for y in terms of x we obtain:

$$y = -\frac{10 - x^3}{3x^2}.$$

Using this expression, the value of y corresponding to an arbitrary value of x can be found. For example:

When $x = 1$, then $y = -3$;

When $x = 2$, then $y = -\frac{1}{3}$;

When $x = -1$, then $y = -\frac{11}{3}$.

When two equations involving x and y are given simultaneously, the problem is to find numerical values of x and y satisfying both equations; or we can say simply that our problem is to *solve the simultaneous system of equations*. For example, the system of simultaneous equations

$$x^3 - y^2 = 23,$$

$$x^2 - y^3 = 1$$

has the solution $x = 3$, $y = 2$. This solution has been obtained by inspection and can be checked by direct substitution since $3^3 - 2^2 = 23$ and $3^2 - 2^3 = 1$. We do not know, however, whether or not there are other solutions. The general solution of the above system of equations is rather difficult.

When the equations are not linear, it is often very difficult to find solutions of a system of simultaneous equations in two unknowns. A few cases where the solution can be found are discussed in the next section.

14-10. Systems of Equations Involving Quadratics. Some systems of two simultaneous equations can be solved by eliminating one of the variables. To do this, solve one of the equations for one of the unknowns in terms of the other and substitute this solution into the second equation. Thus an equation with only one unknown is obtained. This procedure is illustrated by the following example.

Example 1. Solve the following system of equations:

$$y^2 = 4x + 20,$$

$$xy = 1.$$

Substituting $\frac{1}{x}$ in place of y in the first of the given equations, we obtain:

$$\frac{1}{x^2} = 4x + 20$$

or

$$4x^3 + 20x - 1 = 0.$$

The last equation is of the third degree and cannot be solved by the methods developed so far. Thus we are unable to solve the given system of equations.

This example shows that in general it is very difficult to find solutions of a system of simultaneous equations in two unknowns. There are, however, a few types where the solution can be found. In the remaining portion of this section we will discuss systems of simultaneous equations which can be reduced to quadratics in one unknown.

Type A. One equation of the system is linear, the other is of the second degree. In this case the linear equation is solved for one of the unknowns and the result substituted in the second equation. The following example illustrates this method.

Example 2. Solve the system of equations

$$(1) \quad x^2 + 2y^2 = 6,$$

$$(2) \quad 2x + 3y = 7.$$

Solving the linear equation 2 we obtain:

$$(3) \quad x = \frac{7 - 3y}{2}$$

which when substituted in (1) yields

$$\left(\frac{7 - 3y}{2}\right)^2 + 2y^2 = 6,$$

$$49 - 42y + 9y^2 + 8y^2 = 24,$$

$$(4) \quad 17y^2 - 42y + 25 = 0,$$

a quadratic in one unknown y . Solving equation 4 with the aid of the quadratic formula we obtain

$$y = \frac{42 \pm \sqrt{42^2 - 4 \times 17 \times 25}}{34} = \frac{42 \pm 8}{34},$$

$$y_1 = \frac{50}{34} = \frac{25}{17}, \quad y_2 = \frac{34}{34} = 1.$$

The corresponding values of x are found by substituting y_1 and y_2 in (3), so that

$$x_1 = \frac{7 - 3 \times \frac{25}{17}}{2} = \frac{7 - \frac{75}{17}}{2} = \frac{119 - 75}{34} = \frac{22}{17},$$

$$x_2 = \frac{7 - 3}{2} = 2.$$

The given system of equations has, therefore, two sets of solutions:

$$x_1 = \frac{22}{17}, \quad y_1 = \frac{25}{17},$$

and

$$x_2 = 2, \quad y_2 = 1.$$

These solutions should be checked by substituting them in the given equations.

Type B. Both equations are linear in x^2 and y^2 , that is, when both equations are of the form $ax^2 + by^2 = c$. Such a system of equations can always be solved by the methods used to solve linear equations (see Sec. 13-1). This will now be illustrated.

Example 3. Solve

$$x^2 + 2y^2 = 17,$$

$$2x^2 - 3y^2 = 6.$$

Multiplying the first equation by 2 and subtracting the second equation from the product, we obtain

$$7y^2 = 28 \quad \text{or} \quad y^2 = 4.$$

Therefore

$$y = \pm 2.$$

Using the first of the given equations we obtain

$$x = \pm 3.$$

The system has four sets of solutions, because each of the y -values can be combined with each of the x -values. The solutions are:

$$x_1 = 3, \quad y_1 = 2,$$

$$x_2 = 3, \quad y_2 = -2,$$

$$x_3 = -3, \quad y_3 = 2,$$

$$x_4 = -3, \quad y_4 = -2.$$

They can be checked by substituting in the original equations.

EXERCISES

Solve the following systems of equations and check the solutions.

$$1. \quad 3x^2 - 2xy + y - 1 = 0, \\ 5y - 3x = 7.$$

$$3. \quad y^2 = 5x, \\ y = 3x - 10.$$

$$5. \quad E_1 + E_2 = 5.80, \\ E_1 E_2 = 8.32.$$

$$7. \quad 6u^2 + 5v^2 = 58, \\ u^2 - 2v^2 = 4.$$

$$9. \quad 3.76p^2 - 5.38q^2 = 9.65,$$

$$2.89p + 8.75q = 25.$$

$$11. \quad m + n = 20, \\ mn = 40.$$

$$13. \quad 2x^2 - 6xy - 9y^2 = 11, \\ 2x + 3y + 1 = 0.$$

$$15. \quad z_1^2 + z_2^2 = 5, \\ z_1 z_2 = 2.$$

$$17. \quad 5x^2 + y^2 = 9,$$

$$7x^2 - 2y^2 = -1.$$

$$19. \quad p^2 + q^2 = 40, \\ p^2 - q^2 = 32.$$

$$21. \quad 5x^2 - y^2 = 19, \\ y^2 = 2x - 3.$$

$$2. \quad 5x^2 + 2y^2 = 38, \\ 7x - 4y = 2.$$

$$4. \quad 5uv = 7u - 12, \\ u = v.$$

$$6. \quad R_1^2 + R_1 R_2 = 40, \\ 2R_1 = 3R_2 + 1.$$

$$8. \quad I_1^2 - 5I_2 = -15, \\ 2I_1^2 + 4I_2^2 = -2.$$

$$10. \quad \frac{x^2}{25} + \frac{y^2}{16} = 1,$$

$$\frac{x}{4} - \frac{y}{3} = 1.$$

$$12. \quad 5A^2 - 6AB + 2B^2 = 65, \\ 4A - 3B = 10.$$

$$14. \quad p^2 + 4q^2 = 20, \\ 4p^2 + q^2 = 20.$$

$$16. \quad 2x + y + 3 = 0, \\ 2x^2 + y^2 - 6y = 9.$$

$$18. \quad \frac{x^2}{16} + \frac{y^2}{9} = 1,$$

$$\frac{x^2}{12} + \frac{y^2}{20} = 1.$$

$$20. \quad 3.76A^2 + 2.51B^2 = 6.37, \\ 8.25A^2 - 5.32B^2 = 2.14.$$

$$22. \quad m^2 + n^2 = 20, \\ mn = 8.$$

Solve the following systems for x and y and find the values of k for which there is only one system of solutions.

$$23. \quad x^2 + y^2 = 25, \\ 3x + 4y = k.$$

$$25. \quad xy = k, \\ x + y = 1.$$

$$24. \quad 2x^2 - 5y^2 = 10, \\ y = 2x + k.$$

$$26. \quad 2x + 3y = k, \\ xy = 5.$$

Solve the following systems for x and y .

$$\begin{array}{ll} 27. \quad x^2 + y^2 = r^2, & 28. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2}, \\ & Ax + By + C = 0. \\ 29. \quad \begin{array}{l} y = mx + b. \\ x^2 - y^2 = k^2, \\ xy = k^2. \end{array} & 30. \quad \begin{array}{l} p^2x^2 - 9y^2 = 0, \\ x = 10y - 15y^2. \end{array} \end{array}$$

14-11. Engineering Problems. Many problems arising in engineering involve formulas which have quadratic forms. In working with such formulas one uses the methods developed in this chapter.

Example 1. The inductance of a single layer coil is given by the formula $L = 0.0251d^2n^2lk$. Solve this equation for n .

Dividing both sides of this equation by $0.0251d^2lk$ gives

$$\frac{L}{0.0251d^2lk} = n^2.$$

The value of n is obtained by computing the square root of both sides of this equation.

$$\begin{aligned} n &= \sqrt{\frac{L}{0.0251d^2lk}} = \frac{1}{d} \sqrt{\frac{L}{0.0251lk}} \\ &= \frac{1}{0.158d} \sqrt{\frac{L}{lk}}. \end{aligned}$$

The negative value of the square root has no meaning in this problem and is therefore omitted.

Example 2. Two resistors have to be selected which when combined in parallel have a resistance of 1.82 ohms, and when combined in series have a resistance of 10 ohms.

Let R_1 and R_2 be the resistances of the two resistors to be selected. In the theory of electricity it is shown that the series resistance R_s is given by

$$R_s = R_1 + R_2$$

and the parallel resistance R_p by the equation

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Therefore, to determine R_1 and R_2 , the following system of simultaneous equations has to be solved.

$$\begin{aligned} (1) \quad & R_1 + R_2 = 10, \\ (2) \quad & \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{1.82} = 0.55. \end{aligned}$$

Multiplying equation 2 by the product R_1R_2 of the denominators gives

$$R_2 + R_1 = 0.55R_1R_2,$$

and making use of equation 1 we obtain

$$0.55R_1R_2 = 10.$$

From equation 1,

$$R_2 = 10 - R_1,$$

and therefore

$$0.55R_1(10 - R_1) = 10,$$

or

$$5.5R_1 - 0.55R_1^2 = 10,$$

or

$$0.55R_1^2 - 5.5R_1 + 10 = 0.$$

This is a quadratic equation which can be solved by the quadratic formula, in which

$$A = 0.55, \quad B = -5.5, \quad C = 10,$$

$$D = B^2 - 4AC = 30.25 - 22 = 8.25.$$

$$\sqrt{D} = 2.87.$$

$$R_1 = \frac{5.5 \pm 2.87}{1.1}.$$

$$R_2 = 10 - R_1 = 10 - \frac{5.5 \pm 2.87}{1.1} = \frac{5.5 \mp 2.87}{1.1}.$$

If the upper signs are used,

$$R_1 = \frac{8.37}{1.1} = 7.6 \text{ ohms},$$

$$R_2 = \frac{2.63}{1.1} = 2.4 \text{ ohms}.$$

The lower signs give the same values except that the order is changed.

EXERCISES

Solve the following equations for the variable indicated in each problem.

1. $Fd = \frac{1}{2}Mv^2$; solve for v .

2. $R = \frac{kl}{d^2}$; solve for d .

3. $F = \frac{8.94B^2A}{10^8}$; solve for B .

4. $L = \frac{1.26n^2A\mu}{10^8l}$; solve for n .

5. $I_x = M \left(\frac{r^2}{4} + \frac{h^2}{12} \right)$; solve for r and h .

6. $m_1 = \frac{m_0}{1 - \frac{v^2}{c^2}}$; solve for v , when $m_1 = 1.1m_0$ and $c = 3 \times 10^{10}$.

7. $S = 145.5\omega^2 + 120\omega$; solve for ω .

8. $r = \frac{b^2 + 4h^2}{8h}$; solve for h , when $r = 58$ in. and $b = 72$ in. (circular arch).

9. $S = 2\pi r(r + h)$; solve for r when $S = 1$ sq. ft. and $h = 6$ in. (surface of circular cylinder).

10. $Z = \frac{RX}{\sqrt{R^2 + X^2}}$; solve for R and X .

11. $s = 0.1gt^2 + 2t + c$; solve for t .

12. $a = 5(k^2 + 3k)$; solve for k .

13. $T = a + bt + ct^2$; solve for t .

14. $am^2 + bm + \frac{1}{c} = 0$; solve for m .

15. The height h of a projectile t seconds after firing is given by

$$h = kv_0t - \frac{1}{2}gt^2,$$

where k depends on the angle of elevation of the cannon, v_0 is the initial velocity, and g the acceleration of gravity. Solve for t .

16. The formula

$$s = v_0t + \frac{g}{2}t^2$$

gives the distance s in feet passed over in t seconds by a falling body whose initial velocity was v_0 feet per second. If $g = 32$ ft. per second per second and $v_0 = 80$ ft. per second, how long will it take the object to fall 800 ft.?

17. The formula for the reactance necessary to tune a transmission line is

$$X_l = \frac{z_0 R_e}{\pm \sqrt{R_e(z_0 - R_e)}}.$$

Solve this for z_0 in terms of the other quantities.

18. The formula

$$h = v_0t - 16t^2$$

gives the height h in feet which is reached in t seconds by a body projected vertically upward with an initial velocity of v_0 feet per second. Solve for t . Also find t for $v_0 = 79.5$ ft. per second and $h = 94.6$ ft.

19. For a simple beam loaded and supported in a certain way, the *bending moment* at any distance of x feet from one end is $M = 20x - x^2$. For what value of x will $M = 70$?

20. If a square wooden column x inches on each side is to carry a certain load, the smallest safe value of x is a root of $x^4 - 125x^2 - 10,368 = 0$. Find that root.

21. The formula for the surface S of a right circular cone of altitude h and base-radius r is given by $S = \pi r\sqrt{r^2 + h^2}$. Solve for r^2 .

22. The reactance of an electric circuit is

$$X = 2\pi fL - \frac{1}{2\pi fC}.$$

Compute f , if $L = 30 \times 10^{-6}$, $C = 200 \times 10^{-12}$, and $X = 150$.

23. If a lens of focal length f is used to produce an image of an object, the following relation exists between the distances of object and image from the lens.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}.$$

Compute a and b if $f = 5$ in., and $a + b = 2$ ft.

14-12. Operations with Polynomials in One Variable. In order to solve problems involving algebraic equations, a certain amount of skill in dealing with polynomials is necessary. The student will acquire this skill in the present section.

In almost all problems involving polynomials it is advantageous to arrange all the polynomials according to descending powers of the variable, that is, starting with the term of highest degree.

Addition and subtraction. If polynomials have to be combined by addition and subtraction, all the terms of like degree are collected. One starts with the terms of highest degree and then proceeds to terms of the next highest degree and so on until all terms of like degree are collected.

Example 1.

$$\begin{aligned} 5(2x^3 - 4x^2 + 7x + 9) - 7(x^3 + x) + 5(x^2 + 7) \\ = (5 \cdot 2 - 7)x^3 + [5(-4) + 5]x^2 + (5 \cdot 7 - 7)x + 5 \cdot 9 + 5 \cdot 7 \\ = 3x^3 - 15x^2 + 28x + 80. \end{aligned}$$

Multiplication. To multiply two polynomials, first arrange them according to descending powers of the variable. Then multiply all of the terms of the first polynomial successively by the first term, second term, etc., of the second factor. When all the terms of like degree are collected the work is finished.

Example 2. To find the product $(2x^3 - 3x^2 - x + 5)(3x^2 + 2x - 4)$ we arrange our work in the following way.

	$(2x^3 - 3x^2 - x + 5)(3x^2 + 2x - 4)$
The product of the first factor by $3x^2$:	$6x^5 - 9x^4 - 3x^3 + 15x^2$
The product of the first factor by $2x$:	$4x^4 - 6x^3 - 2x^2 + 10x$
The product of the first factor by -4 :	$- 8x^3 + 12x^2 + 4x - 20$
Sum of all the terms:	$6x^5 - 5x^4 - 17x^3 + 25x^2 + 14x - 20$

Example 3. Similarly, to find the product $(x^4 + 3x^2 - 2)(x^2 + 5x + 1)$ we write:

$(x^4 + 3x^2 - 2)(x^2 + 5x + 1)$				
x^6	$+ 3x^4$	$- 2x^2$		
$+ 5x^5$		$+ 15x^3$	$- 10x$	
	$+ x^4$	$+ 3x^2$	$- 2$	
$x^6 + 5x^5$	$+ 4x^4$	$+ 15x^3$	$+ x^2$	$- 10x - 2$

Division. Each division may be regarded as a repeated subtraction, the divisor being subtracted from the dividend as many times as possible. Thus the statement: "If 20 is divided by 3, the quotient is 6 and

the remainder is 2" means that 3 can be subtracted 6 times from 20 without getting a negative difference. This is written in the form,

$$(1) \quad 20 - 3 \cdot 6 = 2,$$

or

$$(2) \quad 20 = 3 \cdot 6 + 2.$$

In this example 20 is the dividend, 3 is the divisor, 6 is the quotient and 2 is the remainder. Equation 1 is a special case of the relation

$$(3) \quad \text{Dividend} - \text{Divisor} \times \text{Quotient} = \text{Remainder}$$

which holds whenever two numbers are divided by one another.

Using relation (3) we now give the following definition for the division of two polynomials.

Dividing a polynomial $F(x)$ by a polynomial $G(x)$ means finding a third polynomial $Q(x)$ such that the difference

$$(4) \quad F(x) - Q(x) \cdot G(x) = R(x)$$

where $R(x)$ is a polynomial whose degree is smaller than the degree of $F(x)$.

Relation (3) may be also written as

$$(5) \quad \text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

of which (2) is a special case. Using (5), the above definition for division of polynomials takes on the following form.

*If two polynomials $F(x)$ and $G(x)$ are given, a third polynomial $Q(x)$, called the **quotient** can be found such that*

$$(6) \quad F(x) = Q(x) \cdot G(x) + R(x)$$

*where the polynomial $R(x)$, called **remainder**, is of lower degree than $G(x)$.*

The process of division of polynomials is then the same as long division in arithmetic. This process consists in finding a series of multiples of the divisor and subtracting these from the dividend until the last remainder is smaller than the divisor. The mechanics of division of polynomials will become clear from the following illustrations.

Example 4. Divide the polynomial $6x^4 + x^3 + 5x^2 + 9x + 4$ by $3x^2 + 2x - 1$.

The two given polynomials are already arranged according to the descending powers of x . If this were not the case, they would have to be arranged in this way.

To start the division, the highest term of the dividend, $6x^4$, is divided by the highest term of the divisor, $3x^2$. The quotient is $2x^2$. The divisor is now multiplied by $2x^2$ and the product subtracted from the dividend. The remainder found in this way is used as a dividend for the next step and this process is repeated until the highest term of the last remainder is of lower degree than the divisor.

The computations described above can be arranged in the following way:

$$\begin{array}{r|l}
 \text{(Dividend)} \quad 6x^4 + x^3 + 5x^2 + 9x + 4 & 3x^2 + 2x - 1 \text{ (Divisor)} \\
 & \hline
 & 2x^2 - x + 3 \text{ (Quotient)} \\
 \hline
 6x^4 + 4x^3 - 2x^2 & \\
 \hline
 -3x^3 + 7x^2 + 9x & \\
 -3x^3 - 2x^2 + x & \\
 \hline
 9x^2 + 8x + 4 & \\
 9x^2 + 6x - 3 & \\
 \hline
 2x + 7 \text{ (Remainder)} &
 \end{array}$$

The quotient is $2x^2 - x + 3$ and the remainder $2x + 7$. The result can be checked by the fundamental relation (6) given above. We obtain,

$$6x^4 + x^3 + 5x^2 + 9x + 4 = (3x^2 + 2x - 1)(2x^2 - x + 3) + 2x + 7.$$

The result of the operations on the right side of this equation is equal to the left side and therefore the division was performed without mistake.

Example 5. Divide $2x^3 - x + 5$ by $x - 2$.

The dividend does not contain a term of the second degree. In such a case we write the dividend with space left free for any missing terms. The required division may be arranged in the following way.

$$\begin{array}{r|l}
 2x^3 & -x + 5 \\
 & x - 2 \\
 \hline
 & 2x^2 + 4x + 7 \\
 \hline
 2x^3 - 4x^2 & \\
 \hline
 4x^2 - x & \\
 4x^2 - 8x & \\
 \hline
 7x + 5 & \\
 7x - 14 & \\
 \hline
 19 &
 \end{array}$$

A check for this computation is obtained by testing the relation

$$2x^3 - x + 5 = (x - 2)(2x^2 + 4x + 7) + 19.$$

It should also be noted that the divisor $x - 2$ is of degree one, hence the remainder, being of lower degree, has no terms containing x and is therefore a constant, in this case 19.

From relation (6) it follows that *the dividend is divisible by the quotient or that the quotient is a factor of the dividend, if the remainder is zero.* Since in this case $R(x) = 0$, we have

$$(7) \quad F(x) = Q(x) \cdot G(x).$$

Example 6. Divide $x^3 + 8x^2 + 3x - 18$ by $x + 2$.

$$\begin{array}{r|l}
 x^3 + 8x^2 + 3x - 18 & x + 2 \\
 \hline
 x^3 + 2x^2 & \\
 \hline
 6x^2 + 3x & \\
 6x^2 + 12x & \\
 \hline
 -9x - 18 & \\
 -9x - 18 & \\
 \hline
 0 &
 \end{array}$$

A check on these computations is obtained by showing that

$$x^3 + 8x^2 + 3x - 18 = (x + 2)(x^2 + 6x - 9).$$

EXERCISES

In the following problems find the quotient and the remainder. Check the results by using relation (6).

1. $(3x^3 - 5x^2 + 7x - 2) \div (x - 1)$.
2. $(3x^3 - 5x^2 + 7x - 2) \div (x + 1)$.
3. $(x^4 + 3x^3 + x + 5) \div (x - 3)$.
4. $(x^3 - x^2 - x - 2) \div (x - 2)$.
5. $(6r^2 - 19r + 10) \div (3r - 2)$.
6. $(m^4 - 3m^2 + 2m - 1) \div (m^2 - 3m + 1)$.
7. $(x^3 - 1) \div (x - 1)$.
8. $(x^3 + 1) \div (x + 1)$.
9. $(x^3 - 1) \div (x + 1)$.
10. $(x^3 + 1) \div (x - 1)$.
11. $(x^4 - 1) \div (x - 1)$.
12. $(x^4 + 1) \div (x + 1)$.
13. $(x^4 - 1) \div (x + 1)$.
14. $(x^4 + 1) \div (x - 1)$.
15. $(x^4 + 1) \div (x + 2)$.
16. $(x^4 + 3x^2 - 5) \div (x^2 - 2)$.
17. $(8x^3 + 27y^3) \div (2x - 3y)$.
18. $(x^3 + 8y^3) \div (x + 2y)$.
19. $(x^4 + 3x^2 - 4) \div (x^2 - 1)$.
20. $(x^3 - y^3) \div (x - y)$.
21. $(2x^4 - 3x^3 - 2x^2 + 5x - 2) \div (x^2 - 2x + 1)$.
22. $(6E^4 - 3E^3 - 5E^2 + 24E - 16) \div (3E^2 + 3E - 4)$.

14-13. Remainder Theorem and Factor Theorem. Let us restate relation (6) of the section just preceding, namely,

$$(1) \quad F(x) = Q(x) \cdot G(x) + R(x)$$

where $R(x)$ is of lower degree than $G(x)$. Now if the divisor $G(x)$ is of the first degree, then it follows that $R(x)$ is a constant. Hence when the dividend $F(x)$ is a given polynomial and the divisor $G(x) = x - r$ (a linear expression), then (1) becomes

$$(2) \quad F(x) = (x - r) Q(x) + R$$

where $Q(x)$ is the quotient of the division $\frac{F(x)}{x-r}$ and the *constant* R is the remainder.

The equation 2 is an identity, true for all values of the variable x . If an equation is true for all values of x , it remains true if a particular value is substituted for x . Substituting $x = r$ on both sides of (2) we obtain

$$F(r) = R.$$

Hence (2) can be written in the form

$$(3) \quad F(x) = (x - r) \cdot Q(x) + F(r).$$

The result given in (3) states symbolically the following theorem.

*If a polynomial $F(x)$ is divided by $x - r$, the remainder is $F(r)$; that is, the remainder is the number obtained by replacing x by r in $F(x)$. This result is known as the **remainder theorem**.*

Example 1. In Example 5 of Sec. 14-12 it was seen that when $F(x) = 2x^3 - x + 5$ is divided by $x - 2$ the remainder is 19. This can now be verified by the remainder theorem, for replacing x by 2 in $F(x) = 2x^3 - x + 5$ we obtain

$$R = F(2) = 2 \cdot 8 - 2 + 5 = 19.$$

Example 2. What is the remainder when $x^{59} - 1$ is divided by $x + 1$?

In this case $F(x) = x^{59} - 1$ and $r = -1$; hence, by the remainder theorem:

$$R = F(r) = F(-1) = (-1)^{59} - 1 = -1 - 1 = -2.$$

The remainder theorem has enabled us to find the remainder $R = -2$ without performing the extremely long division of $x^{59} - 1$ by $x + 1$.

If, in particular, $F(r) = 0$, then we obtain from (3)

$$(4) \quad F(x) = (x - r) \cdot Q(x).$$

This result can be stated in the following theorem.

*If the polynomial $F(x)$ is equal to zero when $x = r$, then $(x - r)$ is a factor of $F(x)$ and conversely, if $(x - r)$ is a factor of $F(x)$, then $F(r) = 0$. This result is known as the **factor theorem**. This theorem enables us to find out whether or not a polynomial $F(x)$ is divisible by a linear factor $x - r$ without actually carrying out the division.*

Example 3. The polynomial $F(x) = x^3 - 8$ vanishes for $x = 2$. Hence by the factor theorem, $F(x)$ is divisible by $x - 2$; in fact,

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4).$$

Example 4. Decide whether or not $x + 3$ is a factor of $x^3 - 3x + 9$.

In this case $F(x) = x^3 - 3x + 9$ and $x - r = x + 3$; therefore $r = -3$ and

$$F(-3) = (-3)^3 - 3(-3) + 9 = -27 + 9 + 9 = -9 \neq 0,$$

which shows that $x + 3$ is not a factor of $x^3 - 3x + 9$.

EXERCISES

Perform the following divisions and check the remainder using the remainder theorem.

1. $(x^4 + 7) \div (x + 1)$.
2. $(x^3 - 10) \div (x - 3)$.
3. $(x^3 - 3x^2 - 7x - 15) \div (x - 5)$.
4. $(2x^4 - 15x^2 + 2) \div (x + 3)$.
5. $(2E^3 - E^2 + 6E + 5) \div (E - 2)$.
6. $(R^4 + 4R^3 + 3R^2 - 7R - 28) \div (R + 3)$.
7. $(y^4 - 2y^2 + 3y - 5) \div (y + 2)$.
8. $(x^5 - 2x^4 - 2x^2 + 6x + 3) \div (x + 1)$.
9. $(2z^4 - 5z^2 - 2z + 5) \div (z + 5)$.
10. $(5L^4 - 2L^3 + 3L - 7) \div (L + 2)$.

Find the remainders of the following divisions without actually dividing.

11. $(2x^{27} + 27) \div (x - 1)$.
12. $(2x^{27} + 27) \div (x + 1)$.
13. $(3E^3 - 2E^2 + 4E - 31) \div (E - 2)$.
14. $(5R^4 - 2R^3 + 3R + 7) \div (R + 2)$.
15. $(8r^9 + 2r^5 - 7) \div (r + 3)$.
16. $3(2y^7 - 8y^4 + y) \div (y - \frac{1}{2})$.
17. $(3 \cos^2 \theta - 5 \cos \theta + 4) \div (\cos \theta - 0.5)$.
18. $(\tan^3 \theta + 2) \div (\tan \theta + 1)$.
19. $(\sin^5 \theta + 3) \div (\sin \theta - 1)$.
20. $(5 \cos^3 \alpha + 2 \cos \alpha + 1) \div (\cos \alpha + 2)$.

Decide whether $x - 2$ is a factor of the following polynomials.

21. $2x^5 - 4x^4 - 3x^2 + 9x - 17$.
22. $x^4 - 16$.
23. $x^4 + 16$.
24. $x^2 + x + 4$.
25. $x^5 - x^3 - 24$.
26. $7x^3 - 55$.
27. $x^5 + x^2 - 3x + 3$.
28. $x^6 - 2x^3 + 2x - 52$.

Decide whether $x + 3$ is a factor of the polynomials in Exercises 29-36.

29. $4x^3 + 2x^2 - 51x - 63$.
30. $x^4 + 81$.
31. $x^4 - 81$.
32. $2x^2 - 5x - 1$.
33. $2x^5 + 5x^4 + 81$.
34. $x^5 - 5x^3 + 12x^2$.
35. $x^4 - 4x^3 + 13$.
36. $5x^3 + 7x^2 + 1$.

37. Is $4 \sin^2 \theta - 3 \sin \theta + 0.5$ divisible by $\sin \theta - 0.5$?

38. Is $\tan^3 A + 1$ divisible by $\tan A - 1$?

39. Find the value of k so that $4x^3 + 3kx^2 + 5x - 1$ is divisible by $x - 1$.

40. Find the value of m so that $x - 2$ will divide $5x^2 - 3m + 4x - x^3 + 15$ with a remainder equal to 5.

14-14. The Fundamental Theorem of Algebra. Let $F(x)$ be any polynomial of degree n in the variable x . Then the equation

$$F(x) = 0$$

is called an algebraic equation of n th degree. Every number, real or imaginary, which when substituted for x satisfies this equation is called a root of $F(x) = 0$.

Thus, for example,

$$2x^3 + x^2 - 8x - 4 = 0$$

is an algebraic equation of the third degree. The left-hand side vanishes for $x = 2$. Therefore, $x = 2$ is a root of the given equation.

Every algebraic equation has at least one root. This statement, even though it may seem obvious to the student, is one of the most important theorems in algebra and is therefore known as the **fundamental theorem of algebra**. This theorem was proved the first time by the mathematician Gauss in 1799. A proof may be found in advanced books on algebra.

Obviously this theorem is true only when the complex numbers are admitted as roots. For example, the equation $x^2 + 1 = 0$ has only the roots $x = \pm j$.

14-15. Number of Roots of an Algebraic Equation. We can now restate the factor theorem given in Sec. 14-13 in the following way.

If r is a root of the algebraic equation $F(x) = 0$, then $(x - r)$ is a factor of $F(x)$.

The linear expression $x - r$ is called the **root factor** corresponding to the root $x = r$.

We shall now use the factor theorem to determine the number of roots of an algebraic equation.

Let $F(x)$ be a given polynomial. By the fundamental theorem of algebra stated in Sec. 14-14, the equation

$$F(x) = 0$$

has at least one root which we may call $x = r_1$. The polynomial $F(x)$ is then divisible by $x - r_1$ and can be factored, so that

$$F(x) = (x - r_1) F_1(x),$$

where $F_1(x)$ is found by dividing $F(x)$ by $x - r_1$. The degree of $F_1(x)$ is one less than that of $F(x)$.

The same procedure can be repeated with $F_1(x)$. The equation $F_1(x) = 0$ has at least one root $x = r_2$ and the polynomial $F_1(x)$ can be factored, so that

$$F_1(x) = (x - r_2) F_2(x),$$

and therefore

$$F(x) = (x - r_1)(x - r_2) F_2(x),$$

where the degree of $F_2(x)$ is two less than the degree of $F(x)$.

This procedure can be repeated until the degree of one of the quotients $F_1(x)$, $F_2(x)$, $F_3(x)$, \dots is zero, that is, until a quotient is a constant A . If $F(x)$ is a polynomial of degree n , we have then

$$(1) \quad F(x) = A(x - r_1)(x - r_2) \cdots (x - r_n).$$

This proves the following theorem.

Every polynomial of the n th degree can be factored in n linear factors. Symbolically this is given in (1).

The equation $F(x) = 0$ when written in factored form

$$F(x) = A(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

is easily seen to be satisfied when $x = r_1, x = r_2, \cdots, x = r_n$. The equation $F(x) = 0$, therefore, has n different roots, provided that all the numbers r_1, r_2, \cdots, r_n are different. If some of these numbers are equal, then the corresponding root is said to be a **multiple root**. For example, when $r_1 = r_2$, then the root r_1 is said to have multiplicity two or to be a double root; when $r_1 = r_2 = r_3$, then the root r_1 is said to have multiplicity three or to be a triple root, and so on. This leads to the following theorem.

Given an algebraic equation of degree n :

$$F(x) = 0.$$

The left-hand member of this equation is the product of n linear factors, and the equation has exactly n roots if each root is counted as often as its multiplicity indicates.

Example. Consider the equation $x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2 = 0$.

The left-hand side can be factored and written in the form

$$(x - 1)^2(x + 1)^2(x + 2) = 0,$$

so that it is the product of five linear root factors. The corresponding roots are, therefore,

$$r_1 = r_2 = 1, \quad r_3 = r_4 = -1, \quad r_5 = -2.$$

The equation has three different roots, 1, -1, -2. Each of the first two of these roots is a double root so that the total number of roots is five.

14-16. Reduced Equation. If $x = r$ is a root of the algebraic equation $F(x) = 0$, then this equation can be written in the form

$$(x - r_1) F_1(x) = 0.$$

Here the left-hand member can be zero only if either $x - r_1 = 0$ or $F_1(x) = 0$. The first possibility gives $x = r_1$, the root already known. All the other roots of $F(x) = 0$ are, therefore, roots of the **reduced equation**

$$F_1(x) = 0.$$

The degree of this reduced equation is one less than the degree of the original equation $F(x) = 0$.

The problem of finding the roots of the equation $F(x) = 0$ which is of degree n is therefore reduced to finding the roots of the equation $F_1(x) = 0$, which is simpler, being of degree $n - 1$.

Example. Solve the equation $x^3 + x^2 - 8x - 6 = 0$, given that one of the roots is $r_1 = 1$.

That this equation has the root $r_1 = 1$ can easily be checked by substitution in the given equation for $1 + 1 - 8 \cdot 1 + 6 = 0$. To find the reduced equation, divide $(x^3 + x^2 - 8x + 6)$ by $(x - 1)$ as follows.

$$\begin{array}{r|l}
 x^3 + x^2 - 8x + 6 & x - 1 \\
 \hline
 x^3 - x^2 & \\
 \hline
 2x^2 - 8x & \\
 2x^2 - 2x & \\
 \hline
 -6x + 6 & \\
 -6x + 6 & \\
 \hline
 0 &
 \end{array}$$

The reduced equation is

$$x^2 + 2x - 6 = 0,$$

and its roots are

$$x = \frac{-2 \pm \sqrt{4 + 24}}{2} = \frac{-2 \pm \sqrt{28}}{2} = -1 \pm \sqrt{7}.$$

The given equation has, therefore, the following three roots:

$$x_1 = 1,$$

$$x_2 = -1 + \sqrt{7} = -1 + 2.646 = 1.646,$$

$$x_3 = -1 - \sqrt{7} = -1 - 2.646 = -3.646.$$

EXERCISES

In the following equation one root is given. Find all the other roots. Express all irrational roots in decimal form, correct to three significant figures.

1. $2x^3 + 3x^2 - 5x - 18 = 0$; $r_1 = 2$. 2. $x^3 - 5x^2 - 20x + 300 = 0$; $r_1 = 10$.

3. $x^3 - 1 = 0$; $r_1 = 1$.

4. $x^3 + 1 = 0$; $r_1 = -1$.

5. $x^3 + x^2 - 8x - 12 = 0$; $r_1 = -2$.

6. $\tan^3 x - 3 \tan x - 2 = 0$; $\tan x = 2$.

7. $4x^3 - 10x^2 - 7x + 3 = 0$; $r_1 = 3$.

8. $x^3 - 2x^2 - 13x - 10 = 0$; $r_1 = 5$.

9. $2 \tan^3 \theta + 3 \tan^2 \theta - 4 \tan \theta + 1 = 0$; $\tan \theta = \frac{1}{2}$.

10. $x^3 - 2x^2 - x + 2 = 0$; $r_1 = 1$.

In the following equations two roots are given. Find all the other roots.

11. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$; $r_1 = 1$, $r_2 = 4$.

12. $x^4 - 6x^3 - 19x^2 + 144x - 180 = 0$; $r_1 = 6$, $r_2 = 2$.

13. $6x^4 - 5x^3 - 39x^2 - 4x + 12 = 0$; $r_1 = 3$, $r_2 = -2$.

14. $15x^4 + 43x^3 - 157x^2 + 17x + 10 = 0$; $r_1 = 2$, $r_2 = -5$.

15. $x^4 + x^3 - 2x^2 + 4x - 24 = 0$; $r_1 = 2$, $r_2 = -3$.

16. $x^4 - 4x^3 + 4x^2 - 36x - 45 = 0$; $r_1 = 5$, $r_2 = -1$.

14-17. Rational Roots. We shall first consider an algebraic equation in which the coefficient of the highest power of the unknown is unity and all the other coefficients are integers. For example,

$$(1) \quad x^3 + 2x^2 - 3x - 6 = 0$$

is such an equation. It can be proved that the real roots of such an equation are either irrational numbers or integers (they cannot be fractions). In the present section we shall explain the method for finding the integral roots which such an equation may have. In this connection the following theorem is true.

For an algebraic equation in which the coefficient of the highest power of the unknown is unity and all of those other coefficients are integers, any existing integral roots are factors of the absolute term.

Applying this result to the equation given in (1) we see that if integral roots exist they have to be factors of the absolute term 6. In order to find these roots, the factors of 6 have to be tested. All the possible integral factors of 6 are:

$$-1, +1, -2, +2, -3, +3, -6, +6.$$

Substituting in the given equation it is found that of these numbers only -2 satisfies (1) and is, therefore, a root of this equation. Since (1) is an equation of the third degree and has therefore three roots, the other two roots are either irrational or complex. These can be determined by dividing $x^3 + 2x^2 - 3x - 6$ by $x + 2$ and solving the resulting reduced equation. Carrying out the proper computations we get

$$x^3 + 2x^2 - 3x - 6 = (x + 2)(x^2 - 3);$$

so that the three roots of equation 1 are

$$r_1 = -2; \quad r_2 = \sqrt{3}; \quad r_3 = -\sqrt{3}.$$

From this example the following rule can be derived.

In order to find all existing integral roots of an algebraic equation with integral coefficients and the highest coefficient 1, test all factors of the absolute term.

If the highest coefficient is not unity, but all the coefficients are integers, the equation may have rational roots which are fractions. The method for finding them will be shown in the following example.

Let

$$(2) \quad 2x^3 - 5x^2 + 7x - 6 = 0$$

be the given equation. The coefficient of the highest power of x is 2,

while all the other coefficients are integers. Introduce a new unknown quantity y by the substitution

$$(3) \quad x = \frac{y}{2},$$

where the denominator on the right-hand side is equal to the coefficient of the highest power of x in the given equation. The result of replacing x by $\frac{y}{2}$ in the given equation 2 is

$$2 \frac{y^3}{8} - 5 \frac{y^2}{4} + 7 \frac{y}{2} - 6 = 0,$$

from which

$$(4) \quad y^3 - 5y^2 + 14y - 24 = 0.$$

This is an equation with integral coefficients and the highest coefficient equal to 1. Hence, by the method discussed previously, each integral root of this equation is a factor of 24. Testing the various factors of 24 we find that $y = 3$ is a root of equation 4, and therefore by (3), $x = \frac{y}{2} = \frac{3}{2}$ is a root of the original equation 2.

Once an integer or fractional root has been found it can be used, by the method of Sec. 14-16, to form a reduced equation of lower degree. Thus, in the last example the polynomial on the left-hand side of (2) can be divided by $x - \frac{3}{2}$ and the resulting quotient (a quadratic) solved for the other two roots of the given equation. These two roots will be irrational or complex numbers. An alternative method which will avoid working with fractions is to divide the polynomial on the left-hand side of (4) by $y - 3$ and thus obtain the other two roots of equation 4. Then by (3) we shall have the corresponding roots of (2).

Example. Find all the roots of the equation

$$(5) \quad 5x^3 + 28x^2 + 10x - 3 = 0.$$

Since the coefficient of the highest power of x is 5, we substitute $x = \frac{y}{5}$ in (5) and obtain,

$$5 \cdot \frac{y^3}{125} + 28 \cdot \frac{y^2}{25} + 10 \cdot \frac{y}{5} - 3 = 0,$$

$$(6) \quad y^3 + 28y^2 + 50y - 75 = 0.$$

Each integer root of this equation is a factor of 75. Testing the different factors of 75, it is found that -3 is a root of the given equation. To find the other roots

we form the reduced equation. This is accomplished by the following long division.

$$\begin{array}{r|l}
 y^3 + 28y^2 + 50y - 75 & y + 3 \\
 \hline
 y^3 + 3y^2 & \\
 \hline
 25y^2 + 50y & \\
 25y^2 + 75y & \\
 \hline
 -25y - 75 & \\
 -25y - 75 & \\
 \hline
 0 &
 \end{array}$$

The reduced equation is therefore

$$y^2 + 25y - 25 = 0,$$

and its roots are found by the quadratic formula as follows.

$$D = 25^2 - 4(-25) = 625 + 100 = 725.$$

$$\sqrt{D} = \sqrt{725} = \sqrt{25 \cdot 29} = 5\sqrt{29}.$$

$$y_2 = \frac{-25 + 5\sqrt{29}}{2}, \quad y_3 = \frac{-25 - 5\sqrt{29}}{2},$$

which together with $y_1 = -3$ give us the three roots of equation 6.

Using the relation $x = \frac{y}{5}$, we find the three roots of equation 5:

$$x_1 = \frac{y_1}{5} = -\frac{3}{5} = -0.600,$$

$$x_2 = \frac{y_2}{5} = \frac{-5 + \sqrt{29}}{2} = \frac{-5 + 5.385}{2} = 0.192,$$

$$x_3 = \frac{y_3}{5} = \frac{-5 - \sqrt{29}}{2} = \frac{-5 - 5.385}{2} = -5.192.$$

EXERCISES

Find the rational roots of the following equations.

1. $2x^2 - 5x + 2 = 0.$
2. $x^3 - 6x^2 + 11x - 6 = 0.$
3. $x^3 - 2x^2 - 4x + 8 = 0.$
4. $3x^2 + 2x^2 + 2x - 1 = 0.$
5. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
6. $5x^3 + 24x^2 - 1 = 0.$
7. $E^3 - 3E^2 - 2E + 6 = 0.$
8. $3y^3 + 8y^2 + y - 2 = 0.$
9. $6x^3 + x^2 - 12x + 5 = 0.$
10. $L^3 - 4L^2 - 11L - 6 = 0.$
11. $E^3 + 3E^2 - 24E + 28 = 0.$
12. $R^3 - 8R^2 + 17R - 10 = 0.$
13. $L^3 - 9L^2 + 23L - 15 = 0.$
14. $8y^3 + 4y^2 - 66y + 63 = 0.$
15. $x^4 - 6x^3 + 6x^2 + 5x + 12 = 0.$
16. $x^5 + x^4 - 9x^3 - 5x^2 + 16x + 12 = 0.$

Find all the roots of the following equations.

17. $x^3 + 2x^2 - x - 2 = 0.$

18. $2x^3 + 3x^2 - 2x - 3 = 0.$

19. $3x^3 + 5x^2 - x - 2 = 0.$

20. $x^4 + x^3 - 34x^2 - 4x + 120 = 0.$

21. $x^4 + 3x^3 - 6x^2 - 18x + 20 = 0.$

22. $x^3 - 1 = 0.$

23. $t^3 = 8.$

24. $x^4 = 16.$

25. $(x+1)(x-2)(x+3) = 24.$

26. $E^4 - 2E^2 + E = 0.$

27. $R^3 - 8R - 3 = 0.$

28. $2L^3 + 5L^2 - 13L + 5 = 0.$

29. $2y^3 + 3y^2 - 4y + 1 = 0.$

30. $z^4 + z^3 - 2z^2 - 4z - 8 = 0.$

14-18. Graphical Solution of Equations. After having worked with algebraic equations, we could hardly expect that all the roots of such an equation are rational, or even that an algebraic equation has any rational roots at all. The method for finding rational roots, explained in Sec. 14-17, can therefore be used only occasionally. If we are to find all the real roots of an algebraic equation, we are still faced with the problem of finding the irrational roots. A method for finding the approximate values of the irrational roots will be developed in the remaining sections of this chapter. The student should be reminded that an algebraic equation may also have roots which are complex numbers. Finding the complex roots of an equation of a degree higher than the quadratic is rather difficult and will therefore not be given in this book. This, however, is not a serious omission in view of the fact that in the applications of mathematics one is usually interested only in the real roots.

In the case of a quadratic equation, the irrational roots can be found by the quadratic formula (explained in Sec. 14-6). This fact may suggest the idea of finding corresponding formulas for higher degree equations. Formulas of this kind which permit the computation of the roots are known for equations of the third and fourth degree, but they are too complicated to be used generally for solving such equations. That no such formulas exist for equations of the fifth or higher degree was proved by the Norwegian mathematician Abel in 1827.

Even though formulas for obtaining irrational roots of an algebraic equation are difficult or impossible to derive, there are methods for finding approximate values of these irrational roots to any desired degree of accuracy. Some of these methods will be explained in the remainder of this chapter. We shall start in the present section with the graphical solution of equations.

The first method for the graphical solution of equations has been already explained in Sec. 3-12. In order to locate approximately the real roots of the equation

$$F(x) = 0,$$

plot the graph of the function

$$y = F(x),$$

and find its x -intercepts, the abscissas of the points where the graph meets the x -axis.

Example 1. Locate approximately the roots of the equation $x^3 - 3x + 1 = 0$.
In order to plot the graph of

$$y = x^3 - 3x + 1$$

the following table of corresponding values is computed,

x	-3	-2	-1	0	1	2	3
y	-17	-1	3	1	-1	3	19

and the graph is plotted on cross-section paper (Fig. 14-6). From the graph it can be concluded that the given equation has three real roots, approximately

$$x_1 = -1.9, \quad x_2 = 0.3, \quad x_3 = 1.5.$$

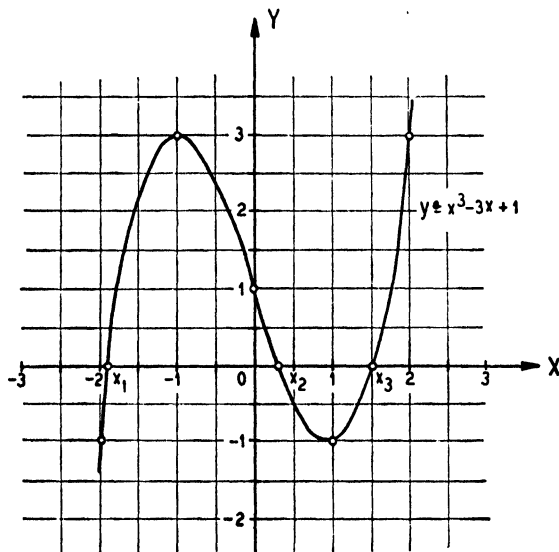


FIG. 14-6.

To plot the graph of the function $y = F(x)$ is often troublesome. Another method, therefore, will be given here, which can be used occasionally and which requires less work. The method will be explained in connection with the equation of the previous example,

(1)
$$x^3 - 3x + 1 = 0.$$

Write this equation in the form

$$x^3 = 3x - 1,$$

and plot the graphs of the two functions

$$y = x^3, \quad y = 3x - 1.$$

These graphs can be easily plotted. Moreover, if other cubic equations have to be solved, the graph of $y = x^3$ can always be used. The graph of $y = x^3$ has been discussed in Sec. 8-13, while the graph of $y = 3x - 1$ is a straight line (Fig. 14-7).

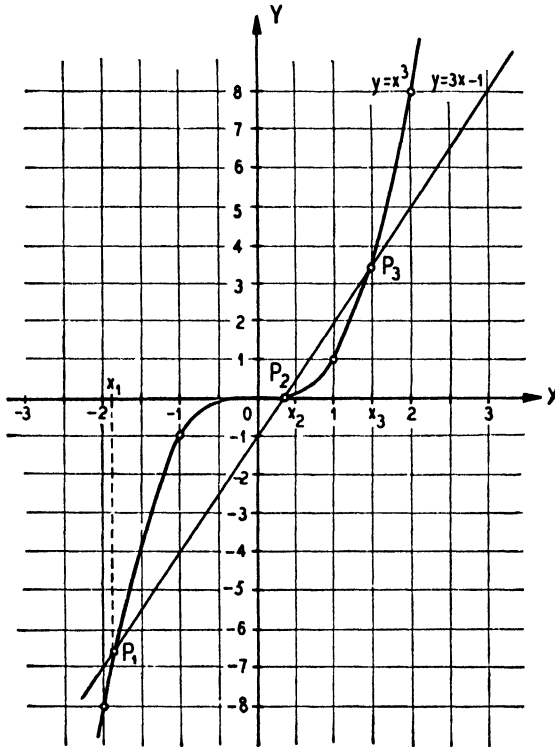


FIG. 14-7.

If the two graphs intersect in a point P , the two functions have the same values at P , namely,

$$y = x^3 = 3x - 1.$$

The abscissa x of P , therefore, satisfies the given equation 1. From Fig. 14-7 it can be inferred that there are three points of intersection, P_1 , P_2 , P_3 whose abscissas are approximately:

$$x_1 = -1.9 \quad x_2 = 0.3 \quad x_3 = 1.5.$$

This method can always be used when a few terms of the given equation $F(x) = 0$ can be transposed so that both sides of the equation can be plotted with less effort than the graph of $F(x)$. The method can also be applied to non-algebraic equations.

Example 2. Solve approximately the equation $x^x = 10$.

Computing the logarithms of both sides of the equation we have

$$x \log x = 1$$

or

$$\log x = \frac{1}{x}.$$

The graphs of the two functions $y = \log x$ and $y = \frac{1}{x}$ are easier to plot than the graph of $x^x = 10$.

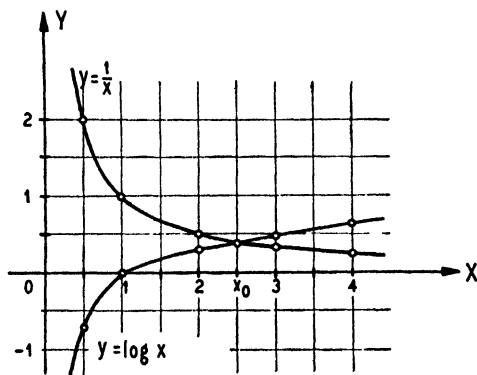


FIG. 14-8.

Figure 14-8 shows that the two graphs intersect at a point whose abscissa is roughly 2.5. Hence, $x_0 = 2.5$ is an approximate value satisfying the equation $x^x = 10$. In fact, $2.5^{2.5} = 9.88$.

EXERCISES

Find by the methods of this section approximate values for the real roots of the following equations.

1. $x^3 - 5x^2 - 10 = 0$.
2. $x^3 + 5x^2 - 10 = 0$.
3. $x^3 + 4x^2 - 7 = 0$.
4. $x^3 - 3x + 1 = 0$.
5. $x^3 + 2x + 20 = 0$.
6. $x^3 - 7x + 7 = 0$.
7. $3x^4 - 4x^2 + 8 = 0$.
8. $x^3 + 4x - 5 = 0$.
9. $2.3x^3 - 21x + 1.2 = 0$.
10. $x^4 - 2.7x - 3.8 = 0$.
11. $x^3 - 3x + 2 = 0$.
12. $x^4 - x + 5 = 0$.
13. $x + \log x = 3$.
14. $x - \log x = 7$.
15. $2x - \log x = 8$.
16. $4x - \log x = 9$.
17. $e^x - 3x = 0$.
18. $2^x - 4x = 6$.

In the following equations the unknown is an angle measured in radians.

19. $x - \cos x = 0.$

20. $x + \sin 2x = 1.4.$

21. $x - \sin x = 2.$

22. $x - \sin x - \frac{\pi}{4} = 0.$

23. The angle of a circular sector whose area is bisected by its chord satisfies the equation $\theta = 2 \sin \theta$. Find an approximate value for θ .

14-19. Method of Repeated Linear Interpolation. Various methods have been developed which permit us to find the irrational roots of an

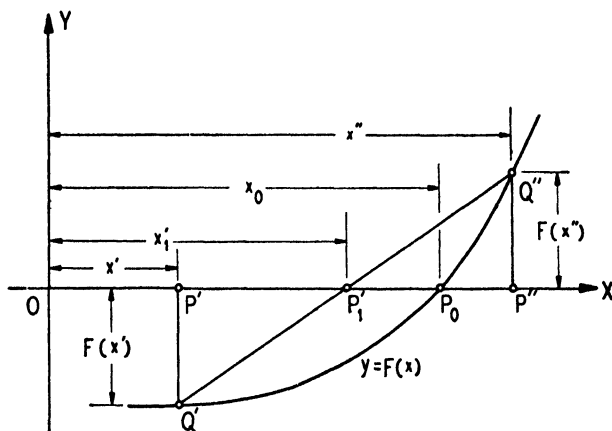


FIG. 14-9.

equation with greater precision than it is possible to find them by the graphical methods of Sec. 14-18. The method which is known as the **method of trial and error** or the **method of linear interpolation** will be discussed in this section. This method is simple and applies to all cases in which the graph of the function $y = F(x)$ intersects the x -axis so that y is positive on one side and negative on the other side of the point of intersection of the curve with the x -axis, as in Fig. 14-9. This method cannot be used when the curve touches the x -axis as in Fig. 14-10. This latter case, however, occurs seldom and will not be discussed here.

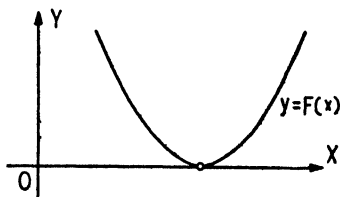


FIG. 14-10.

The method of repeated linear interpolation is based on the following two statements.

1. If x' and x'' are two values sufficiently close to one another (Fig. 14-9) so that $F(x')$ and $F(x'')$ have different signs, then the graph of

$y = F(x)$ intersects the x -axis at a point P_0 whose abscissa X_0 is between x' and x'' . This statement means that there is a value $x = x_0$ between x' and x'' such that $F(x_0) = 0$.

2. Let Q' and Q'' (Fig. 14-9) be the two points on the graph of $y = F(x)$ with abscissas x' and x'' respectively. Also let P' and P'' be the projections on the x -axis of the points Q' and Q'' respectively. Now if $F(x')$ and $F(x'')$ have different signs and the points Q' and Q'' are sufficiently close to one another, then the straight line $Q'Q''$ crosses the x -axis at the point P_1' which, as a rule, will be nearer to the point P_0

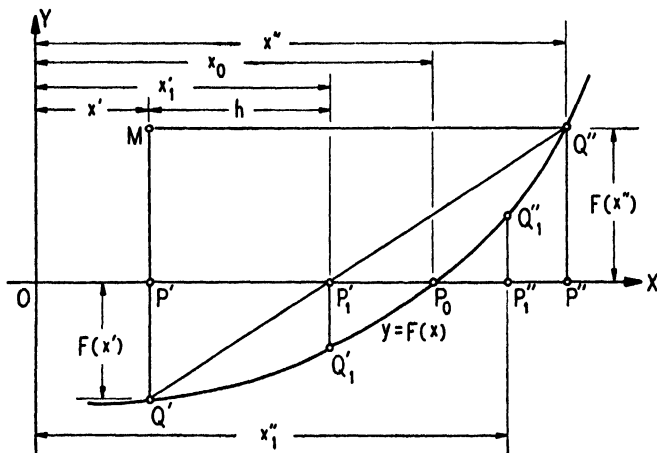


FIG. 14-11.

than either of the points P' and P'' . Thus the line $Q'Q''$ becomes a close approximation of the section of curve of $y = F(x)$ between the points Q' and Q'' .

Our object now is to find the abscissa x_1' of the point P_1' . This value x_1' will give us the first approximation to the abscissa x_0 of the point P_0 , which is the root in question.

The abscissa x_1' of the point P_1' can be found very easily from the fact that the two triangles $P'P_1'Q'$ and $MQ''Q'$ are similar (see Figs 14-11 and 14-12). Therefore it follows that

$$\frac{P'P_1'}{MQ''} = \frac{P'Q'}{MQ'}.$$

Now since we have $P'P_1' = h$ and $MQ'' = |x'' - x'|$,

$$h = \frac{P'Q'}{MQ'} |x'' - x'|.$$

In the last expression $P'Q'$ is the absolute value of $F(x')$ and MQ' is the sum of the absolute values of $F(x')$ and $F(x'')$, so that

$$(1) \quad h = \frac{|F(x')|}{|F(x')| + |F(x'')|} |x'' - x'|$$

and

$$(2) \quad x_1' = x' + h.$$

We can now summarize our work.

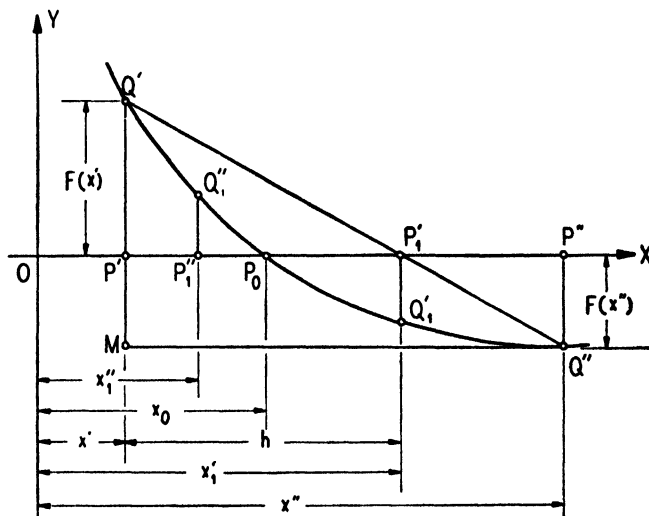


FIG. 14-12.

To find a first approximation to a real root of the equation

$$F(x) = 0,$$

proceed as follows:

1. Find two values x' and x'' sufficiently close to one another so that $F(x')$ and $F(x'')$ have different signs.
2. Having found the values of $F(x')$ and $F(x'')$, use (1) to compute h .
3. Using (2) find the value of $x' + h$ which gives us the first approximation to the required root.

This procedure of replacing a curve by a straight line in order to find an approximation to its x -intercept is the process of **linear interpolation**. In the examples below it will be explained how a second value x_1'' can be found such that $F(x_1')$ and $F(x_1'')$ have different signs. Using the values x_1' and x_1'' instead of the original values x' and x'' the whole procedure can be repeated in order to obtain a closer approximation to the root.

Thus, by successive repetition of this process, the root can be found with any desired degree of precision.

Example 1. Find correct to three significant digits, the greater positive root of the equation

$$F(x) = x^3 - 3x + 1 = 0.$$

This equation has already been investigated in Sec. 14-18. It was seen there that the greater positive root is roughly $x' = 1.5$ and that the corresponding graph (Fig. 14-6) crosses the x -axis at this point from negative to positive values of $F(x)$.

Step 1. Compute

$$F(x') = F(1.5) = 1.5^3 - 3(1.5) + 1 = 3.375 - 4.5 + 1 = -0.125.$$

From the graph it can be inferred that x' is to the left of the unknown root. A second value x'' to the right of this root is found by trials, increasing x' by one or more units of its right-most place. Trying the value $x = 1.6$, we obtain

$$F(1.6) = 1.6^3 - 3(1.6) + 1 = 4.096 - 4.8 + 1 = 0.296.$$

Since this value of the function is positive, we choose $x'' = 1.6$. The required root lies between $x' = 1.5$ and $x'' = 1.6$. These two values can now be used to find a

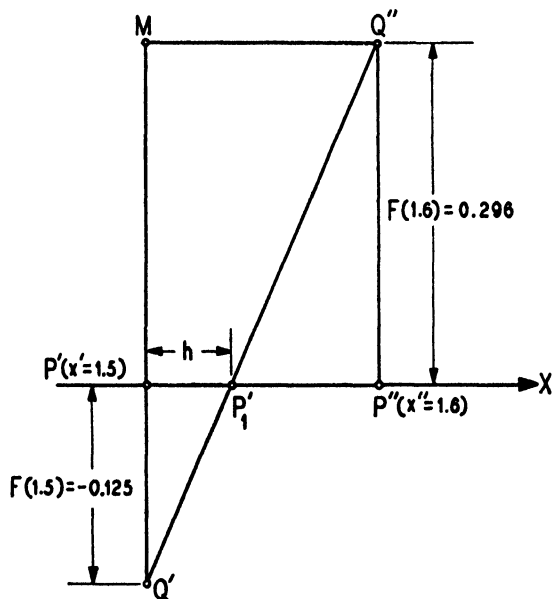


FIG. 14-13.

better approximation by linear interpolation. For this purpose it is always advisable to plot a graph, as shown in Fig. 14-13, which is large enough so that all the necessary arguments can be made in connection with the figure.

Substituting in (1) we obtain

$$h = \frac{0.125}{0.125 + 0.296} (1.6 - 1.5) = \frac{0.125}{0.421} (0.1) = 0.03.$$

It is generally sufficient to compute only one digit of h . The new approximate value for the root is the abscissa of P_1' ,

$$x_1' = x' + h = 1.5 + 0.03 = 1.53.$$

In order to test the precision of this value and to obtain a closer approximation, the whole procedure is now repeated, starting with the new value $x_1' = 1.53$.

Step 2. Compute

$$F(1.53) = 1.53^3 - 3(1.53) + 1 = 3.582 - 4.59 + 1 = -0.008.$$

The value $x_1' = 1.53$, therefore, is smaller than the correct root value. In order to find a value greater than the root, increase the value 1.53 and compute the corresponding value of $F(x)$.

$$F(1.54) = 1.54^3 - 3(1.54) + 1 = 3.652 - 4.62 + 1 = 0.032.$$

We, therefore, take $x_1'' = 1.54$, and x_1' and x_1'' are two values which can be used for the linear interpolation. The corresponding part of the graph is plotted on a much enlarged scale in Fig. 14-14.

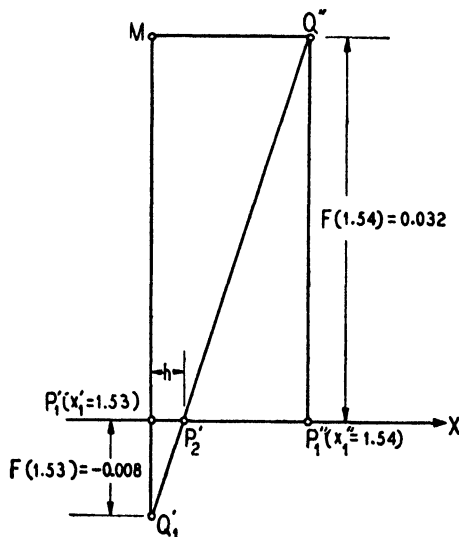


FIG. 14-14.

Computing h as before, we find it to be

$$h = \frac{0.008}{0.008 + 0.032} (0.01) = \frac{8}{40} (0.01) = 0.002,$$

and therefore

$$x_2' = 1.53 + 0.002 = 1.532.$$

Since the value $x_2' = 1.532$ is a better approximation than the previous value $x_1' = 1.53$, it can, therefore, be concluded that $x = 1.53$ is the value of the root to three significant digits.

If a greater precision is required, the procedure can be repeated as often as necessary.

Example 2. Find the negative root of the equation

$$F(x) = 2x^4 - x - 5 = 0.$$

We first construct a table of values for the function

$$y = 2x^4 - x - 5,$$

obtaining

x	-2	-1	0	1	2
y	29	-2	-5	-4	25

There is a negative root between -2 and -1, for as the table shows $F(-2) = 29$ and $F(-1) = -2$ have opposite signs. In order to find closer approximations of this root, the process of linear interpolation is applied starting with the values

$$x' = -2, \quad x'' = -1.$$

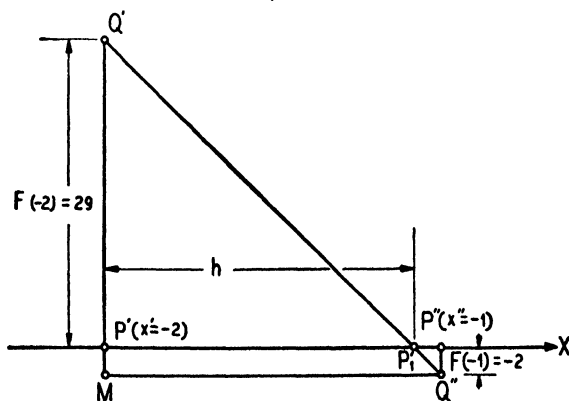


FIG. 14-15.

Step 1. As in the previous example, we have from Fig. 14-15,

$$h = \frac{29}{29 + 2} (-1 + 2) = 0.9$$

whence

$$x_1' = -2 + 0.9 = -1.1.$$

Step 2. From the table of functional values it can be concluded that the function is positive to the left and negative to the right of the required root. The value $x_1' = -1.1$ is too large, because

$$F(-1.1) = 2(-1.1)^4 - (-1.1) - 5 = 2(1.46) + 1.1 - 5 = -0.98$$

is negative. In order to have a value x_1'' on the other side of the root, a smaller value is tested:

$$F(-1.2) = 2(-1.2)^4 - (-1.2) - 5 = 2(2.07) + 1.2 - 5 = 0.34.$$

The unknown root, therefore, is located between -1.2 and -1.1 , and these two values are used for the next linear interpolation (Fig. 14-16):

$$h = \frac{0.34}{0.34 + 0.98} (0.1) = \frac{34}{132} (0.1) = 0.03$$

whence

$$x_2' = -1.2 + 0.03 = -1.17.$$

If the accuracy of the digit 7 needs to be tested, the whole procedure can be repeated

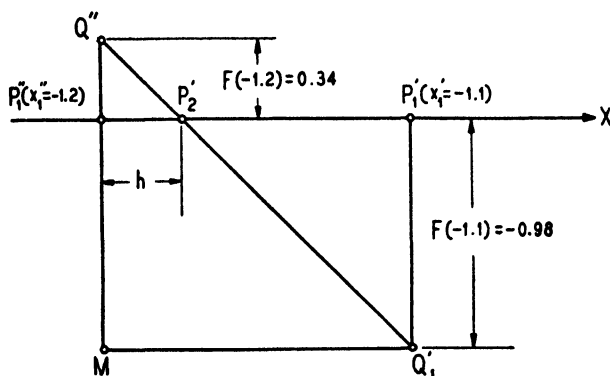


FIG. 14-16.

The fact that $F(x)$ is a polynomial is not essential for the application of the method of linear interpolation. The same procedure, therefore, can be applied to non-algebraic equations.

EXERCISES

Find the real roots of the following equations correct to three significant figures.

1. $x^3 - 4x - 5 = 0$.
2. $x^3 + 7x + 3 = 0$.
3. $x^3 + 3x - 5 = 0$.
4. $x^3 + 2x^2 + 3x + 4 = 0$.
5. $x^3 + 5x^2 - 10 = 0$.
6. $x^4 - 2.7x - 5.2 = 0$.
7. $r^2 - \frac{20}{r} = 2$.
8. $(1 + h)^3 - 1 = \frac{1}{10}$.
9. $x^3 + 4x^2 - 2x - 8 = 0$.
10. $x^3 - 3x^2 + 3x - 10 = 0$.
11. $x^4 + x^3 - 4x^2 - 16 = 0$.
12. $x^3 + x - 65 = 0$.
13. Find the cube root of 50.
14. Find the cube root of 30.

15. Find two fourth roots of 30.

16. A sphere of ice, 1 ft. in diameter, floating in water, sinks to a depth of h feet, given by the equation $2h^3 - 3h^2 + 0.9 = 0$. Find the depth correct to three significant digits.

17. An open tank is to be made from a rectangle of sheet metal, 20 by 30 in., by cutting out equal squares from each corner and turning up the sides. What size squares should be cut out in order that the volume of the tank is 900 cu. in.? Give your result correct to two significant digits.

18. What is the thickness of a hollow spherical vessel whose outer diameter is 10 in. and which can hold 400 cu. in.? For a sphere of radius r the volume is $\frac{4}{3}\pi r^3$. Give your result correct to three significant figures.

19. The values of $\sin \theta$ and $\sin \frac{\theta}{3}$ satisfy the following equation

$$\sin \theta = 3 \sin \frac{\theta}{3} - 4 \sin^3 \frac{\theta}{3}.$$

If $\sin \theta = 0.7$, find $\sin \frac{\theta}{3}$ correct to two significant digits.

20. The diameter d in inches of the bolts needed in certain cylindrical shafts is the positive root of the equation $d^4 + 800d^2 - 18d - 360 = 0$. Find d to two decimal places.

PROGRESS REPORT

In this chapter the quadratic equation was considered, and methods for finding its roots were discussed. A precise formula for the roots was also derived. It was seen that these methods could be used to find the roots of certain more complicated equations of quadratic type. Methods were developed for solving some simple systems of quadratic equations in two unknowns. After discussing operations with polynomials, an exact method was developed, using the factor theorem, for finding the real rational roots of a polynomial equation with rational coefficients. Finally we discussed the method of linear interpolation for finding approximately the values of the real irrational roots of a polynomial equation.

CHAPTER 15

THE STRAIGHT LINE

In plane geometry problems were solved by construction and by geometrical reasonings. However, by introducing a coordinate system as described in Chapter 3, it is possible to apply the algebraic processes to the solution of geometrical problems. Since the methods of algebra are more direct and usually easier than geometrical reasonings, much is gained in the solving of problems in this way. Thus by means of a coordinate system algebra and geometry are united and the resulting subject is **analytic geometry**. This subject was first introduced by René Descartes (1596–1650), a French mathematician.

15-1. The Distance between Two Points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points in the plane, as shown in Fig. 15-1. They

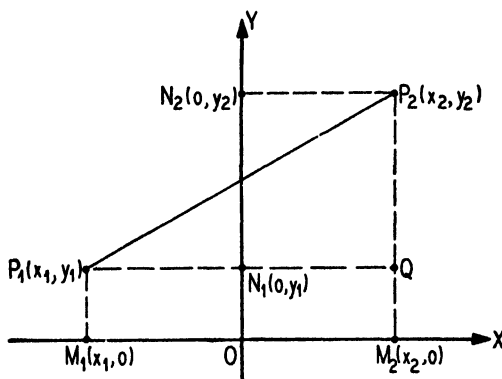


FIG. 15-1.

determine the segment joining them, which will be denoted by P_1P_2 . We shall also let P_1P_2 represent the length of the segment, a positive number.

The perpendiculars from P_1 and P_2 to the x -axis cut that axis at $M_1(x_1, 0)$ and $M_2(x_2, 0)$, respectively. Since M_2 is to the right of M_1 , $x_2 > x_1$, and $M_1M_2 = x_2 - x_1$. Similarly, if the perpendiculars from P_1 and P_2 to the y -axis cut that axis at $N_1(0, y_1)$ and $N_2(0, y_2)$ respectively, then $N_1N_2 = y_2 - y_1$. Both of these statements depend, of course, on how Fig. 15-1 has been drawn. If it were drawn differently, the order of the subscripts might be changed.

From Fig. 15-1 and the Pythagorean theorem we have:

$$\begin{aligned}(P_1P_2)^2 &= (P_1Q)^2 + (QP_2)^2 \\ &= (M_1M_2)^2 + (N_1N_2)^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2.\end{aligned}$$

If we denote the distance P_1P_2 by d we have that

$$(1) \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since the order of subscripts in (1) does not matter, the formula is valid for any two points in the plane.

Example. Find the distance between the points $(-2, 5)$ and $(7, -3)$. Let P_2 be $(-2, 5)$ and P_1 be $(7, -3)$. Then

$$d = \sqrt{(-2 - 7)^2 + (5 + 3)^2} = \sqrt{81 + 64} = \sqrt{145}.$$

EXERCISES

Find the distance between the points of each of the following pairs.

- | | |
|-------------------------|--------------------------|
| 1. $(2, 4), (1, -6)$. | 2. $(3, 1), (-4, -2)$. |
| 3. $(-3, 5), (-6, 2)$. | 4. $(5, 10), (-3, 2)$. |
| 5. $(6, 3), (-5, 3)$. | 6. $(3, 2), (3, -8)$. |
| 7. $(5, 3), (-4, 2)$. | 8. $(15, 8), (-8, 12)$. |
| 9. $(3, 5), (-6, -3)$. | 10. $(3, 6), (-8, 5)$. |

Prove by using the distance formula that the three points given in each exercise form the vertices of a right triangle.

- | | |
|-------------------------------------|------------------------------------|
| 11. $(0, 0), (4, 2), (2, 6)$. | 12. $(-6, -1), (8, 7), (-10, 6)$. |
| 13. $(-3, 0), (-1, -4), (-5, -6)$. | 14. $(0, 9), (6, 3), (-8, 1)$. |
| 15. $(0, 3), (2, 1), (6, 5)$. | |

Prove by using the distance formula that the three points given in each exercise form the vertices of an isosceles triangle.

- | | |
|------------------------------------|----------------------------------|
| 16. $(2, -2), (8, -2), (5, 6)$. | 17. $(4, 4), (2, 1), (1, 2)$. |
| 18. $(0, 0), (-9, 6), (6, 9)$. | 19. $(-3, 3), (3, 5), (2, -2)$. |
| 20. $(-2, -2), (2, -1), (-1, 2)$. | |

Prove by using the distance formula that the four points given in each exercise form the vertices of a parallelogram. By finding the lengths of the diagonals, find out whether or not the parallelogram is a rectangle.

21. $(-2, 0), (-2, -2), (2, 0), (2, 2)$.
22. $(-2, 1), (1, -2), (5, 2), (2, 5)$.
23. $(-5, -3), (1, -11), (7, -6), (1, 2)$.
24. $(0, 6), (-2, 10), (-4, 6), (-2, 2)$.
25. $(-3, 1), (3, 3), (4, 0), (-2, -2)$.

26. Show that the points $(3, 2)$, $(0, -1)$, $(0, 5)$, and $(-3, 2)$ form the vertices of a rhombus. Show that the figure is a square.

27. Show that the points $(4, 1)$, $(4, -3)$, and $(0, 5)$ are equidistant from $(-2, -1)$. What is the center and radius of the circle determined by the first three points given?

28. Find a point on the y -axis equidistant from $(-1, 4)$ and $(3, 2)$.

29. Find a point on the x -axis equidistant from $(1, 2)$ and $(4, 1)$.

15-2. The Midpoint of a Segment. Let $P(x, y)$ be the midpoint of the segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as shown in Fig. 15-2.

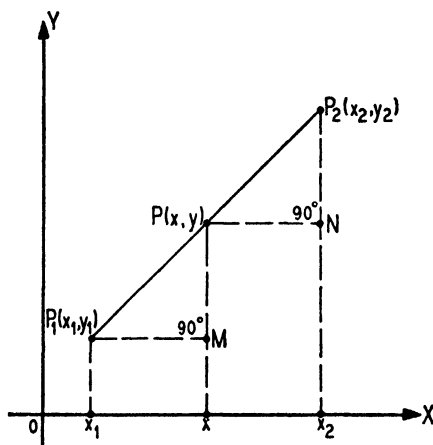


FIG. 15-2.

With the constructions shown in this figure, it can be shown easily that triangle P_1MP is congruent to triangle PNP_2 . Then $P_1M = PN$. Obviously $P_1M = x - x_1$ and $PN = x_2 - x$. Then

$$x - x_1 = x_2 - x$$

whence

$$(1) \quad x = \frac{x_1 + x_2}{2}.$$

In like manner

$$(2) \quad y = \frac{y_1 + y_2}{2}.$$

This discussion, of course, depends upon the way the figure was drawn, but the results (1) and (2) are the same no matter what figure is drawn, as the reader may easily verify.

Example. Find the midpoint of the segment joining $(1, 2)$ and $(-5, 4)$.

By formula (1), $x = \frac{1 - 5}{2} = -2$, and by formula (2), $y = \frac{2 + 4}{2} = 3$. Thus, the midpoint is $(-2, 3)$.

EXERCISES

Find the coordinates of the midpoint of the segment joining:

- | | |
|-----------------------|----------------------|
| 1. (4, 6), (2, -4). | 2. (4, -6), (-2, 4). |
| 3. (-5, 3), (-3, -6). | 4. (2, -5), (4, 7). |
| 5. (0, 3), (-5, 9). | 6. (-5, -8), (3, 4). |

7. Find the coordinates of the midpoints of the sides of a triangle whose vertices are (1, 1), (5, 7), and (-3, 6).

8. Find the coordinates of the midpoints of the sides of a triangle whose vertices are (3, 2), (-5, 8), and (-3, 7).

9. The midpoint of a segment is (2, 3), and one extremity is (-6, 5). Find the coordinates of the other extremity.

10. The midpoint of a segment is (3, 4), and one extremity is (4, 5). Find the coordinates of the other extremity.

11. The points (6, 8), (-4, 0), (0, -4), and (6, -2) form the vertices of a quadrilateral. Prove that the segments joining the midpoints of opposite sides bisect each other.

12. The points (8, 2), (4, 10), (-2, 4), and (-4, -4) form the vertices of a quadrilateral. Prove that the segments joining the midpoints of opposite sides bisect each other.

13. For the quadrilateral of Exercise 11, show that the segment joining the midpoints of a pair of opposite sides bisects the segment joining the midpoints of the diagonals.

14. For the quadrilateral of Exercise 12, show that the segment joining the midpoints of a pair of opposite sides bisects the segment joining the midpoints of the diagonals.

15-3. The Inclination and Slope of a Line. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points in the plane. By the line P_1P_2 we mean the line determined by P_1 and P_2 and extending infinitely far in both direc-

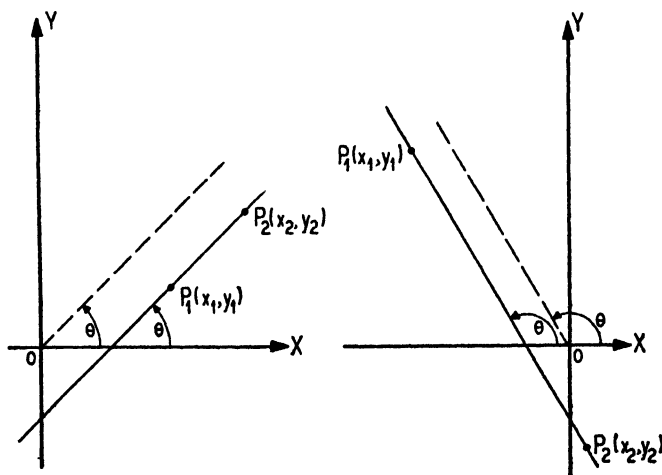


FIG. 15-3.

tions, and by the segment P_1P_2 we mean the portion of the line included between P_1 and P_2 .

The smallest angle through which the positive half of the x -axis must be rotated counterclockwise to bring it parallel to a given line P_1P_2 is called the angle of inclination of P_1P_2 . In Fig. 15-3, θ is the angle of inclination.

Any line P_1P_2 will either cut the x -axis or be parallel to it. If it cuts the x -axis, the angle of inclination is less than 180° ; if it is parallel to the x -axis, the angle of inclination is taken as 0° .

Another description of the direction of a line is given by its slope m , which is defined by the relation

$$(1) \quad m = \tan \theta$$

where θ is the angle of inclination of the given line.

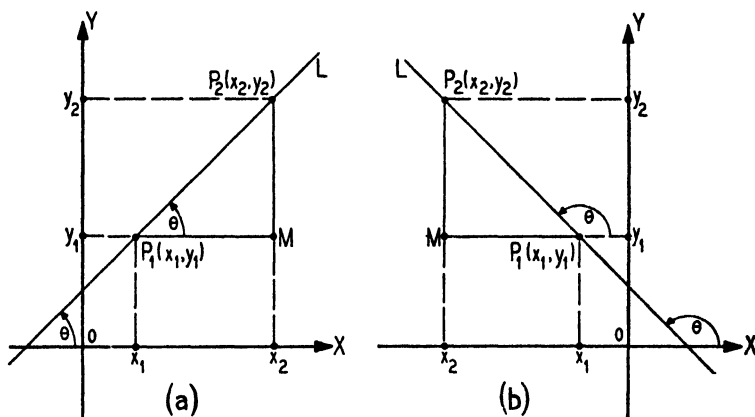


FIG. 15-4.

Figure 15-4 shows two possible positions of the line P_1P_2 , with θ between 0° and 90° as in (a), and with θ between 90° and 180° as in (b). We see then that

$$|m| = |\tan \theta| = \frac{MP_2}{P_1M} = \left| \frac{y_2 - y_1}{x_2 - x_1} \right|.$$

Now, when θ is between 0° and 90° , $m = \tan \theta$ is a positive number, and we see that in Fig. 15-4a the expression $\frac{y_2 - y_1}{x_2 - x_1}$ is positive. When θ is between 90° and 180° , $m = \tan \theta$ is negative, and $\frac{y_2 - y_1}{x_2 - x_1}$ for Fig. 15-4b is negative. Hence we state simply that

$$(2) \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, the order in which the points are chosen in (2) is immaterial, and hence (2) is valid for any line P_1P_2 . However, when P_1P_2 is parallel to the y -axis, $x_1 = x_2$ and the denominator of (2) is zero. Thus (2) has no meaning and we say that the line has no slope. Since the angle of inclination of a line parallel to the x -axis is 90° , and $\tan 90^\circ = \infty$, we can say that *the slope of a line parallel to the y -axis is infinite*. Also, *the slope of a line parallel to the x -axis is zero*.

Example 1. Find the slope and angle of inclination of the line through $(1, 0)$ and $(4, -2)$. By formula (2),

$$m = \frac{0 - (-2)}{1 - 4} = \frac{2}{-3} = -\frac{2}{3} = -0.6667$$

whence $\theta = 146.3^\circ$.

Example 2. Draw the line through $P(2, 4)$ with slope $-\frac{3}{4}$.

From $P(2, 4)$ lay off a horizontal segment PQ four units to the left. Then from Q lay off a vertical segment upward three units as shown in Fig. 15-5. Then the

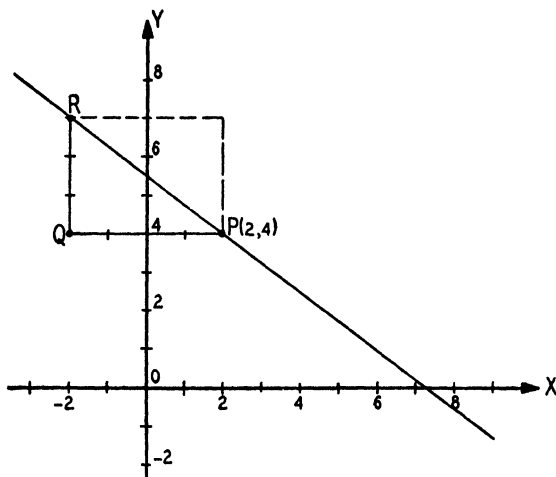


FIG. 15-5.

line RP is the required line. The same result could have been achieved by making the construction indicated by the dotted lines in Fig. 15-5.

EXERCISES

Find the slope of the line joining each pair of points. Find also the angle of inclination of each line, accurate to the nearest tenth of a degree.

- | | |
|--------------------------|--------------------------|
| 1. $(3, 6), (6, 3)$. | 2. $(2, 4), (-5, 6)$. |
| 3. $(-3, 2), (-5, 8)$. | 4. $(6, 7), (-5, 3)$. |
| 5. $(-2, 8), (12, -4)$. | 6. $(8, 3), (-5, -6)$. |
| 7. $(-2, -3), (4, 6)$. | 8. $(10, 7), (-8, -3)$. |

If three points P_1 , P_2 , and P_3 are given, and the slope of P_1P_2 is equal to the slope of P_1P_3 , then these two lines coincide. Use this fact to show that the points in each of the following sets lie on a straight line.

- | | |
|--------------------------------|---------------------------------|
| 9. (0, 0), (3, 4), (6, 8). | 10. (0, 0), (-1, 2), (-3, 6). |
| 11. (2, 1), (0, -1), (-2, -3). | 12. (-2, 2), (1, 1), (4, 0). |
| 13. (-4, -1), (0, 1), (2, 2). | 14. (0, -4), (-2, -2), (-4, 0). |
| 15. (-2, 1), (0, 0), (4, -2). | 16. (-1, 4), (3, 3), (7, 2). |

Determine k so that the three points in each of the following sets lie on the same line.

- | | |
|-------------------------------------|-------------------------------------|
| 17. (2, 3), (4, 6), (k , 9). | 18. (1, 4), (2, 3), (k , 1). |
| 19. (0, 0), (-1, -2), (3, k). | 20. (3, 1), (5, 4), (1, k). |
| 21. (1, -1), (-2, 5), (2, k). | 22. (0, 2), (1, 1), (k , 3). |
| 23. (1, -1), (2, 1), (k , k). | 24. (-3, 8), (4, 6), (k , k). |

Construct the line through the given point with the given slope.

- | | |
|-------------------------------------|--------------------------------------|
| 25. $P(6, 2)$, $m = \frac{1}{2}$. | 26. $P(-2, 1)$, $m = \frac{3}{4}$. |
| 27. $P(3, 4)$, $m = 3$. | 28. $P(-2, 4)$, $m = 5$. |
| 29. $P(-3, -4)$, $m = -3$. | 30. $P(4, 2)$, $m = -\frac{1}{5}$. |
| 31. $P(3, 1)$, $m = -5$. | 32. $P(2, 3)$, $m = -2$. |

15-4. Parallel and Perpendicular Lines. When two lines L_1 and L_2 are parallel, they have the same angles of inclination θ_1 and θ_2 , respectively. Then, since $\theta_1 = \theta_2$, $\tan \theta_1 = \tan \theta_2$, and

$$(1) \quad m_1 = m_2.$$

It follows that, *if two lines are parallel, their slopes are equal. Conversely, if the slopes of two lines are equal, the lines are parallel.*

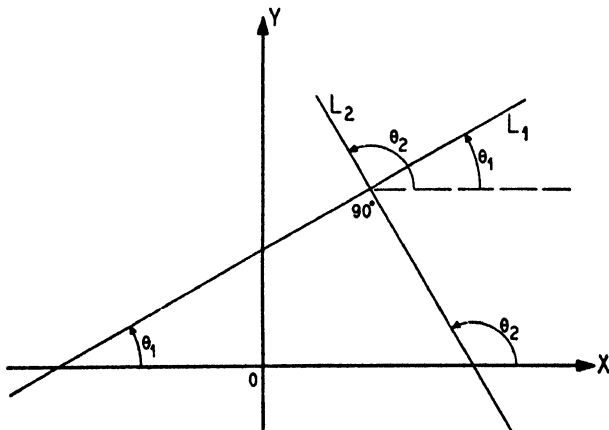


FIG. 15-6.

Let L_1 and L_2 be lines with angles of inclination of θ_1 and θ_2 respectively, and let L_1 be perpendicular to L_2 . Now either $\theta_2 = \theta_1 + 90^\circ$

as shown in Fig. 15-6, or $\theta_1 = \theta_2 + 90^\circ$. In the first case, if we denote the slopes of L_1 and L_2 by m_1 and m_2 respectively,

$$\begin{aligned} m_2 &= \tan \theta_2 = \tan (\theta_1 + 90^\circ) \\ &= -\cot \theta_1 = -\frac{1}{\tan \theta_1} = -\frac{1}{m_1}. \end{aligned}$$

In the second case,

$$\begin{aligned} m_1 &= \tan \theta_1 = \tan (\theta_2 + 90^\circ) \\ &= -\cot \theta_2 = -\frac{1}{\tan \theta_2} = -\frac{1}{m_2}. \end{aligned}$$

Hence, in either case,

$$(2) \quad m_1 m_2 = -1 \quad \text{or} \quad m_1 = -\frac{1}{m_2}.$$

That is to say, *if two lines are perpendicular, their slopes are negative reciprocals. Conversely, if two lines have slopes which are negative reciprocals, the lines are perpendicular.*

EXERCISES

The three points given in each exercise form the vertices of a triangle. Find the slopes of the sides of the triangle, and using this data show that each is a right triangle.

1. (6, 2), (5, 7), (3, 5).
2. (0, 0), (4, 2), (2, 6).
3. (0, 9), (6, 3), (-8, 1).
4. (0, 3), (2, 1), (6, 5).
5. (-3, 0), (-1, -4), (-5, -6).
6. (-6, -1), (8, 7), (-10, 6).

7. Show that the line through (-2, 0) and (0, 1) is parallel to the line through (1, -1) and (5, 1).

8. Show that the line through (1, -2) and (6, 5) is parallel to the line through (-2, -4) and (3, 3).

9. Show that the segment joining (2, 0) and (0, 3) is parallel to and half as long as the segment joining (4, 0) and (0, 6).

10. Show that the line through (-1, -2) and (3, 0) is perpendicular to the line through (2, 0) and (0, 4).

11. Show that the line through (-2, -3) and (2, -2) is perpendicular to the line through (1, 0) and (0, 4).

12. Prove that the points (0, 3), (3, 4), (5, -2), and (2, -3) form the vertices of a rectangle.

13. Using the idea of slope, show that the points (3, 2), (0, -1), (0, 5), and (-3, 2) form the vertices of a square.

14. Prove that the points (2, 8), (6, 4), (3, -2), and (-1, 2) form the vertices of a quadrilateral whose opposite sides are parallel.

Using the slope, show that the four points given in each exercise below form the vertices of a parallelogram. Which of these parallelograms are rectangles?

15. $(-2, 0), (-2, 2), (2, 0), (2, 2)$. 16. $(-3, 1), (3, 3), (4, 0), (-2, -2)$.
 17. $(0, 6), (-2, 10), (-4, 6), (-2, 2)$. 18. $(-5, -3), (1, -11), (7, -6), (1, 2)$.
 19. $(-2, 1), (1, -2), (5, 2), (2, 5)$.

20. The points $(6, 8), (-4, 0), (0, -4)$, and $(6, -2)$ form the vertices of a quadrilateral. Prove that the figure formed by joining in order the midpoints of the sides of the quadrilateral is a parallelogram.

21. The points $(8, 2), (4, 10), (-2, 3)$, and $(-4, -4)$ form the vertices of a quadrilateral. Prove that the figure formed by joining in order the midpoints of the sides of the quadrilateral is a parallelogram.

22. Show that the points $(-2, 0), (0, -4), (4, -2)$, and $(2, 2)$ form the vertices of a square. Then prove that the midpoints of the sides of this square form the vertices of a square.

23. Show that the points $(3, 2), (0, -1), (0, 5)$, and $(-3, 2)$ form the vertices of a square. Then prove that the midpoints of the sides of this square form the vertices of a square.

15-5. The Angle between Two Intersecting Lines. Let L_1 and L_2 be two intersecting lines with angles of inclination θ_1 and θ_2 respectively.

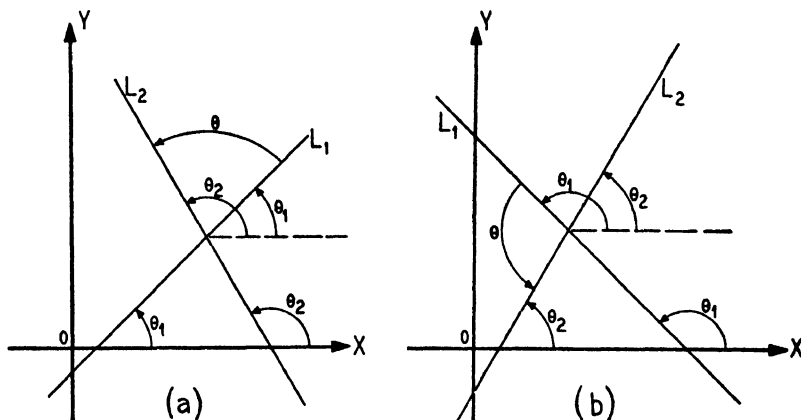


FIG. 15-7.

Then, either $\theta_1 < \theta_2$ as in Fig. 15-7a, or $\theta_1 > \theta_2$ as in Fig. 15-7b. In order to specify one angle between L_1 and L_2 , we adopt the following definition.

The angle θ from L_1 to L_2 is the positive angle less than 180° through which L_1 must be rotated in counterclockwise direction in order to coincide with L_2 .

Now in Fig. 15-7a, $\theta = \theta_2 - \theta_1$ and

$$(1) \quad \tan \theta = \tan (\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2}.$$

Since $\tan \theta_1$ is the slope m_1 of L_1 , and $\tan \theta_2$ is the slope m_2 of L_2 , we may write (1) as

$$(2) \quad \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

In the case of Fig. 15-7b, $\theta_1 - \theta_2 + \theta = 180^\circ$ whence $\theta = 180^\circ + \theta_2 - \theta_1$. Then

$$(3) \quad \tan \theta = \tan [180^\circ + (\theta_2 - \theta_1)] = \tan (\theta_2 - \theta_1)$$

by the reduction formulas in Sec. 4-11, and (2) holds as in the previous case.

Hence, if L_1 and L_2 are two intersecting lines with slopes m_1 and m_2 respectively, the angle θ from L_1 to L_2 is given by

$$(3) \quad \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

Formula (3) requires that neither m_1 nor m_2 be infinite. But this exceptional case occurs when L_1 or L_2 are parallel to the y -axis, and θ can then be found without using formula (3) as will be shown in the example which follows.

Example 1. Find the interior angles of the triangle whose vertices are $A(-3, 5)$, $B(3, 3)$, and $C(3, -2)$.

The slope of AB is $\frac{5-3}{-3-3} = -\frac{1}{3}$, and the slope of AC is $-\frac{7}{6}$.

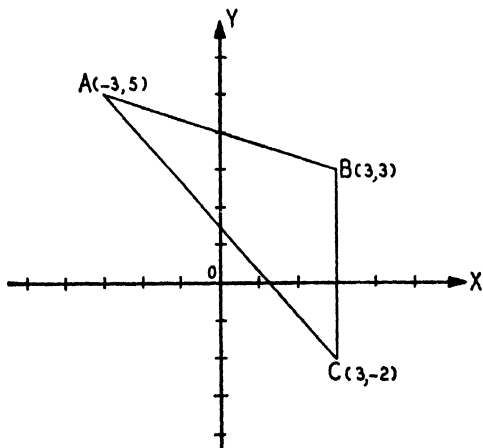


FIG. 15-8.

The angle A is the angle from AC to AB , whence from (3)

$$\begin{aligned}\tan A &= \frac{\text{Slope } AB - \text{Slope } AC}{1 + (\text{Slope } AB)(\text{Slope } AC)} \\ &= \frac{-\frac{1}{3} - (-\frac{7}{6})}{1 + (-\frac{1}{3})(-\frac{7}{6})} = \frac{3}{5} = 0.6000.\end{aligned}$$

From Table 3,

$$A = 31.0^\circ.$$

In this case BC is parallel to the y -axis so that formula (3) cannot be used to find angles C and B . Now if θ_1 and θ_2 denote the angles of inclination of AC and AB respectively, from Fig. 15-8 it is obvious that

$$C = \theta_1 - 90^\circ, \quad B = 270^\circ - \theta_2.$$

Now since

$$\tan \theta_1 = -\frac{7}{6} = -1.1667 \quad \text{and} \quad \tan \theta_2 = -\frac{1}{3} = -0.3333,$$

we have

$$\theta_1 = 130.6^\circ \quad \text{and} \quad \theta_2 = 161.6^\circ.$$

Hence

$$C = 40.6^\circ \quad \text{and} \quad B = 108.4^\circ.$$

As a check we have

$$31.0^\circ + 40.6^\circ + 108.4^\circ = 180^\circ.$$

Example 2. If L_1 has the slope $\frac{1}{3}$, find the slope of a line L_2 such that the angle from L_1 to L_2 is 45° .

If the slopes of L_1 and L_2 are m_1 and m_2 , then $m_1 = \frac{1}{3}$, and

$$\tan 45^\circ = \frac{m_2 - \frac{1}{3}}{1 + \frac{1}{3}m_2},$$

whence

$$1 = \frac{3m_2 - 1}{3 + m_2},$$

$$3 + m_2 = 3m_2 - 1,$$

$$m_2 = 2.$$

EXERCISES

Find the angle from L_1 to L_2 when the following data are given.

1. L_1 has slope $\frac{3}{4}$, L_2 has the slope 1.
2. L_1 has the slope $-\frac{1}{2}$, L_2 has the slope 3.
3. L_1 has the slope 1, L_2 has the slope 2.
4. L_1 has the slope $\frac{1}{3}$, L_2 has the slope 3.
5. L_1 has the slope $\frac{3}{4}$, L_2 has the slope -4 .
6. L_1 has the slope 2, L_2 has the slope -1 .
7. L_1 passes through $(1, 1)$ and $(2, 3)$; L_2 passes through $(-1, 2)$ and $(0, 0)$.
8. L_1 passes through $(-1, 2)$ and $(5, 6)$; L_2 passes through $(0, 0)$ and $(4, -1)$.
9. L_1 passes through $(-3, -2)$ and $(-1, 3)$; L_2 passes through $(5, 0)$ and $(0, 2)$.
10. L_1 passes through $(4, -3)$ and $(0, 0)$; L_2 passes through $(-4, -4)$ and $(2, 2)$.

Find the interior angles of each of the triangles whose vertices are given.

- | | |
|----------------------------------|-----------------------------------|
| 11. $(-4, -1), (0, 1), (2, 2)$. | 12. $(-1, 4), (3, 3), (7, 2)$. |
| 13. $(-3, 3), (3, 5), (2, -2)$. | 14. $(1, 2), (-3, 5), (-1, -1)$. |
| 15. $(5, 6), (-2, -1), (4, 3)$. | 16. $(-4, 2), (3, 2), (0, -2)$. |
| 17. $(-2, 0), (-3, 2), (3, 1)$. | 18. $(4, 4), (2, 1), (1, 2)$. |

In each of the following problems find m_1 , the slope of L_1 , when m_2 , the slope of L_2 , and θ , the angle from L_1 to L_2 , are given.

- | | |
|-------------------------------------|---|
| 19. $m_2 = 2, \theta = 45^\circ$. | 20. $m_2 = \frac{1}{2}, \theta = 60^\circ$. |
| 21. $m_2 = 1, \theta = 135^\circ$. | 22. $m_2 = 0, \theta = 150^\circ$. |
| 23. $m_2 = -2, \theta = 30^\circ$. | 24. $m_2 = -\frac{1}{2}, \theta = 60^\circ$. |

15-6. The Area of a Triangle. Consider the triangle shown in Fig. 15-9 with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$. Start with P_1 ,

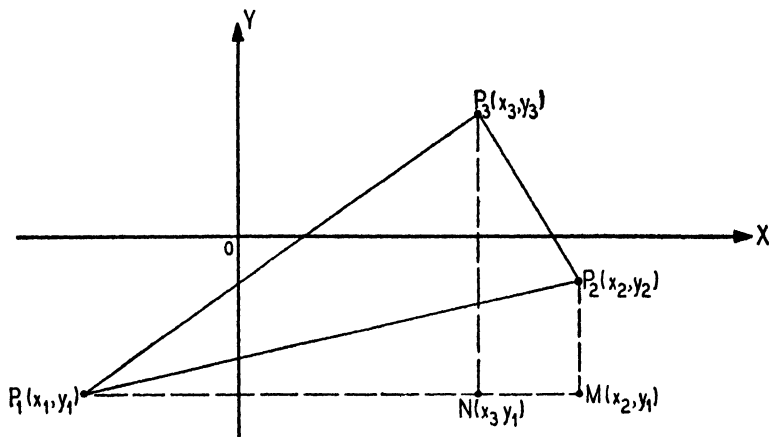


FIG. 15-9.

which is the lowest vertex, and locate the points $M(x_2, y_1)$ and $N(x_3, y_1)$, and construct the dotted segments P_1N , NM , P_3N , and P_2M . Now the area $P_1P_2P_3$ equals the area P_1P_3N plus the area NP_3P_2M minus the area P_1P_2M . Since the area of a triangle is one-half the base times the altitude and the area of a trapezoid is one-half the sum of the parallel sides times the distance between those sides, we have

$$\begin{aligned}\text{Area } P_1P_3N &= \frac{1}{2}(x_3 - x_1)(y_3 - y_1), \\ \text{Area } NP_3P_2M &= \frac{1}{2}(x_2 - x_3)(y_2 - y_1 + y_3 - y_1), \\ \text{Area } P_1P_2M &= \frac{1}{2}(x_2 - x_1)(y_2 - y_1).\end{aligned}$$

Then

$$\begin{aligned}\text{Area } P_1P_2P_3 &= \frac{1}{2}[(x_3 - x_1)(y_3 - y_1) + (x_2 - x_3)(y_2 + y_3 - 2y_1) \\ &\quad - (x_2 - x_1)(y_2 - y_1)] \\ &= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\end{aligned}$$

which we may write as

$$\frac{1}{2} \left[x_1 \begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} \right].$$

Since this is obviously the expansion of a third-order determinant, we have that

$$(1) \quad \text{Area } P_1P_2P_3 = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The proof given here has been based upon Fig. 15-9. However, similar proofs cover all other cases, and the results obtained are the same as (1) with the possible difference of a sign. Thus for any triangle we may state the following.

The area A of a triangle given by its three vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ is given by the formula

$$(2) \quad A = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the sign is chosen so that A is positive.

Example. Find the area of the triangle whose vertices are $(3, 5)$, $(-2, 2)$, and $(1, -2)$.

By (2) we have

$$\begin{aligned} \frac{1}{2} \begin{vmatrix} 3 & 5 & 1 \\ -2 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} &= \frac{1}{2} \begin{vmatrix} 2 & 7 & 0 \\ -3 & 4 & 0 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} 2 & 7 \\ -3 & 4 \end{vmatrix} = \frac{1}{2}(8 + 21) = \frac{29}{2} = 14.5. \end{aligned}$$

It follows that the area of the triangle is 14.5.

In this example the points were chosen for insertion in (2) in the order in which they are met by starting with $(3, 5)$ and traversing the perimeter of the triangle in the counterclockwise direction (shown by the arrows in Fig. 15-10). It can be shown in general that whenever the points are chosen in this counterclockwise order the determinant in (2) is positive, and, further, that this is true regardless of which point is used as a starting point. If the points are chosen in the opposite, or clockwise, order the determinant in (2) will be negative regardless of

which point is used as a starting point. The reader may easily verify that these general statements are correct for this example.

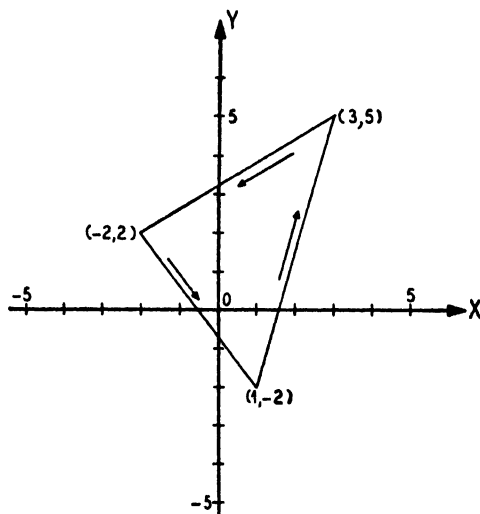


FIG. 15-10.

EXERCISES

Find the areas of the triangles whose vertices are given.

1. (2, 1), (0, -3), (4, 1).
2. (4, 4), (2, 1), (1, 2).
3. (-3, 3), (3, 5), (2, -2).
4. (-3, 0), (-1, -4), (-5, -6).
5. (-6, -1), (14, 8), (-10, 6).
6. (8, 2), (4, 10), (-2, 3).
7. (-5, 3), (1, -1), (7, -6).
8. (-2, 0), (-2, 2), (2, 0).
9. (-4, -1), (0, 1), (2, 2).
10. (-1, 4), (3, 3), (7, 2).

Prove that the points in each exercise form the vertices of a parallelogram, and then find its area.

11. (1, 1), (-1, 1), (2, 4), (-2, -2).
12. (2, 0), (2, 8), (6, 4), (-2, 4).
13. (-6, -4), (6, 4), (0, 2), (0, -2).
14. (2, 8), (-4, 4), (8, 4), (2, 0).
15. (5, 6), (6, 2), (-2, 0), (-3, 4).

Form a quadrilateral by joining the points in the order given. Then find the area of the quadrilateral.

16. (6, 8), (-4, 0), (0, -4), (6, -2).
17. (8, 2), (4, 10), (-2, 3), (-4, -4).
18. (5, 2), (0, 4), (-2, 2), (-3, -5).
19. (5, 0), (1, 4), (-4, -2), (0, -3).
20. (6, 3), (1, 6), (-3, 2), (0, -5).
21. Find the area of the polygon formed by joining in order the points (5, 0), (4, 3), (0, 4), (-2, 0), (2, 2).
22. Find the area of the polygon formed by joining in order the points (5, -2), (6, 2), (1, -1), (-3, 1), (0, -4).
23. Find the area of the polygon formed by joining in order the points (4, 0), (6, 2), (0, 3), (-2, 0), (0, -4), (1, -1).

15-7. The Equation of a Straight Line. By definition, the equation of a straight line is an equation in x and y of the form $f(x, y) = 0$ which is satisfied by the coordinates of every point on the line and which is not satisfied by the coordinates of any point not on the line.

A line may be determined by giving its slope and a point through which it passes or by giving two points through which it passes. In the next few sections we shall find the equations of lines given by these geometrical conditions. When these geometrical conditions are stated in general literal notation, the resulting equations are called **standard forms**. These standard forms will enable us to show that the points satisfying a first degree equation form a straight line, and conversely.

15-8. The Point-Slope Form. Let L be a line passing through the point $P_1(x_1, y_1)$ and having a slope m . If $P(x, y)$ is any point on the line different from P_1 , then the slope of PP_1 is the same as the slope of L , and we have

$$(1) \quad m = \frac{y - y_1}{x - x_1},$$

and hence,

$$(2) \quad y - y_1 = m(x - x_1).$$

Since (1) holds for any point on L different from $P_1(x_1, y_1)$, (2) does also. In addition (2) is satisfied by (x_1, y_1) , whence (2) holds for every point on L .

On the other hand, if (x, y) is not on L , (1) and therefore (2) does not hold. Thus (2) is satisfied by the coordinates of the points on L and by the coordinates of no other points, and therefore it is the equation of L .

The line passing through $P_1(x_1, y_1)$ with slope m has the equation

$$(2) \quad y - y_1 = m(x - x_1).$$

This equation is called the **point-slope form**. It enables us to write very readily the equation of a line given by its slope and a point through which it passes. Also, any equation of first degree that can be put into the form (2) is an equation of a line whose slope is m and which passes through the point (x_1, y_1) .

Example 1. Find an equation of the line through $(2, -3)$ with slope $\frac{1}{2}$.

By (2) we have

$$y + 3 = \frac{1}{2}(x - 2)$$

which can be written

$$x - 2y - 8 = 0.$$

This is an equation of the given line.

Example 2. Show that $2x - y + 2 = 0$ is an equation of a straight line and find the slope of the line.

The equation can be written as

$$y = 2x + 2$$

or

$$y - 0 = 2[x - (-1)].$$

It follows that the line passes through $(-1, 0)$ and has slope 2.

15-9. The Slope-Intercept Form. If a line intersects the x -axis, the abscissa of the point of intersection is called the **x -intercept** of the line. Similarly the **y -intercept** is the ordinate of the point of intersection of the line and the y -axis.

As a special case of the point-slope form, let the point be given by the y -intercept b . Then the line passes through $(0, b)$, and has slope m and has the equation

$$y - b = m(x - 0),$$

or

$$y = mx + b.$$

Hence, the line with slope m and y -intercept b has the equation

$$(1) \quad y = mx + b.$$

This equation is called the **slope-intercept form**. Since a line with a finite slope is not parallel to the y -axis, it intersects the y -axis, and therefore has a y -intercept. Thus any line not parallel to the y -axis has an equation of the form (1). Also any first degree equation whose y -coefficient is not zero can be put in the form (1) by solving for y . Thus we see that *if a first degree equation is solved for y , the coefficient of x is the slope of the line given by the equation.*

Example 1. Find the slope of the line whose equation is

$$3x - 2y + 6 = 0.$$

Solving the equation for y , we obtain

$$2y = 3x + 6, \quad y = \frac{3}{2}x + 3$$

whence the line has slope $\frac{3}{2}$ and y -intercept 3.

Example 2. Find the equation of the straight line through $(3, 4)$ and perpendicular to $3x - 2y + 6 = 0$.

The equation of the line can be written as $y = \frac{3}{2}x + 3$, whence its slope is $\frac{3}{2}$. Then the slope of any perpendicular line is $-\frac{2}{3}$. Using the point-slope form, the required line has the equation

$$(y - 4) = -\frac{2}{3}(x - 3)$$

or

$$2x + 3y - 18 = 0.$$

15-10. Lines Parallel to the Axes. Since a line parallel to the y -axis has no finite slope, neither of the preceding forms of the equation of a line apply. However, it is obvious that any line parallel to the y -axis has an equation of the form

$$(1) \quad x = a$$

where a is a constant and equal to the abscissa of any point on the line. Thus the line through $(5, -3)$ and parallel to the y -axis is $x = 5$.

Any line parallel to the x -axis has slope zero and hence has the form

$$(2) \quad y = b$$

where b is the y -intercept of the line. Thus b is the ordinate of any point on the line. For example, the line through $(5, -3)$ and parallel to the x -axis is $y = -3$.

15-11. The General Linear Equation. As we have just seen, every line not parallel to the y -axis has an equation of the form $y = mx + b$. A line parallel to the y -axis has an equation of the form $x = a$. Hence the following theorem holds.

Every straight line has an equation of first degree.

The converse theorem is also true.

Every equation of first degree is satisfied by the coordinates of the points on a straight line.

To prove this statement we start with the general equation of first degree.

$$(1) \quad Ax + By + C = 0.$$

Since (1) would be trivial if A and B were both zero, we assume that A and B are not both zero. We shall consider two cases in connection with (1), (a) when $B = 0$, and (b) when $B \neq 0$.

(a) If $B = 0$, then $A \neq 0$ by hypothesis, and we have $Ax + C = 0$ which can be written as

$$x = -\frac{C}{A},$$

which is the equation of a line parallel to the y -axis.

(b) If $B \neq 0$, we can solve (1) for y and obtain

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is the equation of a line with slope $-\frac{A}{B}$ and y -intercept $-\frac{C}{B}$.

EXERCISES

Find an equation of the line passing through the given point with the given slope

1. $(5, 6)$, $m = \frac{1}{2}$.
2. $(2, -3)$, $m = \frac{3}{4}$.
3. $(1, 4)$, $m = -\frac{1}{3}$.
4. $(-3, 1)$, $m = \frac{2}{5}$.
5. $(-1, -2)$, $m = -3$.
6. $(2, -3)$, $m = 4$.
7. $(5, 4)$, $m = -\frac{1}{2}$.
8. $(7, -6)$, $m = \frac{2}{3}$.
9. $(-5, 8)$, $m = 0$.
10. $(2, 2)$, $m = -\frac{3}{4}$.

Find the slope-intercept form of the equation of the line whose slope m and y -intercept b are given.

11. $m = \frac{1}{2}$, $b = 3$.
12. $m = \frac{3}{4}$, $b = -2$.
13. $m = -5$, $b = 4$.
14. $m = -\frac{3}{8}$, $b = -5$.
15. $m = \frac{2}{5}$, $b = 2$.
16. $m = -\frac{1}{4}$, $b = 1$.
17. $m = 4$, $b = 0$.
18. $m = 5$, $b = -3$.
19. $m = -3$, $b = -2$.
20. $m = \frac{1}{5}$, $b = 1$.

Find an equation of the line passing through the given point with the given angle of inclination θ .

21. $(1, 1)$, $\theta = 45^\circ$.
22. $(0, 2)$, $\theta = 60^\circ$.
23. $(3, -4)$, $\theta = 135^\circ$.
24. $(0, 0)$, $\theta = 120^\circ$.
25. $(-3, 2)$, $\theta = 30^\circ$.
26. $(-1, -3)$, $\theta = 150^\circ$.

Find the slope-intercept form of the equation of the line whose y -intercept b and angle of inclination θ are given.

27. $b = 5$, $\theta = 45^\circ$.
28. $b = -3$, $\theta = 135^\circ$.
29. $b = -4$, $\theta = 60^\circ$.
30. $b = -2$, $\theta = 30^\circ$.
31. $b = 0$, $\theta = 120^\circ$.
32. $b = -1$, $\theta = 150^\circ$.

Find the slope and y -intercept of each of the following lines.

33. $3x + y = 6$.
34. $2x + 3y = 12$.
35. $5x + 2y - 10 = 0$.
36. $3x + 5y - 10 = 0$.
37. $2x - y + 4 = 0$.
38. $y - 3x + 12 = 0$.
39. $2x + 3y - 6 = 0$.
40. $7x + 5y + 14 = 0$.
41. $3x - 5y - 20 = 0$.
42. $4x + 3y - 12 = 0$.

Find an equation of the line through the given point (a) parallel, and (b) perpendicular to the given line.

43. $(1, 2)$, $x - y = 3$.
44. $(-2, 3)$, $2x + y = 5$.
45. $(4, 6)$, $x - 2y = 4$.
46. $(-1, -1)$, $3x + 4y = 12$.
47. $(0, 0)$, $3x - 5y = 15$.
48. $(-3, -5)$, $6x - 8y = 13$.
49. $(6, 8)$, $x = 5y + 6$.
50. $(-4, 2)$, $y = 6x + 10$.

Find an equation of each of the lines determined by the following pairs of points.

51. $(3, 2)$, $(6, 1)$.
52. $(-1, 2)$, $(5, 3)$.
53. $(-5, 6)$, $(0, 0)$.
54. $(6, 2)$, $(-3, 2)$.
55. $(8, -3)$, $(-2, 1)$.
56. $(4, 3)$, $(-6, 2)$.

57. Find the equation of the y -axis.

58. Find the equation of the x -axis.

59. Find the equation of the line through $(3, -2)$ parallel to the y -axis.

60. Find the equation of the line through $(3, -2)$ parallel to the x -axis.

61. Find the equation of the line through $(5, -3)$ perpendicular to the y -axis.

62. Find the equation of the line through $(5, -3)$ perpendicular to the y -axis.

Find the angle from L_1 to L_2 .

63. $L_1: x + y = 4,$

$L_2: 2x + y = 8.$

64. $L_1: x - 3y = 3,$

$L_2: 2x + y = 2.$

65. $L_1: 2x + 3y = 6,$

$L_2: 5x - 3y = 15.$

66. $L_1: 4x + 3y = 12,$

$L_2: 2x + 3y = 6.$

67. $L_1: 5x - y = 5,$

$L_2: x + y = 4.$

68. $L_1: 3x + 4y = 12,$

$L_2: 4x - 3y = 6.$

15-12. The Two-Point Form. If two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given on a line not parallel to the y -axis, the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting this for m in the point-slope form we have

$$(1) \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

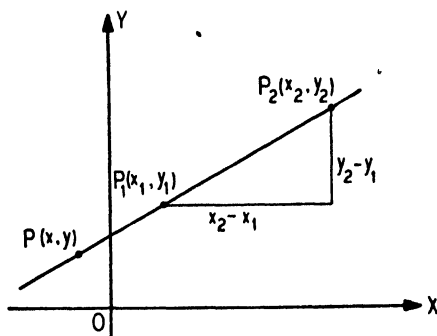


FIG. 15-11.

An equation of the straight line passing through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by (1). This is called the **two-point form** of the equation of a line. Writing (1) as

$$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1),$$

we have a second form of the equation of the line. This form has no exceptions since it is also valid in the case $x_1 = x_2$.

The reader may easily verify that (1) can be written as a determinant

$$(2) \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

This determinant is not as useful as (1) for actually finding an equation of a line, but it is sometimes convenient in theoretical discussions.

Example. Find an equation of the line through (3, 4) and (-1, 2). From (1) we have

$$(y - 4) = \frac{(2 - 4)}{(-1 - 3)}(x - 3)$$

or

$$x - 2y + 5 = 0.$$

EXERCISES

Find an equation of the line passing through each of the following pairs of points.

- | | | |
|----------------------|-----------------------|----------------------|
| 1. (0, 0), (4, 5). | 2. (1, 2), (3, 4). | 3. (-2, 3), (4, -1). |
| 4. (1, 3), (3, 6). | 5. (0, 0), (-1, 4). | 6. (-2, -2), (3, 5). |
| 7. (-3, -2), (1, 1). | 8. (-5, 6), (2, 3). | 9. (-3, -2), (5, 7). |
| 10. (5, 6), (-9, 3). | 11. (6, 2), (-5, 3). | 12. (0, 0), (1, 2). |
| 13. (0, 0), (-1, 3). | 14. (2, 3), (-1, -5). | 15. (1, 2), (1, 5). |

Find the equations of the medians of the triangles whose vertices are as follows.

- | | |
|---------------------------------|--------------------------------|
| 16. (0, 0), (1, 5), (6, 2). | 17. (2, 3), (6, -3), (0, -2). |
| 18. (4, 4), (-2, 2), (0, -4). | 19. (6, 10), (-3, 5), (-5, 3). |
| 20. (1, 1), (-3, -3), (-1, -5). | |

21. Find the equation of the line through (1, 1) parallel to the line through (3, 4) and (5, -8).

22. Find the equation of the line through (3, -3) parallel to the line through (1, 1) and (3, 3).

23. Find the equation of the line through (1, 1) perpendicular to the line through (3, 4) and (5, -8).

24. Find the equation of the line through (3, -3) perpendicular to the line through (3, 4) and (5, -8).

25. Show that (1, 1), (3, 4), and (5, 7) lie on a straight line.

26. Show that (1, -3), (2, -1), and (-2, -9) lie on a straight line.

27. Show that (2, 4), (3, 8), and (0, -4) lie on a straight line.

15-13. The Intercept Form. One of the easiest ways of plotting the line corresponding to a given equation is to find its intercepts on the axes. We shall now use the intercepts of a line to find its equation.

Let a be the x -intercept and b the y -intercept (Fig. 15-12), neither of which is zero. Since the line thus contains the points $(a, 0)$ and $(0, b)$, its equation, found from the two-point form, is

$$y - 0 = -\frac{b}{a}(x - a),$$

or

$$(1) \quad \frac{x}{a} + \frac{y}{b} = 1.$$

This is called the **intercept form** of the equation of a line.

An equation of the form $Ax + By + C = 0$ where A , B , and C are not 0, can be reduced to the intercept form after dividing by $-C$, obtaining

$$\left(-\frac{A}{C}\right)x + \left(-\frac{B}{C}\right)y = 1$$

whence

$$a = -\frac{C}{A}, \quad b = -\frac{C}{B}.$$

If either A , B , or C is 0, the equation cannot be written in intercept form. (Why?)

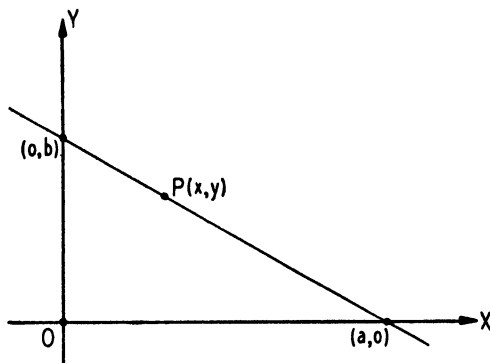


FIG. 15-12.

Example 1. Find the equation of the line with x -intercept 5 and y -intercept -6 . From (1), $a = 5$ and $b = -6$, whence the equation of the line is

$$\frac{x}{5} + \frac{y}{-6} = 1,$$

which we may write as

$$6x - 5y = 30.$$

Example 2. Find the intercepts of the line $2x - 3y = 6$.

Dividing by 6 to make the right member unity we obtain

$$\frac{x}{3} + \frac{y}{-2} = 1,$$

whence the x -intercept is 3 and the y -intercept is -2 .

EXERCISES

Find an equation of each of the lines determined by the following data.

- | | |
|--|--|
| 1. x -intercept 3, y -intercept 4. | 2. x -intercept -5 , y -intercept 2. |
| 3. x -intercept -2 , y -intercept 6. | 4. x -intercept 3, y -intercept -4 . |
| 5. x -intercept 2, y -intercept -7 . | 6. x -intercept -6 , y -intercept -2 . |
| 7. x -intercept -8 , y -intercept 3. | 8. x -intercept 4, y -intercept 4. |
| 9. x -intercept 5, y -intercept 1. | |

Find the intercepts of the following lines by reducing each of them to the form (1).

10. $3x - 4y - 8 = 0$.

11. $2x - 3y + 3 = 0$.

12. $3x - 2y = 6$.

13. $5x - 3y = 15$.

14. $5x + 4y = 20$.

15. $2x - 7y = 14$.

16. $3x + 5y = 10$.

17. $8x - 3y = 12$.

18. $6y - 5x + 30 = 0$.

19. $2y + x - 8 = 0$.

20. Find the equation of the line passing through (4, 3) having equal intercepts.

21. Find the equation of the line passing through (1, 3) and forming with the axes a triangle of area 8.

22. Find the equation of the line passing through (2, 3) and forming with the axes a triangle of area 12.

23. Find the equation of the line passing through (3, 1) and having equal intercepts.

24. Find the equation of the line through (5, 12) having equal intercepts.

25. Find the equation of the line passing through (3, 3) and forming with the axes a triangle of area 24.

15-14. Intersection of Two Lines. The coordinates of the point of intersection of two non-parallel lines satisfy the equations of both lines. The coordinates of this point of intersection are the values of x and y obtained by solving the two equations simultaneously. This has been done algebraically and graphically in Chapters 3 and 13.

MISCELLANEOUS EXERCISES

1. Find the equation of the line through (2, -3) parallel to the line through (2, 4), (-4, 1).

2. Find the coordinates of the vertices of the triangle the equations of whose sides are $x + y = 2$, $2x - y = 1$, $3x + y = 9$.

3. A diagonal of a square joins the points (2, 1) and (3, 6). Find the coordinates of the other vertices.

4. Find the distance from the origin to the point of intersection of the lines $x + 3y - 1 = 0$ and $2x - 4y + 5 = 0$.

5. Find the distance from the point (1, 1) to the point of intersection of the lines $5x + y - 4 = 0$ and $3x + y + 1 = 0$.

6. Show that the lines $3x - 2y + 3 = 0$, $5x + y - 8 = 0$, and $x - 4y + 11 = 0$ intersect in a common point.

7. Find the equation of the line perpendicular to $3x - y = 5$ and bisecting the segment joining (-2, 4) and (4, 6).

8. Find an equation of the perpendicular bisector of the segment whose ends are (1, -2) and (3, 6).

9. Find an equation of a line through the point (-2, 0) bisecting the segment whose ends are (0, 2) and (3, -1).

10. Find the point of intersection of the two diagonals of the rectangle whose vertices are (0, 0), (8, 0), (8, 4), and (0, 4).

11. Write the equations of the sides of the triangle whose vertices are (3, 3), (-3, 3), and (0, 0).

12. Find the area of the triangle formed by the lines $x - y = 0$, $x + y = 0$, and $2x + y - 3 = 0$.

13. Find the area of the triangle whose sides lie along the lines $2x + y - 6 = 0$, $x - y + 3 = 0$, and $x - 2y - 8 = 0$.

14. Find the equation of the line whose intercepts are twice those of the line $2x + y - 4 = 0$.

15. Show that the two points $(5, 2)$ and $(6, -15)$ subtend a right angle at the origin.

16. Find the equation of a line through the point $(1, -3)$ and parallel to the line $3x - y - 5 = 0$.

17. Find the coordinates of the foot of the perpendicular from the origin to the line whose equations is $x + 2y - 5 = 0$.

18. Find the coordinates of the foot of the perpendicular from the point $(1, -1)$ to the line $3x + 4y - 12 = 0$.

19. Find the equation of the line through the origin whose slope is twice that of the line $3x - y = 5$.

20. Determine p so that the line $3x - py = 5$ shall be parallel to the line $3x - 4y = 3$.

21. Find the form of formula (2) in Sec. 15-6 when one of the vertices, say (x_3, y_3) , is in the origin.

22. The vertices of a triangle are $(-5, 3)$, $(5, -3)$, and $(7, 3)$. Find:

(a) The equations of the sides.

(b) The equations of the lines through the vertices parallel to the opposite sides.

(c) The equations of the perpendicular bisectors of the sides.

(d) The equations of the lines through the vertices perpendicular to the opposite sides.

(e) The area of the triangle.

23. Find the equation of the line which makes equal intercepts on the axes and passes through the point $(2, 5)$.

24. Find the equation of the line whose y -intercepts are twice its x -intercepts and which passes through the point $(3, 4)$.

25. Find the point on the x -axis which is equidistant from the points $(3, 2)$, $(-2, 3)$.

26. Find the slope of each of the following straight lines:

$$(a) \begin{vmatrix} x & y & 1 \\ 5 & 6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0, \quad (b) \begin{vmatrix} x & y & 1 \\ 0 & 9 & 1 \\ -3 & 0 & 1 \end{vmatrix} = 0.$$

27. For the triangle whose sides are

$$x - y + 2 = 0, \quad 2x + 9y + 15 = 0, \quad 7x + 4y - 30 = 0$$

find:

(a) The coordinates of the vertices.

(b) The equations of lines through the vertices perpendicular to the opposite sides.

(c) The equations of the lines through the vertices parallel to the opposite sides.

(d) The equations of the perpendicular bisectors of the sides.

(e) The area of the triangle.

28. The electrical resistance R of a bismuth spiral changes when placed in a magnetic field and varies linearly with the field density B . Two observations were made in the laboratory. At densities of 14,000 and 6000 lines per sq. cm. the resistances of the spiral were 9.6 and 7.4 ohms respectively. Find the equation of the line. What is the resistance of the bismuth spiral at zero field density?

29. The charging current in amperes of an automobile storage battery is given by the equation $I = \frac{E}{R} + E_1$, where E is the generator voltage, E_1 the battery voltage which is considered constant, and R the resistance in ohms of the circuit. In a specific case the storage battery voltage is 6.4 volts and the circuit resistance is 0.5 ohm. What is the current increase through the battery per volt increase of the generator?

30. The resistance of Nichrome wire changes linearly with temperature. Two readings were taken in the laboratory. At a temperature of 100°C . the resistance of the sample was 24 ohms; at 200°C . the resistance was 50.4 ohms. What is the resistance of the sample at 0°C ?

31. The voltage E of a storage battery cell varies with the concentration Z in grams of sulphuric acid per liter of electrolyte throughout a given range according to the equation $E = 1.850 + 0.00057Z$. Determine the voltage increase when the acid concentration changes from 0.8 to 1.4 kilograms per liter of electrolyte. What is the slope of the line given by the equation?

PROGRESS REPORT

This chapter was devoted to the study of those parts of geometry which involve the straight line. In this study we have translated certain geometrical concepts into algebra. In doing so, a geometry-algebra dictionary has been created in which the geometrical concepts of point, distance, angle, line, etc., were written as algebraic symbols, expressions, and equations. Anyone who works in the field of mathematics and its applications should be well acquainted with the results of this chapter. We shall therefore conclude with a summary of the principal results in the form of a geometry-algebra dictionary.

GEOMETRY-ALGEBRA DICTIONARY

Geometry	Algebra
A point	(x, y)
Distance between two points	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint of a line segment	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Slope of a line segment	$\frac{y_2 - y_1}{x_2 - x_1}$ or $\tan \theta$
Parallel lines	Slopes are equals: $m_1 = m_2$
Perpendicular lines	Slopes are negative reciprocals: $m_2 = -\frac{1}{m_1}$
Angle between two intersecting lines	$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$
Line parallel to the x -axis	$y = b$
Line parallel to the y -axis	$x = a$

Geometry

Algebra

Line through a given point and with a given slope $y - y_1 = m(x - x_1)$

Line with a given slope and a given y -intercept $y = mx + b$

Line through two given points $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

Line with given intercepts $\frac{x}{a} + \frac{y}{b} = 1$

Any straight line $Ax + By + C = 0$

Point of intersection of two lines Solution of two equations

CHAPTER 16

THE CIRCLE, EQUATIONS, AND LOCI

In this chapter the method of studying geometry by means of algebra will be applied to the circle. Knowledge of this study of the circle is indispensable in the study of calculus and in many engineering problems. Thus in electrical engineering, the current of an induction motor, depending on the load, may be represented as a vector from the origin with varying length and angular position; with the voltage constant, the terminal point of the varying vector will describe a circle. This diagram is commonly known as the circle diagram. Again, a circuit containing a fixed resistance and variable inductance establishes a relation between the current and voltage which assumes the form of an equation of a circle.

In addition, we shall study in this chapter how to plot graphs on polar coordinates and how to plot graphs from data in parametric form.

16-1. The Circle. From his study of geometry the student knows that a circle is a figure consisting of all the points in a plane which are at a constant distance from a fixed point. The fixed point is called the center of the circle, and the constant distance is called the radius. To obtain an algebraic representation of the circle we shall translate the geometrical definition of the circle into algebra.

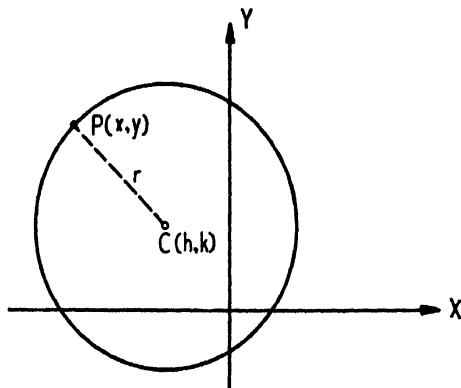


FIG. 16-1.

Let $C(h, k)$ be the fixed point, and let r be the radius (Fig. 16-1). If $P(x, y)$ is any point of the circle, then from the formula for the dis-

tance between two points we have

$$(1) \quad \sqrt{(x-h)^2 + (y-k)^2} = r,$$

which becomes, after squaring,

$$(2) \quad (x-h)^2 + (y-k)^2 = r^2.$$

Thus, if a point P is on the circle, its coordinates satisfy (2). The equation 2 is called the **standard form** of the equation of the circle with center at (h, k) and radius r .

If the center of the circle is at the origin then h and k are both zero, and (2) reduces to

$$(3) \quad x^2 + y^2 = r^2.$$

Example. Find the equation of the circle with center at $(-2, 1)$ and radius 3. We have

$$h = -2, \quad k = 1, \quad r = 3.$$

Therefore the equation of the required circle is

$$(x+2)^2 + (y-1)^2 = 9,$$

or

$$x^2 + 4x + y^2 - 2y - 4 = 0.$$

EXERCISES

Find the equations of the following circles.

1. Center at $(3, 4)$, radius = 7.
2. Center at $(0, 0)$, radius = 5.
3. Center at $(3, -1)$, radius = 4.
4. Center at $(0, 0)$, radius = 8.
5. Center at $(-2, -3)$, radius = 6.
6. Center at $(-a, -a)$, radius = a .
7. Center at $(a, -a)$, radius = $2a$.
8. Center at (b, b) , radius = $b\sqrt{3}$.
9. Center at $(6, 0)$ and passing through the origin.
10. Center at $(2, 3)$ and touching the x -axis.
11. Center at $(-2, -3)$ and touching the y -axis.
12. Center at $(2, 2)$ and touching both axes.
13. Center at (a, a) and touching both axes.
14. Center at $(-3, 5)$ and passing through the point $(3, -3)$.
15. Diameter, the segment from $(6, 2)$ to $(8, 4)$.
16. Diameter, the segment from $(12, -8)$ to $(-4, 2)$.
17. Diameter, the segment from $(0, 0)$ to $(4a, 4a)$.
18. Touching the y -axis at the origin and radius = 3 (two solutions).
19. Touching the x -axis at the origin and radius = 2 (two solutions).
20. Touching both axes, radius = 2 (four solutions).

16-2. Reduction of the General Equation of the Circle to Standard Form. The standard form of the equation of the circle, given in (2) of the previous section, when expanded is an equation of the form

$$(1) \quad x^2 + y^2 + Dx + Ey + F = 0,$$

where D , E , and F are constants.

The question now arises: Is every equation of the form (1) the equation of a circle? To answer this question we complete the squares in x and y , so that (1) becomes:

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F$$

or

$$(2) \quad \left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{1}{4}(D^2 + E^2 - 4F).$$

This equation has the standard form of a circle with center at $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$, provided the right member of (2) is positive. If the right member of (2) is negative, no real numbers x , y can satisfy the equation, since the sum of the squares of two real numbers cannot equal a negative number. Therefore in this case there is no circle. If the right member of (2) is zero the center is the only point whose coordinates satisfy the equation, and hence the circle is a single point.

Consider now the equation

$$(3) \quad Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

When this equation is divided by A , it reduces to the form of (1). But division of both members of an equation by a constant does not affect the geometrical figure it represents. Therefore (3) is also a general equation of a circle.

Example. Find the center and the radius, and draw the circle

$$x^2 + y^2 - 2x - 4y - 20 = 0.$$

The equation may be written

$$(x^2 - 2x) + (y^2 - 4y) = 20.$$

Completing the square in each parenthesis we have

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) = 20 + 1 + 4$$

or

$$(x - 1)^2 + (y - 2)^2 = 25.$$

From this equation it follows that the center of the circle is at the point (1, 2) and the radius is 5 (Fig. 16-2).

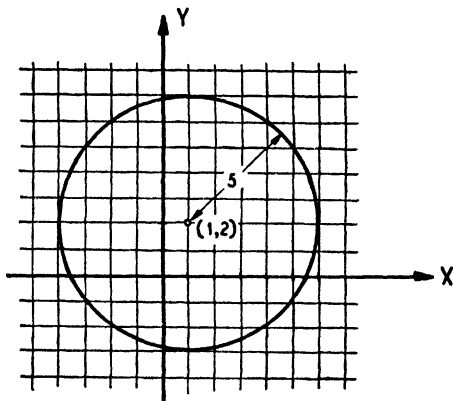


FIG. 16-2.

EXERCISES

By reducing each equation to standard form, determine whether it is the equation of a circle. If it is, find the center and radius of the circle, and draw the circle.

1. $x^2 + y^2 - 6x + 10y - 2 = 0$.
2. $x^2 + y^2 - 6x + 8y + 30 = 0$.
3. $x^2 + y^2 - 6x + 8y = 0$.
4. $x^2 + y^2 + 2x + 8y + 8 = 0$.
5. $2x^2 + 2y^2 + 8x - 4y + 10 = 0$.
6. $3x^2 + 3y^2 + 2x - 4y = 0$.
7. $x^2 + y^2 - 2x - 6y + 10 = 0$.
8. $x^2 + y^2 + 10x - 24y = 0$.
9. $x^2 + y^2 + 10x - 24y + 175 = 0$.
10. $x^2 + y^2 = 8x$.
11. $x^2 + y^2 - 4x - 12 = 0$.
12. $2x^2 + 2y^2 - 3x - 5y + 3 = 0$.

Find the equation of the circle which satisfies the given conditions.

13. Center at (1, 2) and passing through (4, 6).
14. Center at (-5, 3) and radius $\sqrt{2}$.
15. Center at (3, -2) and radius $\sqrt{13}$.
16. Center at (5, -2) and passing through (4, 3).
17. Center at (-1, 3) and passing through the origin.
18. Center at (0, 2) and tangent to the x -axis.
19. Center on the line $x = 3$ and tangent to both axes.
20. Center on the y -axis and passing through (4, 6) and (6, 10).
21. Center on the x -axis and passing through (6, 4) and (8, -4).
22. Whose diameter is the segment $(-2, 3), (5, -2)$.
23. Passing through the points (1, -2) and (3, 2) and with center on the y -axis.
24. The extremities of whose diameter are (4, -2) and (-2, 6).
25. A diameter of the circle is the line segment joining the points (-2, 4) and (6, -2).

16-3. The Circle through Three Points. It was shown in the preceding section that every circle has an equation of the form

$$(1) \quad x^2 + y^2 + Dx + Ey + F = 0,$$

where D , E , and F are constants. Thus, if a circle is given by certain conditions, its equation can be found by determining the proper values of D , E , and F in (1).

It is known that only one circle can be drawn through any three given points which do not lie on a straight line. Hence three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) not on a straight line determine a circle. The values of D , E , and F which give the equation of this circle can be found by substituting the coordinates of these points in (1), thereby obtaining three linear equations from which the values of D , E , and F can be found. This process will be illustrated in the following example.

Example. Find the equation of the circle through the three points $(1, -3)$, $(1, 7)$, and $(4, -2)$.

The equation of the circle has the form (1). Since the coordinates of each of the three points must satisfy the equation of the circle, the three equations obtained by substituting the coordinates of these points in (1) must hold simultaneously.

$$1 + 9 + D - 3E + F = 0,$$

$$1 + 49 + D + 7E + F = 0,$$

$$16 + 4 + 4D - 2E + F = 0.$$

Simplifying these equations and then solving for D , E , and F we find

$$D = -2, \quad E = -4, \quad F = -20.$$

Hence the required equation is

$$x^2 + y^2 - 2x - 4y - 20 = 0.$$

The student should find the center and the radius of this circle.

EXERCISES

Find the equations of the circles through the following sets of three points. Draw the corresponding figures.

1. $(1, 0)$, $(0, -1)$, $(0, 0)$.

2. $(0, 2)$, $(-1, 3)$, $(-3, 2)$.

3. $(0, 0)$, $(0, 10)$, $(3, 1)$.

4. $(1, 1)$, $(3, 2)$, $(2, -1)$.

5. $(1, 6)$, $(2, 5)$, $(-6, -1)$.

6. $(0, 0)$, $(8, 4)$, $(1, 3)$.

7. $(2, 2)$, $(2, 4)$, $(10, 2)$.

8. $(5, 3)$, $(3, 1)$, $(-3, -1)$.

9. Find the equation of the circle which passes through the points $(2, 6)$ and $(8, 4)$ and which has its center on the line $2x + y + 4 = 0$.

10. Find the equation of the circle through the points $(3, 5)$ and $(5, -1)$ which has its center on the x -axis.

11. Find the equation of the circle circumscribing the triangle formed by the coordinate axes and the line $2x + 3y = 6$.
12. Find the equation of the circle passing through the origin and the point $(8, 4)$ and the center on the x -axis.
13. Do the points $(2, 0)$, $(0, 4)$, $(2, 2)$, $(1, 1)$ lie on a circle?
14. Find the equation of the circle through the points $(-1, 3)$ and $(3, 1)$ and which has its center on the line $3x + y - 5 = 0$.
15. Find the equation of the circle circumscribing the triangle whose sides are $x = 0$, $y = 0$, and $2x - y + 4 = 0$.

16-4. Equation of a Locus. The totality of points which satisfy a given geometric condition is said to form a locus of a point. The student has studied such problems in plane geometry. But plane geometry gives us no general method by means of which we could determine a locus. Each problem had to be solved by a method peculiar to the problem itself.

In contrast to plane geometry, analytic geometry provides a general method for solving locus problems. We merely translate the given geometrical conditions, describing the locus, into algebra. Two examples of this method have already been given. In Sec. 15-8 we found the equation of a line through two given points by defining it as the locus of a point moving so that the line segment joining it to one of the given points has a constant slope. In Sec. 16-1 the circle was considered as the locus of a point whose distance from the center is constant. In both cases we translated the given conditions into algebra and simplified the resulting expressions algebraically.

In the examples which follow we shall show how we can apply the method of analytic geometry to locus problems.

Example 1. Find the equation of the locus of a point such that the sum of the squares of its distances from $(0, 0)$ and $(4, 6)$ is 76.

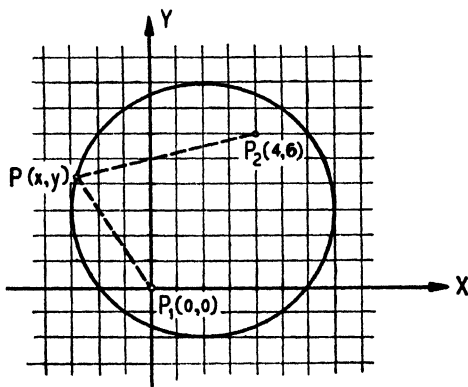


FIG. 16-3.

Let P_1 and P_2 be respectively the given points $(0, 0)$ and $(4, 6)$ and let $P(x, y)$ be any point on the required locus (Fig. 16-3). Our requirement is that

$$(PP_1)^2 + (PP_2)^2 = 76$$

which becomes, when stated algebraically,

$$x^2 + y^2 + (x - 4)^2 + (y - 6)^2 = 76.$$

Simplifying,

$$2x^2 + 2y^2 - 8x - 12y = 24,$$

$$x^2 + y^2 - 4x - 6y = 12,$$

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = 12 + 9 + 4,$$

$$(x - 2)^2 + (y - 3)^2 = 25.$$

This is the equation of a circle with center at $(2, 3)$ and radius 5.

In our work concerning loci we have to consider two types of problems.

Type 1. Given a locus defined geometrically, find the corresponding equation.

Type 2. Given an equation, find the corresponding geometrical locus.

Problems of Type 2 we have already studied in Chapter 3. In the present section we are considering only problems of Type 1. To solve such problems we proceed by the following steps:

1. A figure is drawn, showing the data for a representative point on the locus.
2. If a coordinate system is not given, one must be introduced.
3. The given geometrical conditions, describing the locus, are translated into algebra and the resulting equation is simplified.

Example 2. Find the equation of the locus of points equidistant from two given points.

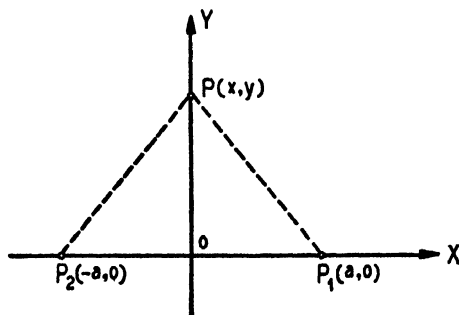


FIG. 16-4.

We construct a figure (Fig. 16-4) and introduce a coordinate system so that the given points have the coordinates $P_1(a, 0)$ and $P_2(-a, 0)$. This choice of coordinates will simplify our work. Now geometrically the requirement is that

$$P_2P = P_1P.$$

When written algebraically, this requirement becomes

$$\sqrt{(x+a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + (y-0)^2}.$$

Simplifying,

$$x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2 + y^2,$$

$$4ax = 0,$$

and since $a \neq 0$, we obtain finally

$$x = 0.$$

This is the equation of the y -axis, which is the perpendicular bisector of P_1P_2 .

EXERCISES

Find an equation of the locus of a point which moves as described below. When-
ever possible, identify the locus.

1. Five units to the right of the y -axis.
2. Six units below the x -axis.
3. Equidistant from $(2, 4)$ and $(6, 8)$.
4. Its distance from $(-2, 3)$ is 7.
5. Its distance from $(0, 0)$ is twice its distance from the line $y = 5$.
6. Its distance from the point $(0, 4)$ is three times its distance from the line $x = -3$.
7. Its distances from $(0, 0)$ and $(4, 8)$ are equal.
8. Its distance from $(0, 0)$ is half its distance from the line $y + 8 = 0$.
9. Its distance from the point $(0, 0)$ is one-half of its distance from the line $x - 2 = 0$.
10. The sum of the squares of its distances from $(0, 0)$ and $(6, 8)$ is 168.
11. The difference of the squares of its distances from $(0, 0)$ and $(3, 5)$ is 12.
12. The sum of the squares of its distances from two fixed points is a constant k .
13. The difference of the squares of its distances from two fixed points is a constant k .
14. The sum of the squares of its distances from the vertices of a square is a constant k .

16-5. Polar Coordinates. In Chapter 3 we introduced the **rectangular coordinate system**, a device for locating points in the plane by setting up a correspondence between pairs of real numbers (x, y) and the points in a plane. In the subsequent chapters this device was used frequently in making geometrical or graphical interpretations of algebraic relationships.

The **polar coordinate system**, a second method for locating points in the plane, is often useful. Let O be a fixed point in the plane and let OA be a fixed line extending from O infinitely far in one direction only (Fig. 16-5), and let the unit of length be chosen along OA as shown. Now, any point P in the plane can be located with respect to O and OA by giving θ , any angle with initial side OA and terminal side OP , and r , the length OP measured on the terminal side of θ . These two values are customarily

written (r, θ) and are called a set of **polar coordinates** of the point P . The point O is called the **pole** of the system; the line OA is called the **polar axis** of the system.

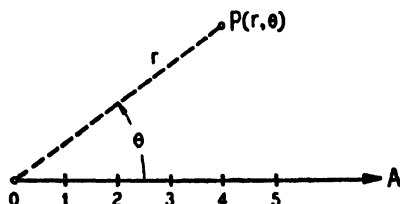


FIG. 16-5.

We shall adopt the following conventions in measuring r and θ :

(a) If θ is measured counterclockwise it is positive; if θ is measured clockwise it is negative.

(b) If r is measured from O to P along the terminal side of θ , r is positive; if r is measured along the extension through O of the terminal side of θ then r is negative.

Example. Fig. 16-6 shows the points $P_1(5, 30^\circ)$, $P_2(-5, 60^\circ)$, $P_3(3, -210^\circ)$, and $P_4(-4, -90^\circ)$.

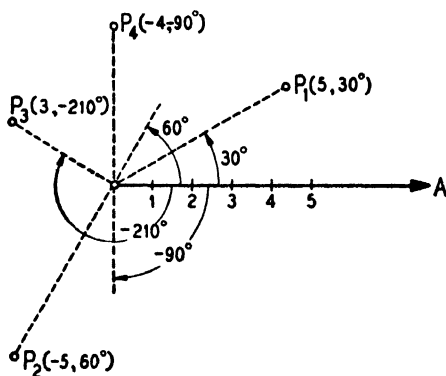


FIG. 16-6.

For a given pair of coordinates (r, θ) only one point in the plane can be located. However, for a given point in the plane many pairs of coordinates can be found. For example, the point P_1 in Fig. 16-6 located by the coordinates $(5, 30^\circ)$ can also be located by the coordinates $(-5, 210^\circ)$, $(5, -330^\circ)$, $(5, 390^\circ)$, $(-5, -150^\circ)$, $(5, 750^\circ)$, etc. For a given point, it is easily seen that there are infinitely many pairs of coordinates. Thus unlike rectangular coordinates, a system of polar coordinates does not establish a one-to-one correspondence between points in the plane and pairs of values (r, θ) .

The student will recall that a complex number could be given in polar form by giving its modulus and argument. In this way the position of the point on the Argand diagram corresponding to the number is given by specifying its distance from the origin and the angle the line joining it to the origin makes with the positive x -axis. Hence the polar form of the complex numbers and the polar coordinate system are based on the same idea.

To facilitate the plotting of points and curves in polar coordinates, especially designed polar coordinate paper should be used.

EXERCISES

Using either a scale and a protractor or polar coordinate paper, plot the following points.

1. $(2, 30^\circ)$, $(3, 120^\circ)$, $(1, 215^\circ)$, $(3.2, 311^\circ)$, $(1.5, -45^\circ)$, $(2.5, -90^\circ)$, $(1.8, 180^\circ)$, $(1.8, -180^\circ)$, $\left(2, \frac{\pi}{6}\right)$, $(1.7, 2\pi)$, $\left(3, \frac{2\pi}{3}\right)$, $\left(5, \frac{13\pi}{8}\right)$.

2. $(3, 170^\circ)$, $(8, 430^\circ)$, $(1.3, -220^\circ)$, $(-3, 65^\circ)$, $(-8, 250^\circ)$, $(-8, 250^\circ)$, $(-6.3, -470^\circ)$, $(6.3, -470^\circ)$, $\left(2, \frac{15\pi}{4}\right)$, $\left(-2, \frac{15\pi}{4}\right)$, $\left(2, \frac{-15\pi}{4}\right)$, $\left(-2, \frac{-15\pi}{4}\right)$.

3. $(3, 70^\circ)$, $(3, 430^\circ)$, $(3, -290^\circ)$, $(3, 650^\circ)$, $(-3, 250^\circ)$, $(-3, -110^\circ)$, $(-3, 610^\circ)$, $(-3, -470^\circ)$.

4. $\left(1.3, \frac{\pi}{6}\right)$, $\left(1.3, \frac{13\pi}{6}\right)$, $\left(-1.3, \frac{7\pi}{6}\right)$, $\left(-1.3, \frac{-5\pi}{6}\right)$, $\left(1.3, \frac{-11\pi}{6}\right)$.

5. $(0, 70^\circ)$, $(0, 0^\circ)$, $(0, 10^\circ)$, $(0, 412^\circ)$, $(0, -80^\circ)$.

6. $(2, 70^\circ)$, $(2, 0^\circ)$, $(2, 120^\circ)$, $(2, 180^\circ)$, $(2, 90^\circ)$, $(2, 270^\circ)$, $(2, -60^\circ)$, $(-2, 45^\circ)$, $(-2, 210^\circ)$, $(2, 0^\circ)$.

Find five other pairs of coordinates for each of the following points:

7. $(2, 90^\circ)$, $(3, 210^\circ)$.

8. $(-3, 180^\circ)$, $(3, 135^\circ)$.

9. $(1, 120^\circ)$, $(1, -120^\circ)$.

10. $(5, 30^\circ)$, $(5, 270^\circ)$.

11. What curve will be obtained from all points (r, θ) where r satisfies the equation $r = a$, a being a constant? Where r satisfies the equation $r = -a$?

12. What locus or curve will be obtained from all points (r, θ) where θ satisfies the equation

$$(a) \theta = \frac{\pi}{3}, \quad (b) \theta = \frac{4\pi}{3}, \quad (c) \theta = -\frac{2\pi}{3}, \quad (d) \theta = 60^\circ$$

16-6. Graphs in Polar Coordinates. In Chapter 3 it was seen that an algebraic relationship between two variables x and y can be represented pictorially by plotting the pairs of values satisfying the relationship on rectangular coordinates. Likewise, we shall now see that an

algebraic relationship between two variables r and θ can be represented by plotting the pairs of values satisfying the relationship on polar coordinates. How this can be done will be illustrated by examples.

Example 1. Plot the curve $r = 5 \sin \theta$ on polar coordinates. To construct a table of values from which to plot this function, we may substitute chosen values for θ and find r , or vice versa. In this case the former presents least difficulty. The choice of values to be substituted for θ is of course at our discretion, but usually the principal angles are the most convenient. If the principal angles do not give

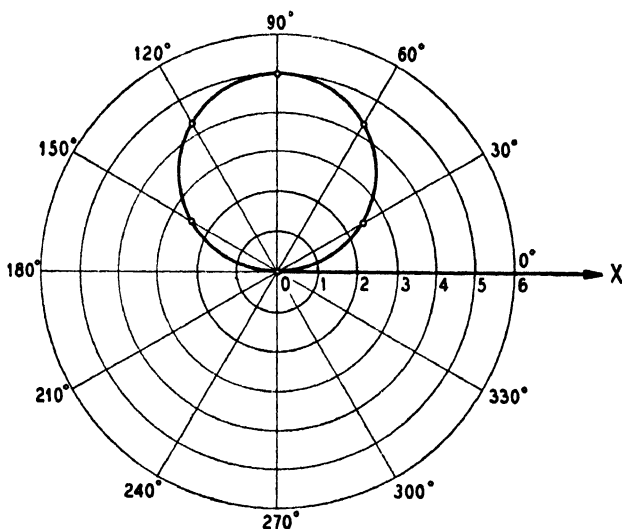


FIG. 16-7.

enough points to determine the general character of the curve, more values of θ should be plotted, using tables of natural functions.

For this example we have the table:

θ	$r = 5 \sin \theta$	θ	$r = 5 \sin \theta$
0°	0.00	150°	2.50
30°	2.50	180°	0.00
60°	4.35	210°	-2.50
90°	5.00	etc.	etc.
120°	4.35		

By analyzing the function $r = 5 \sin \theta$, the general behavior of the curve between the plotted points can be determined. For example, as θ increases between 0° and 90° , the function $5 \sin \theta$ increases from 0 to 5. As θ passes from 90° to 180° , $5 \sin \theta$ decreases from 5 to 0. Now, as θ takes on values from 180° to 360° , $r = 5 \sin \theta$ takes on negative values, giving points already found when θ was less than 180° as the student may verify by extending the table. For example, when $\theta = 30^\circ$ we find the point $(2.5, 30^\circ)$ and for $\theta = 210^\circ$ we obtain the same point located by $(-2.5, 210^\circ)$. Thus, $r = 5 \sin \theta$ gives the circle shown in Fig. 16-7.

In some cases the negative values of r do not give points already found. This is illustrated in what follows.

Example 2. Plot the function $r = 2 \cos 3\theta$.

Substituting values for θ we obtain the following table.

θ	3θ	$r = 2 \cos 3\theta$	θ	3θ	$r = 2 \cos 3\theta$
0°	0°	2.00	140°	420°	1.00
20°	60°	1.00	150°	450°	0.00
30°	90°	0.00	160°	480°	-1.00
40°	120°	-1.00	180°	540°	-2.00
60°	180°	-2.00	200°	600°	-1.00
80°	240°	-1.00	210°	630°	0.00
90°	270°	0.00	220°	660°	1.00
100°	300°	1.00	240°	720°	2.00
120°	360°	2.00	etc.	etc.	etc.

Note that although the angle 3θ is used to evaluate r , the angle θ is the angle used in plotting. Hence the curve represented by $r = 2 \cos 3\theta$ is the three-leaved rose of Fig. 16-8. As in Example 1, the curve retraces itself as θ passes 180° .

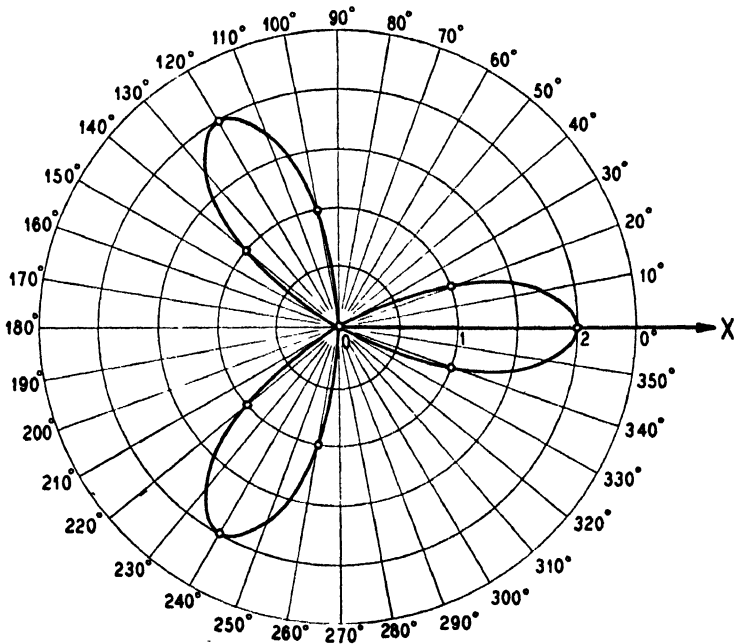


FIG. 16-8.

Curves of this general type plotted on polar coordinate paper are obtained in plotting the field strength of radio energy directed by two or more antennas suitably placed and excited. Advanced treatises on

radio waves and their propagation show many forms of field patterns, of which Fig. 16-8 may be considered a special example. In another engineering application, the horizontal distribution of the intensity of light from an incandescent lamp or fixture is plotted on polar coordinate paper; in this case the pattern is not generally regular or simple mathematically.

EXERCISES

Using polar coordinate paper plot the curves representing the following functions

1. $r = 4 \sin \theta$.
2. $r = -4 \sin \theta$.
3. $r = 6 \sin 2\theta$.
4. $r = -3 \cos 2\theta$.
5. $r = 2 \cos \frac{\theta}{2}$.
6. $r = 5 \sin \frac{\theta}{3}$.
7. $r = 6 \sin 4\theta$.
8. $r = 2 + 2 \cos \theta$.
9. $r = 1 - 2 \sin \theta$.
10. $r = 2(1 - \sin \theta)$.
11. $r \sin \theta = 2$.
12. $r \cos \theta = 2$.
13. $r \cos (\theta - 30^\circ) = 2$.
14. $r \cos (\theta + 60^\circ) = -1$.
15. $r = \frac{5}{1 + 2 \sin \theta}$.
16. $r = \frac{6}{2 - \cos \theta}$.
17. $r = 3\theta$.
18. $r = 2 \sin (\theta - 30^\circ)$.
19. $r = 6 \cos (\theta - 45^\circ)$.
20. $r^2 = 5 + 8r \cos (\theta - 60^\circ)$.
21. Using Fig. 16-9, show that any equation of the type

$$r = a \cos \theta$$

is a circle of diameter a .

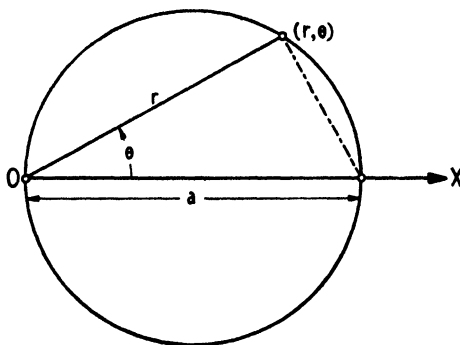


FIG. 16-9.

22. Show that any equation of the type

$$r = a \sin \theta$$

is a circle of diameter a .

16-7. Relations between Polar Coordinates and Rectangular Coordinates. Occasionally it is advantageous to be able to change from rectangular coordinates to polar coordinates, or vice versa. To obtain

a relation between the two coordinate systems, let the pole and polar axis in a polar coordinate system coincide with the origin and positive x -axis of a rectangular coordinate system (Fig. 16-10). Also choose the same unit of length for both systems. Now consider any point P in the plane. If we denote the polar coordinates of P by (r, θ) and the rectangular coordinates of P by (x, y) , then obviously

$$(1) \quad x = r \cos \theta, \quad y = r \sin \theta,$$

$$(2) \quad r^2 = x^2 + y^2, \quad \theta = \arctan \frac{y}{x}.$$

Since $\arctan \frac{y}{x}$ is a multiple-valued

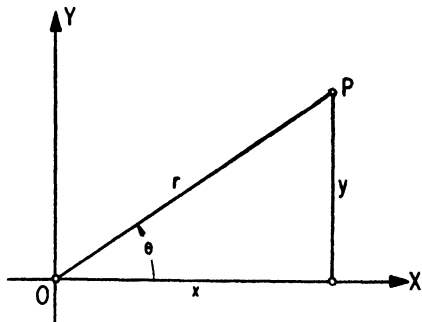


FIG. 16-10.

function, a correct value of θ can be determined by noting the quadrant in which P lies, or the signs of y and x .

Using the formulas (1) it is possible to transform a relationship given in rectangular coordinates to polar coordinates; using the formulas (2) it is possible to transform a relationship given in polar coordinates into rectangular coordinates. The curve which represents the given relationship is the same in both coordinate systems.

Example 1. Transform the equation

$$(3) \quad r \sin \theta = 4$$

to rectangular coordinates.

If the equation 3 is plotted in polar coordinates, it can be shown to be a straight line. However, this fact is made obvious if (3) is transformed to rectangular coordinates, for from (1), $r \sin \theta = y$, whence (3) becomes

$$y = 4.$$

Example 2. Transform the equation

$$(4) \quad x^2 - 4x + y^2 = 0$$

to polar coordinates.

Substituting $x^2 + y^2 = r^2$ from (2) and $x = r \cos \theta$ from (1), the equation becomes

$$r^2 - 4r \cos \theta = 0.$$

Factoring, we obtain

$$r = 0 \quad \text{or} \quad r - 4 \cos \theta = 0.$$

But since $r = 4 \cos \theta$ gives the value $r = 0$ when $\theta = 90^\circ$, we can express (4) in polar form simply as

$$r = 4 \cos \theta.$$

EXERCISES

Transform the following equations into rectangular coordinates.

1. $r = 5$.
2. $r = a$.
3. $\theta = \frac{\pi}{4}$.
4. $\theta = 0$.
5. $\tan \theta = \frac{7}{11}$.
6. $\sin \theta = \frac{1}{2}$.
7. $r \sin \theta = 2$.
8. $r \cos \theta = 3$.
9. $r = 2 \csc \theta$.
10. $r = 7 \csc \theta$.
11. $r + 4 \sec \theta = 0$.
12. $r = 3 \sin \theta$.
13. $r = 2 \sin \theta$.
14. $r = \sin \theta + \cos \theta$.
15. $r = 2(1 - \cos \theta)$.
16. $r = \frac{12}{2 + \cos \theta}$.
17. $r \cos \left(\theta - \frac{\pi}{6} \right) = 2$.
18. $r \sin \left(\frac{\pi}{3} - \theta \right) = 2$.
19. $4 = r^2 + 4 - 4r \cos \left(\theta + \frac{\pi}{3} \right)$.
20. $9 = r^2 + 4 - 4r \cos \left(\theta + \frac{\pi}{3} \right)$.

Transform the following equations into polar coordinates.

21. $y = 0$.
22. $y = x$.
23. $y = \sqrt{3}x$.
24. $x = -\sqrt{3}y$.
25. $x = -1$.
26. $x = 0$.
27. $y = 2$.
28. $x + y = 0$.
29. $x - y = 2$.
30. $2x + y = 4$.
31. $x^2 + y^2 = 25$.
32. $x^2 - 10x + y^2 = 0$.
33. $y = x^2$.
34. $y^2 = 4ax$.
35. $x^2 + y^2 = 36$.
36. $(x - 1)^2 + (y - 2)^2 = 5$.
37. $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$.
38. $x^2 + 2x + y^2 = 2\sqrt{x^2 + y^2}$.
39. $x^2 - x + y^2 + y = 0$.
40. $(x^2 + y^2)^2 = x(x^2 - 3y^2)$.

16-8. Parametric Equations. In general, the graph representing an algebraic relation between two variables is a curve. We have seen that this is true in two coordinate systems, the rectangular and polar systems. In our discussions up to this point the values of one variable corresponding to those of another were specified by a given relation between the two variables. In this way we can think of the pairs of numbers which we plotted as being specified by the given relation. For example, the equation

$$(1) \quad y = x + 1$$

is such a relation specifying pairs of values (x, y) , and its graph is a straight line with y -intercept 1 and slope 1.

However, it is possible to specify pairs of numbers (x, y) by giving each as a function of an auxiliary variable. For example, the relations

$$(2) \quad \begin{aligned} x &= 2t, \\ y &= 2t + 1 \end{aligned}$$

specify a pair of numbers (x, y) for each real value of the auxiliary variable t . The curve representing the pairs (x, y) can be plotted as shown in the example.

Example. Plot the curve representing the equations 2.

Substituting various values of t in (2) we obtain the following table.

t	x	y
0	0	1
1	2	3
2	4	5
-1	-2	-1
-2	-4	-3

The graph, shown in Fig. 16-11, turns out to be a straight line with y -intercept 1 and slope 1.

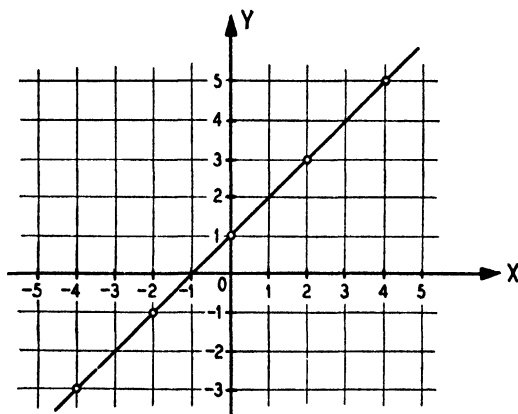


FIG. 16-11.

When the pairs (x, y) are given in terms of a third variable t , as

$$(3) \quad \begin{aligned} x &= f(t), \\ y &= g(t), \end{aligned}$$

the auxiliary variable t is called a **parameter**, and the equations 3 are called **parametric equations**.

In many cases it is possible to eliminate the parameter t , obtaining thus a relationship between x and y . For example, it is easily seen that

eliminating the parameter from (2) we obtain (1), which verifies that Fig. 16-11 is the correct graph of (2). However, if a relationship between two variables is given, many sets of parametric equations representing the same relationship can be found. For instance, the parametric equations

$$x = t^3,$$

$$y = t^3 + 1$$

also represent the relation (1).

In many cases the parameter chosen has no especial significance. However, in applied problems in physics and engineering a parameter is often chosen which has a geometrical or physical meaning. Suppose, for example, that a body is subjected to various forces and as a result moves about in a plane, and that the position of the body is known at every instant. If we then place a rectangular coordinate system in the plane, we know at every instant the coordinates (x, y) of the position of the body. Thus if we choose t as a parameter and let t measure time, then for every value of t we know the corresponding values of x and y . In short, x is a function of the time t , and y is a function of the time t , and we have a relation of the form (3) which describes the path of the moving body.

EXERCISES

Plot the curve corresponding to each set of parametric equations. Find a relation between x and y by eliminating the parameter.

1. $x = \frac{5-t}{2}, y = 2-t.$

2. $x = 2t - \frac{1}{3}, y = 6(2t - 1).$

3. $x = t^2 - 1, y = (t - 1)(t + 1).$

4. $x = 2 \cos 2\theta, y = 2 \sin^2 \theta.$

5. $x = \sin^2 \phi, y = 5 + 2 \cos^2 \phi.$

6. $x = \frac{t}{2} - 5, y = \frac{4}{t - 10}.$

7. $x = 5 \cos \theta, y = 5 \sin \theta.$

8. $x = 2 \cos t, y = 3 \sin t.$

9. $x = t - \frac{1}{2}, y = 4t^2 - 4t + 1.$

10. $x = 2t, y = t^2 - 1.$

11. Show that $x = t, y = 3 + 4t$ are parametric equations of a straight line. Plot the line.

12. Show that $x = 3t + 2, y = 2t - 5$ are parametric equations of a straight line. Plot the line.

13. Show that $x = \frac{2at}{1+t^2}, y = \frac{a(1-t^2)}{1+t^2}$ are parametric equations of a circle. Plot the circle.

14. Show that $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$ are parametric equations of the curve $x^3 + y^3 - 3xy = 0$. Plot the curve.

15. The barrel of a gun is elevated to an angle α with the horizontal and a shell is fired with a muzzle velocity of v_0 feet per second. Assuming that the shell travels in a vertical plane, we shall set up a rectangular coordinate system in this plane, choosing the muzzle of the gun as the origin, the y -axis vertical, and the x -axis horizontal and so that the shell moves in the positive direction. Neglecting the air resistance, wind effects, etc., it can be shown that the position of the shell t seconds after firing is given by

$$x = v_0 t \cos \alpha,$$

$$y = v_0 t \sin \alpha - \frac{gt^2}{2},$$

where $g = 32.2$ ft. per sec. per sec. Plot the path of a projectile fired at 45° with the horizontal with a muzzle velocity of 600 ft. per sec.

PROGRESS REPORT

In this chapter we discussed the circle and obtained a standard form for the equation of the circle. Then locus problems were discussed algebraically. Polar coordinates and parametric equations were also treated.

CHAPTER 17

EQUATIONS OF THE SECOND DEGREE: THE CONICS

For the study of the calculus, a familiarity with the graphs of certain second degree equations is important. In order to establish this familiarity we shall discuss in this chapter the graphs of such equations and some of their analytic and geometrical properties. We shall in this way obtain a partial answer to the question: What is the locus of the general second degree equation in two variables?

The equations of conic sections occur frequently in physical problems. Planetary motion is elliptical, and recent theories of electron motion suggest that electrons travel about the nucleus of an atom in ellipses. The properties of a parabola are utilized in ultra-high frequency techniques; if the source of radio waves is placed at the focus of a paraboloid, the radiations into space will be along parallel, straight lines. This is analogous to the light radiation from a parabolic reflector of an automobile headlight. It is known that a cable supporting a uniform, horizontal load (as cables of a suspension bridge do) assumes the shape of a parabola. But if the cable bears no load, the configuration is that of the catenary, and a solution of this problem hinges on certain functions associated with the hyperbola. These problems are beyond the scope of this book, but the comprehension of such practical applications depends on the fundamental knowledge of the properties of conic sections.

17-1. The Conic Sections. The curves of intersection of a plane and a right circular cone are called **conic sections** or **conics**. If the plane cuts across one nappe of the cone, the curve of intersection is an **ellipse**, as in Fig. 17-1. If the plane is parallel to a straight line in the surface of the cone, the curve of intersection is a **parabola**, as in Fig. 17-2. If the plane cuts both nappes of the cone, as in Fig. 17-3, the curve of intersection is an **hyperbola**. These definitions explain the origin of the term **conic**, which is applied to these curves. A study of these curves can be made from these definitions, but we shall find it more convenient to start from other definitions which will be given in succeeding sections.

We have already made a study of the equation of the first degree, and we found that its locus was a straight line. The conic sections are of

interest because, as we shall see, their equations are equations of second degree. Further, although we shall not prove it in this book, it can be

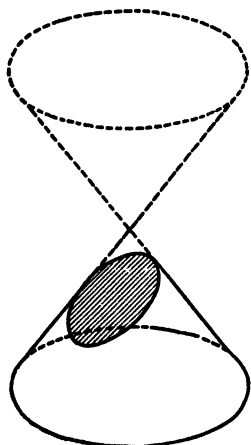


FIG. 17-1.

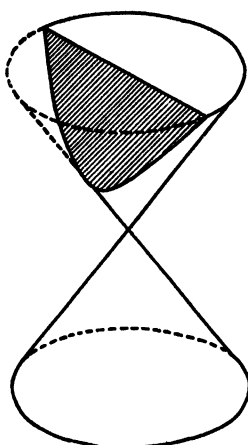


FIG. 17-2.

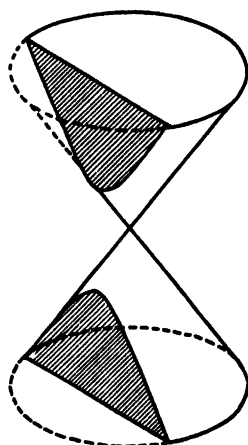


FIG. 17-3.

shown that the locus of the general second-degree equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is a conic.

17-2. The Parabola. The geometrical definition of the parabola which we shall use is the following.

The parabola is the locus of a point which moves in a plane so that its distance from a fixed point is equal to its distance from a fixed line. The fixed point is called the **focus**, the fixed line the **directrix**. The line perpendicular to the directrix through the focus is the **axis** of the parabola. The intersection of the parabola and the axis is the **vertex**.

The equation of a parabola is particularly simple if we choose as focus the point $(p, 0)$, where p is positive, and as directrix the line $x = -p$.

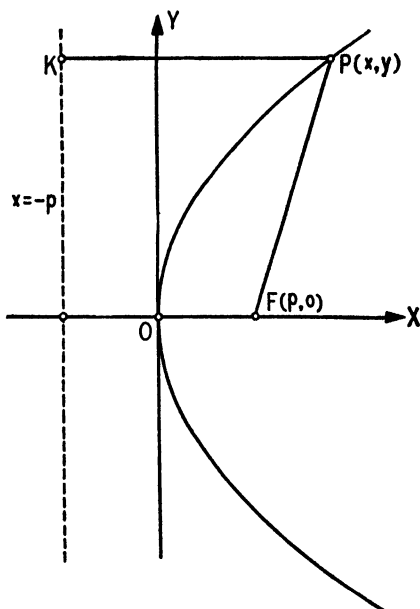


FIG. 17-4.

Then from Fig. 17-4, the definition requires that

$$(1) \quad KP = FP.$$

Obviously,

$$(2) \quad KP = x + p,$$

and

$$(3) \quad FP = \sqrt{(x - p)^2 + (y - 0)^2}.$$

Using (2) and (3) in (1) we obtain

$$x + p = \sqrt{(x - p)^2 + y^2}.$$

Squaring both sides of the equation we obtain

$$(x + p)^2 = (x - p)^2 + y^2$$

which reduces to

$$(4) \quad y^2 = 4px,$$

which is the standard form of the equation of the parabola.

The chord of a parabola drawn through the focus perpendicular to the axis is called the **latus rectum**. When $x = p$, then $y^2 = 4p^2$, and $y = \pm 2p$; hence the length of the latus rectum is $4p$ (Fig. 17-5a).

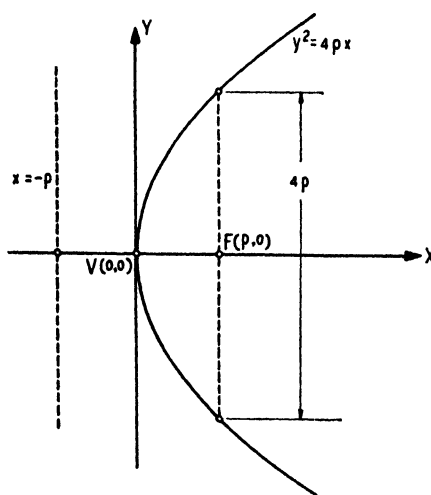


FIG. 17-5a.

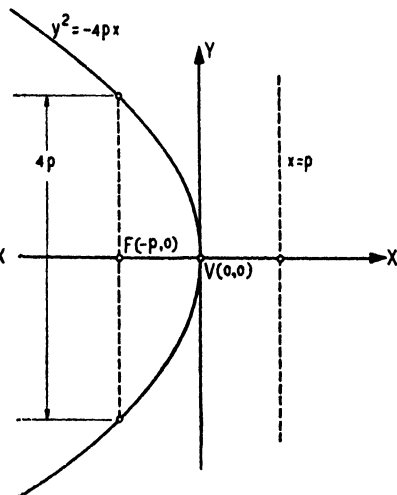
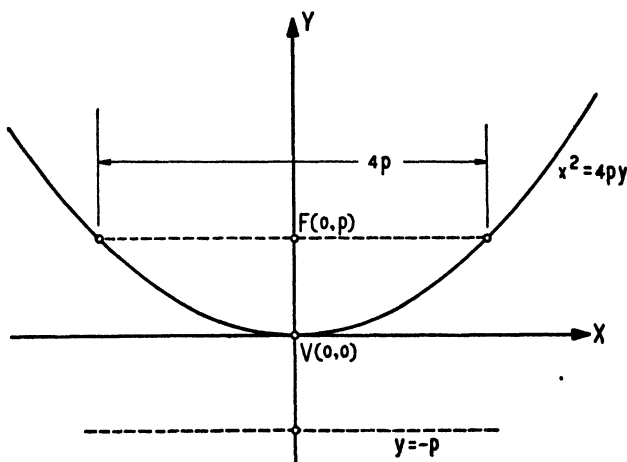
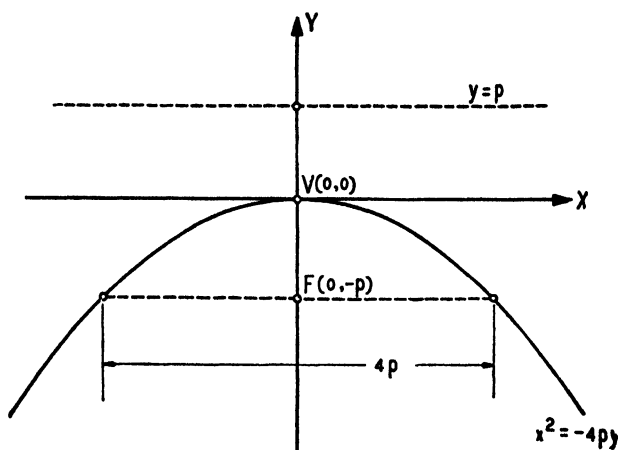


FIG. 17-5b.

It is left as an exercise to the student to show that with the vertex at $(0, 0)$, focus at $(-p, 0)$, and directrix $x = p$, the equation of the parabola is

$$(5) \quad y^2 = -4px.$$

This information is summarized in Fig. 17-5*b*. The student may also verify the information summarized in Fig. 17-6.

FIG. 17-6*a*.FIG. 17-6*b*.

A parabola can be sketched readily by locating the vertex V , the focus F , and drawing the latus rectum, since the general shape of the figure is indicated clearly in Fig. 17-5 and Fig. 17-6. The procedure is to locate these parts and then to sketch in the curve free-hand.

EXERCISES

For each of the following parabolas, find the coordinates of vertex and focus, the equation of the directrix, and the length of latus rectum. Sketch the curve.

- | | |
|--------------------|--------------------|
| 1. $y^2 = 4x$. | 2. $y^2 = -4x$. |
| 3. $x^2 = 4y$. | 4. $x^2 = -4y$. |
| 5. $x^2 = 16y$. | 6. $y^2 = 16x$. |
| 7. $y^2 = -16x$. | 8. $x^2 = -16y$. |
| 9. $y^2 = 64x$. | 10. $3x^2 = 7y$. |
| 11. $5x^2 = 18y$. | 12. $y^2 = -6x$. |
| 13. $x^2 = -10y$. | 14. $2y^2 = 7x$. |
| 15. $y^2 = -8x$. | 16. $3y^2 = 10x$. |
| 17. $y^2 = x$. | 18. $x^2 = -y$. |
| 19. $x^2 = -3y$. | 20. $y^2 = -7x$. |

Find the equations of the following parabolas, and sketch:

21. Vertex at (0, 0), focus at (4, 0).
22. Vertex at (0, 0), focus at (-8, 0).
23. Vertex at (0, 0), focus at (0, -8).
24. Vertex at (0, 0), focus at (0, 6).
25. Vertex at (0, 0), directrix, $3y + 8 = 0$.
26. Vertex at (0, 0), directrix, $2y - 7 = 0$.
27. Vertex at (0, 0), focus at (-6, 0).
28. Vertex at (0, 0), focus at (5, 0).
29. Focus at (4, 0), directrix, $x + 4 = 0$.
30. Vertex at (0, 0), directrix, $y - 4 = 0$.
31. Focus at (0, 10), directrix, $y + 10 = 0$.
32. Vertex at (0, 0), focus at (6, 0).
33. Find the equation of the parabola which has its axis coinciding with the x -axis, its vertex at the origin, and which passes through the point (4, 2).
34. Find the equation of the parabola which has its axis coinciding with the x -axis, its vertex at the origin, and which passes through the point (3, 5).
35. Find the equation of the parabola which has its axis coinciding with the y -axis, its vertex at the origin, and which passes through the point (4, 4).
36. Find the equation of the parabola which has its axis coinciding with the y -axis, its vertex at the origin, and which passes through (-4, -4).
37. Find the equation of the parabola which has its axis coinciding with the y -axis, its vertex at the origin, and which passes through (8, -4).
38. Using the definition of a parabola, derive the equation of the parabola whose vertex is at (4, 0) and whose directrix is the y -axis.
39. Using the definition of a parabola, derive the equation of the parabola whose vertex is at (0, -6) and whose directrix is the x -axis.

17-3. The Ellipse. The geometrical definition of the ellipse which we shall use is the following.

An ellipse is the locus of a point which moves in the plane so that the sum of its distances from two fixed points is a constant greater than the distance between the two points. The student may devise very easily, on the basis of this definition, a mechanical method of constructing an ellipse.

The two fixed points are the **foci** of the ellipse; the midpoint of the segment joining them is the **center** of the ellipse. We shall choose the coordinate system so that the center is at the origin and the foci are $F_1(c, 0)$ and $F_2(-c, 0)$ where c is positive, as in Fig. 17-7. For convenience we shall call the constant length sum $2a$, which must then be

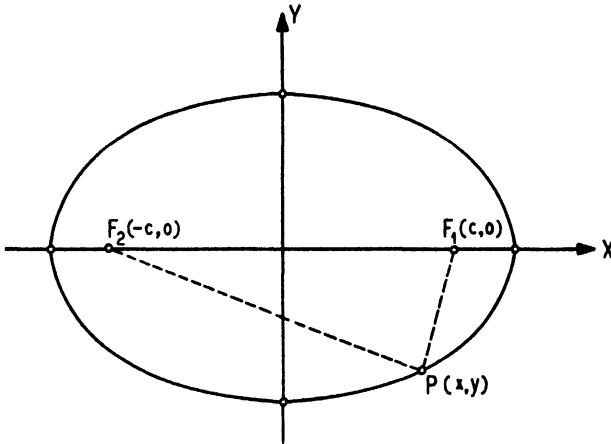


FIG. 17-7.

greater than $2c$ by the definition of the ellipse. Our requirement is that

$$(1) \quad F_1P + F_2P = 2a.$$

Using the distance formula, (1) can be written as

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

or

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2},$$

which becomes, upon squaring both sides of the equation,

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

Simplifying, we obtain

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx.$$

Squaring again, we have

$$a^2(x+c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

or

$$(2) \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Since $a > c$, then $a^2 - c^2$ is positive. Setting

$$(3) \quad b^2 = a^2 - c^2$$

in equation 2 we have

$$(4) \quad b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing both sides of (4) by a^2b^2 we obtain *the standard form of the equation of the ellipse*.

$$(5) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a^2 is greater than b^2 .

When $x = 0$, $y = \pm b$, and when $y = 0$, $x = \pm a$. The locus is plotted as shown in Fig. 17-8.

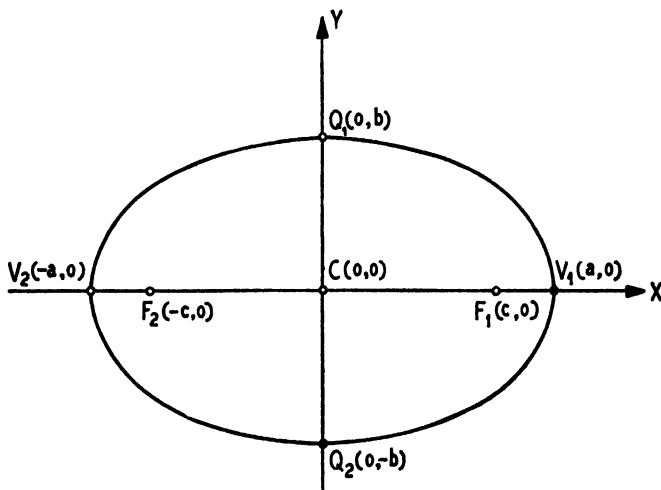


FIG. 17-8.

The segment V_1V_2 is the **major axis**, the segment Q_1Q_2 is the **minor axis**. V_1 and V_2 are the **vertices** of the ellipse. Thus the **semi-major axis** has length a , and the **semi-minor axis** has length b .

The student may verify as an exercise that if we start with foci $F_1(0, c)$ and $F_2(0, -c)$, and proceed as above, we obtain

$$(6) \quad \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1,$$

where a^2 is greater than b^2 . The locus is plotted in Fig. 17-9. In this case the major axis and the vertices are along the y -axis.

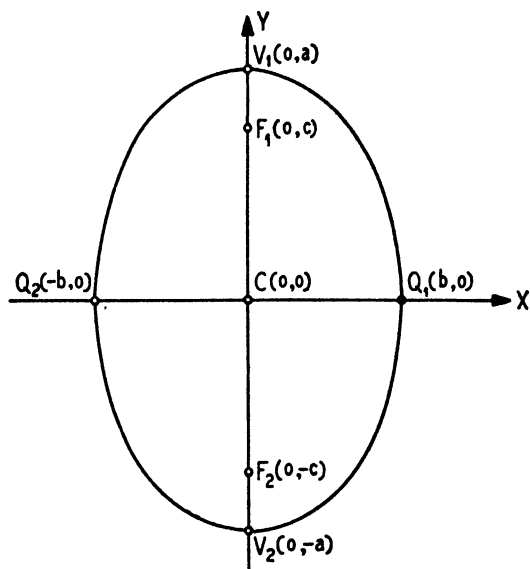


FIG. 17-9.

Example 1. For the ellipse

$$16x^2 + 25y^2 = 400,$$

find (a) the coordinates of the vertices, (b) the coordinates of the foci, (c) the length of the semi-axes. Sketch the ellipse, showing all this data on the figure.

Dividing both sides of the equation by 400, we obtain the standard form

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

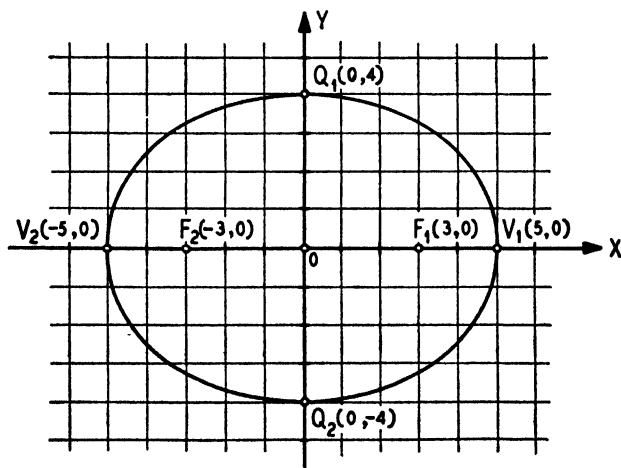


FIG. 17-10.

From this equation and the relation $b^2 = a^2 - c^2$ we obtain the following information:

Semi-major axis: $a = 5$.

Semi-minor axis: $b = 4$.

Coordinates of the vertices: $(5, 0)$, $(-5, 0)$.

Coordinates of the foci: $(3, 0)$, $(-3, 0)$.

The graph is shown in Fig. 17-10.

Example 2. Find the equation of the ellipse whose foci are $(0, 4)$ and $(0, -4)$, and whose vertices are $(0, 5)$ and $(0, -5)$.

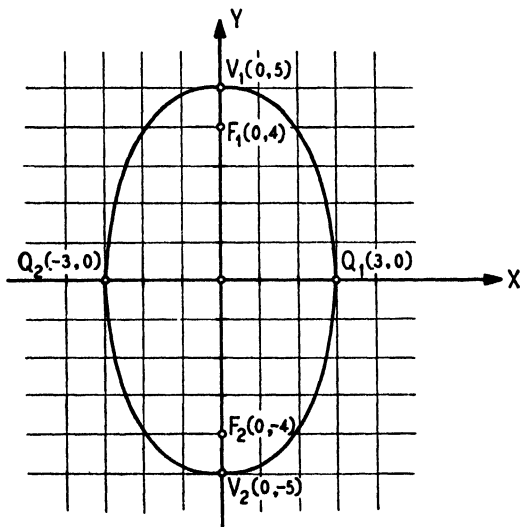


FIG. 17-11.

The midpoint of the line joining the foci is the origin, and hence the origin is the center of the ellipse. The major axis is along the y -axis. Since $a = 5$ and $c = 4$, from $b^2 = a^2 - c^2$ we have $b^2 = 25 - 16 = 9$. Thus the standard form of the equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

The ellipse is sketched in Fig. 17-11.

EXERCISES

For the following ellipses find (a) the coordinates of the vertices, (b) the coordinates of the foci, (c) the lengths of the semi-axes, and sketch, showing all this data on the figure.

1. $4x^2 + 9y^2 = 36$.

3. $25x^2 + 4y^2 = 100$

5. $x^2 + 4y^2 = 16$.

7. $4x^2 + y^2 = 36$.

2. $9x^2 + 4y^2 = 36$.

4. $4x^2 + 25y^2 = 100$.

6. $4x^2 + y^2 = 16$.

8. $25x^2 + 9y^2 = 225$.

- | | |
|-----------------------------|-------------------------------|
| 9. $9x^2 + 25y^2 = 225$. | 10. $x^2 + 4y^2 = 36$. |
| 11. $9x^2 + 16y^2 = 144$. | 12. $16x^2 + 9y^2 = 144$. |
| 13. $4x^2 + 3y^2 = 11$. | 14. $3x^2 + 4y^2 = 9$. |
| 15. $12x^2 + 5y^2 = 23$. | 16. $100x^2 + 225y^2 = 324$. |
| 17. $25x^2 + 36y^2 = 900$. | 18. $36x^2 + 25y^2 = 400$. |
| 19. $9x^2 + 25y^2 = 900$. | 20. $64x^2 + 25y^2 = 1600$. |
| 21. $4x^2 + 25y^2 = 625$. | 22. $49x^2 + 36y^2 = 900$. |
| 23. $x^2 + 49y^2 = 196$. | 24. $16x^2 + 9y^2 = 576$. |

Find the equation of each of the ellipses satisfying the following conditions. Sketch the curves.

25. Foci (5, 0) and (-5, 0); vertices (13, 0) and (-13, 0).
26. Foci (15, 0) and (-15, 0); vertices (17, 0) and (-17, 0).
27. Foci (0, 12) and (0, -12); vertices (0, 15) and (0, -15).
28. Foci (0, 3) and (0, -3); length of major axis 10.
29. Foci (0, 15) and (0, -15); length of minor axis 16.
30. Foci (8, 0) and (-8, 0); length of major axis 34.
31. Foci (12, 0) and (-12, 0); length of major axis 26.
32. Vertices (13, 0) and (-13, 0); length of minor axis 10.
33. Vertices (0, 17) and (0, -17); length of minor axis 16.
34. Vertices (15, 0) and (-15, 0); length of minor axis 24.
35. Find the equation of the ellipse whose vertices are the points (4, 0) and (-4, 0) and which passes through the point (0, 2). Sketch the curve.
36. Find the equation of the ellipse whose vertices are the points (0, 6) and (0, -6) and which passes through the point (4, 0). Sketch the curve.
37. Find the equation of the ellipse whose vertices are the points (0, -4), (0, 4) and which passes through the point (2, 3). Sketch the curve.
38. Find the equation of the ellipse whose foci are the points (-3, 0), (3, 0) and whose semi-major axis has length 5. Sketch the curve.
39. Find the equation of the ellipse whose vertices are (10, 0) and (-10, 0) and which passes through (5, $\sqrt{6}$). Sketch the curve.
40. Find the equation of the locus of a point which moves so that the sum of its distances from (0, 4) and (0, -4) is 10. Sketch the curve.
41. Find the equation of the locus of a point which moves so that the sum of its distances from (8, 0) and (-8, 0) is 20. Sketch the curve.
42. Find the equation of the locus of a point which moves so that the sum of its distances from (-2, 1) and (6, 1) is 10. Sketch the curve.
43. Find the equation of the locus of a point which moves so that the sum of its distances from the points (2, 0) and (-4, 0) is equal to 10. Sketch the curve.

17-4. The Hyperbola. The geometrical definition of the hyperbola which we shall use is the following.

The hyperbola is the locus of a point moving in a plane such that the difference of its distances in either order from two fixed points is a positive constant less than the distance between the two fixed points.

The two fixed points are the **foci** of the hyperbola, the midpoint of the segment joining them is the **center** of the hyperbola. We shall choose the coordinate system so that the center is at the origin and the foci are $F_1(c, 0)$ and $F_2(-c, 0)$ where c is positive, as shown in Fig. 17-12.

For convenience we shall call the positive constant length difference $2a$, which by the definition of the hyperbola is less than $2c$. Then we require, from Fig. 17-12, that either of these two requirements be fulfilled.

$$(1) \quad F_2P - F_1P = 2a$$

$$(2) \quad F_1P - F_2P = 2a.$$

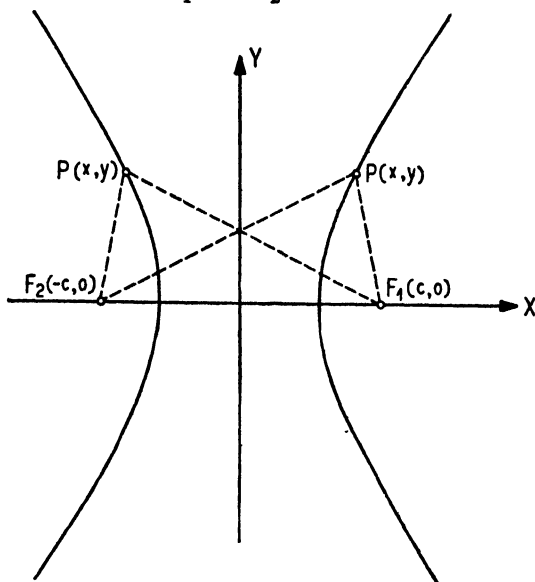


FIG. 17-12.

If (1) is satisfied, since $2a$ is positive, we get the right branch of the hyperbola as shown in Fig. 17-12; if (2) is satisfied, we get the left branch of the hyperbola. Using the distance formula to write (1) and (2) algebraically we obtain respectively:

$$(3) \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$(4) \quad \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$$

The student may simplify both (3) and (4) by the method used to simplify the corresponding equation in the case of the ellipse. Both simplifications result in the equation:

$$(5) \quad (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

Since $c > a$ by definition, $c^2 - a^2 > 0$ and we may write

$$(6) \quad b^2 = c^2 - a^2.$$

Making this substitution in (5) we obtain

$$(7) \quad b^2 x^2 - a^2 y^2 = a^2 b^2.$$

Dividing both members of (7) by $a^2 b^2$ we obtain the result

$$(8) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the *standard form* of the equation of this hyperbola. In this case, unlike that of the ellipse, b^2 can have any size relative to a^2 . The student may verify that (8) is equivalent to (3) and (4) jointly.

When $y = 0$, then $x = \pm a$, and the points $V_1(a, 0)$ and $V_2(-a, 0)$ are the **vertices** of the hyperbola. The segment $V_1 V_2$ is the **transverse axis**, whence the **semi-transverse axis** has length a . When $x = 0$, y has no real values, whence the locus does not cross the y -axis. The segment $B_1 B_2$, where B_1 and B_2 have the coordinates $B_1(0, b)$, $B_2(0, -b)$, is called the **conjugate axis**. The **semi-conjugate axis** has length b .

If a fixed straight line is so related to an infinite branch of a curve that, as a point on the curve recedes indefinitely along the infinite branch, the distance of the point from the line comes as near to zero as we please, but never equals zero, then the line is called an **asymptote** of the curve.

The hyperbola (8) has as asymptotes the lines

$$(9) \quad y = \frac{b}{a} x, \quad y = -\frac{b}{a} x.$$

Proof of this fact is beyond the scope of this book.

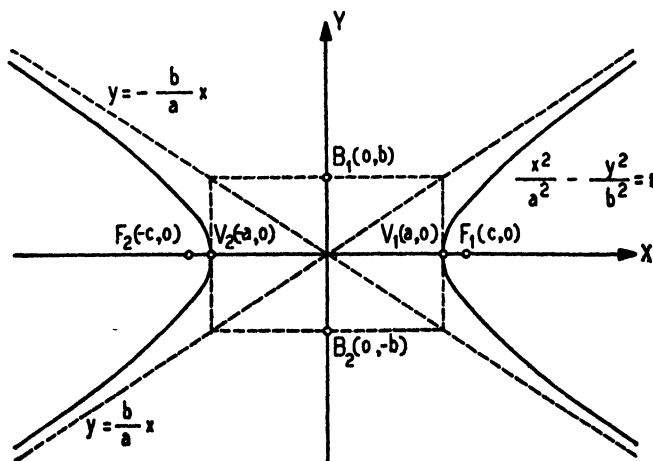


FIG. 17-13.

To sketch the hyperbola, draw the rectangle with sides perpendicular to the x -axis through the points V_1 and V_2 , and with sides perpendicular to the y -axis through the points B_1 and B_2 , as shown in Fig. 17-13. The straight lines through the center of the hyperbola and the opposite corners of the rectangle are the asymptotes (9), as can be readily seen. After these preliminary constructions, the hyperbola can be sketched easily, as shown in Fig. 17-13.

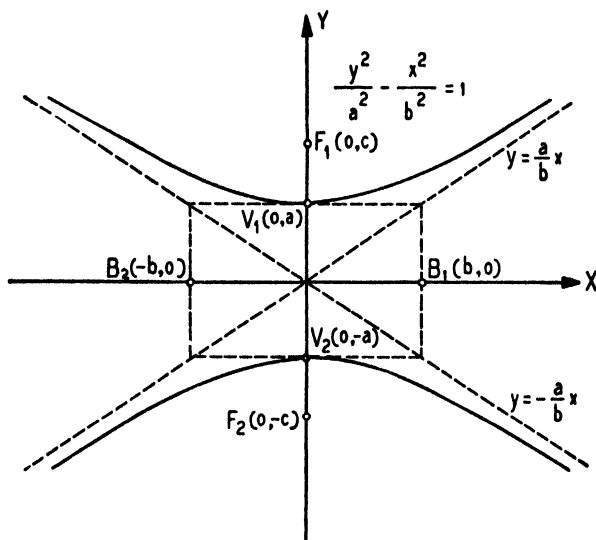


FIG. 17-14.

The student may verify as an exercise that if we start with the foci $F_1(0, c)$ and $F_2(0, -c)$ and proceed as above, we obtain the *standard form*

$$(10) \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where b^2 is given by (6). In this case the asymptotes are

$$(11) \quad y = \frac{a}{b}x, \quad y = -\frac{a}{b}x,$$

and the curve is sketched as shown in Fig. 17-14.

Example. Sketch the curve of

$$16y^2 - 9x^2 = 144,$$

showing on the graph (a) the coordinates of the foci, (b) the coordinates of the vertices, and (c) the asymptotes.

Dividing the equation by 144 we obtain the standard form

$$\frac{y^2}{9} - \frac{x^2}{16} = 1.$$

Thus the semi-transverse axis $a = 3$, and the semi-conjugate axis $b = 4$. The vertices are $V_1(0, 3)$ and $V_2(0, -3)$; the ends of the semi-conjugate axes are $B_1(4, 0)$

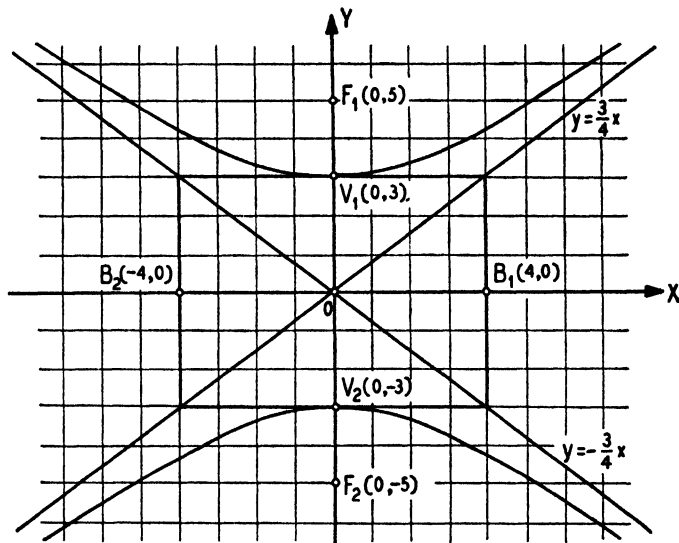


FIG. 17-15.

and $B_2(-4, 0)$. After locating these points, we sketch the rectangular box as shown by Fig. 17-15. Then the asymptotes are drawn through the corners of the box. They are seen to be the lines

$$y = \frac{3}{4}x, \quad y = -\frac{3}{4}x.$$

From (6), $c^2 = a^2 + b^2 = 9 + 16 = 25$, whence $c = 5$, and the foci are at $F_1(0, 5)$ and $F_2(0, -5)$. Finally, the curve can be sketched as shown in Fig. 17-15.

EXERCISES

For the following hyperbolas with center at the origin, find (a) coordinates of the vertices, (b) coordinates of the foci, (c) lengths of the semi-axes, (d) equations of the asymptotes and sketch, showing all this information on the figure.

- $9x^2 - 16y^2 = 144.$
- $4x^2 - 9y^2 = 36.$
- $x^2 - y^2 = 16.$
- $y^2 - 4x^2 = 36.$
- $y^2 - x^2 = 64.$
- $16x^2 - 9y^2 = 144.$
- $y^2 - x^2 = 4.$
- $x^2 - y^2 = 4.$
- $25x^2 - 144y^2 = 3600.$
- $4x^2 - 9y^2 = 25.$
- $9y^2 - 25x^2 = 225.$
- $x^2 - y^2 = 36.$
- $y^2 - x^2 = 36.$
- $4x^2 - 9y^2 = -36.$

15. $9x^2 - 4y^2 = 36$.

17. $16y^2 - x^2 = 16$.

19. $4y^2 - 9x^2 = 25$.

21. $x^2 - 25y^2 = -100$.

23. $25x^2 - y^2 = 100$.

16. $x^2 - 25y^2 = 100$.

18. $y^2 - x^2 = 16$.

20. $y^2 - 64x^2 = -81$.

22. $25x^2 - y^2 = -100$.

24. $x^2 - 49y^2 = -100$.

Find the equations of the hyperbolas given by the following data. Sketch a graph of each.

25. Foci at (5, 0) and (-5, 0); vertices at (3, 0) and (-3, 0).

26. Foci at (0, 5) and (0, -5); vertices at (0, 4) and (0, -4).

27. Foci at (0, 13) and (0, -13); vertices at (0, 5) and (0, -5).

28. Foci at (17, 0) and (-17, 0); length of transverse axis 30.

29. Foci at (0, 17) and (0, -17); length of transverse axis 16.

30. Vertices at (0, 3) and (0, -3); length of conjugate axis 8.

31. Vertices at (15, 0) and (-15, 0); length of conjugate axis 16.

32. Foci at (0, 10) and (0, -10); length of conjugate axis 16.

33. Find the equation of the hyperbola with center at the origin and transverse axis on the x -axis which passes through the points (-6, 2) and (5, 1). Sketch the hyperbola.

34. Find the equation of the hyperbola whose vertices are (4, 0) and (-4, 0) and which passes through (5, 4). Sketch the hyperbola.

35. An hyperbola with center at the origin has its transverse axis along the x -axis of length 24 and conjugate axis of length 10. Find the equation of the hyperbola and sketch its graph.

36. Find the equation of the hyperbola whose foci are (0, 6) and (0, -6) and whose transverse axis is twice its conjugate axis.

17-5. Change of Coordinate Axes. In analytic geometry the solution of a problem can often be simplified by the use of a different set of axes instead of the one employed in the statement of the problem. In fact, we shall devote the next few sections to reducing by a change of axes a number of apparently difficult problems to ones which we have just solved.

If the new axes are parallel, respectively, to the old ones and the positive directions are the same, the transformation is called a **translation of axes**. If the origin remains unchanged and the new axes are obtained by revolving the old ones about the origin through some given angle, then the transformation is called a **rotation of axes**. It is readily seen that, in general, any change of axes can be accomplished by a translation and then a rotation or vice versa. In this book we shall consider only the translation of axes.

17-6. Translation of Coordinate Axes. Let OX and OY of Fig. 17-16 be the original axes and let $O'X'$ and $O'Y'$ be the new axes, parallel, respectively, to the old and having the same positive directions. Let the coordinates of O' referred to the original axes be (h, k) .

Let P be any point in the plane, and let its coordinates referred to the

old axes be (x, y) , and let its coordinates referred to the new axes be (x', y') . From Fig. 17-16 we see that

$$x = NP = NN' + N'P = h + x',$$

$$y = MP = MM' + M'P = k + y',$$

whence

$$(1) \quad \begin{aligned} x &= x' + h, \\ y &= y' + k. \end{aligned}$$

The above discussion depends on the picture as arranged in Fig. 17-16. The student should verify that the same formulas hold for all translations and all positions of P .

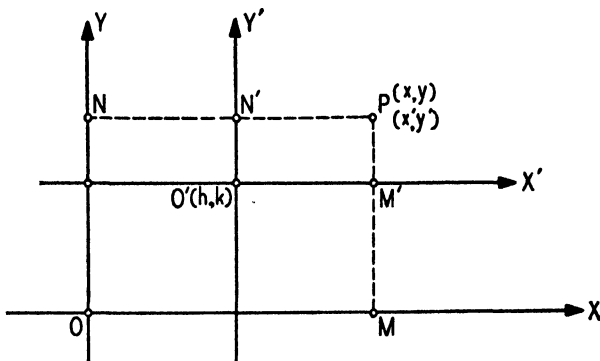


FIG. 17-16.

Example. The equation of a curve is

$$4x^2 + y^2 - 24x + 4y + 24 = 0.$$

Find the equation of the same curve with respect to a new coordinate system obtained by translating the origin of the axes to the point $(3, -2)$, and sketch the curve.

We have from (1),

$$(2) \quad \begin{aligned} x &= x' + 3, \\ y &= y' - 2. \end{aligned}$$

Making this substitution, the given equation becomes

$$4(x' + 3)^2 + (y' - 2)^2 - 24(x' + 3) + 4(y' - 2) + 24 = 0.$$

Simplifying, we obtain

$$4x'^2 + y'^2 = 16,$$

which we can write in standard form as

$$\frac{x'^2}{4} + \frac{y'^2}{16} = 1.$$

This is the equation of an ellipse. By drawing its graph with respect to the new axes we obtain the graph of the original equation with respect to the original axes. With respect to the new axes we have immediately the following information:

Vertices: $V_1: x' = 0, y' = 4; V_2: x' = 0, y' = -4$.

Ends of minor axis: $Q_1: x' = 2, y' = 0; Q_2: x' = -2, y' = 0$.

Foci: $F_1: x' = 0, y' = 2\sqrt{3}; F_2: x' = 0, y' = -2\sqrt{3}$.

From (2) we have immediately the information:

Vertices: $V_1: x = 3, y = 2; V_2: x = 3, y = -6$.

Ends of minor axis: $Q_1: x = 5, y = -2; Q_2: x = 1, y = -2$.

Foci: $F_1: x = 3, y = 2\sqrt{3} - 2; F_2: x = 3, y = -2\sqrt{3} - 2$.

The graph is drawn in Fig. 17-17.

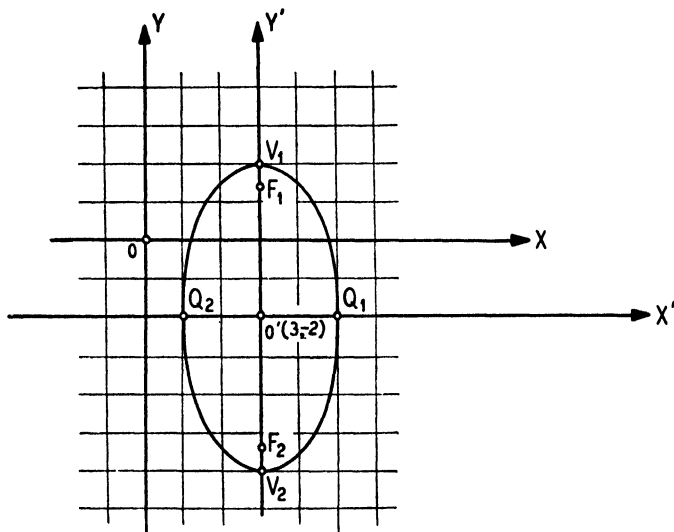


FIG. 17-17.

EXERCISES

Find the coordinates of the following points after translating the axes to the new origin indicated. Draw a figure for each exercise to verify the results.

1. New origin $O' (3, 4)$. Points: $(1, 2), (-3, 2), (3, 5)$.
2. New origin $O' (1, 2)$. Points: $(3, -1), (0, 0), (-1, 2)$.
3. New origin $O' (-3, 1)$. Points: $(4, 2), (0, 0), (5, -7)$.
4. New origin $O' (-1, -2)$. Points: $(3, -4), (-1, -1), (2, 3)$.
5. New origin $O' (3, -4)$. Points: $(5, 4), (-1, 3), (-3, -3)$.
6. New origin $O' (6, 1)$. Points: $(6, 2), (-5, 1), (2, 1)$.

Transform each of the following equations so that the axes are translated to the new origin O' . Plot a graph showing both pairs of axes and the curve.

7. $x - 3y - 5 = 0; O'(2, -1)$.
8. $3x + 4y = 12; O'(1, -2)$.
9. $x - 2y = 5; O'(7, 1)$.
10. $3x - y = 6; O'(2, 0)$.

11. $4x + 6y = -15$; $O'(1, 2)$.
12. $5x - y = 10$; $O'(-2, -1)$.
13. $x^2 + 4y^2 + 4x - 8y - 8 = 0$; $O'(-2, 1)$.
14. $x^2 + y^2 - 4x - 6y - 3 = 0$; $O'(2, 3)$.
15. $x^2 + y^2 - 8x = 0$; $O'(4, 0)$.
16. $x^2 + y^2 + 4x - 8y - 5 = 0$; $O'(-2, 4)$.
17. $3x^2 + 3y^2 + 5x + 12y = 0$; $O'(-\frac{5}{6}, -2)$.
18. $y^2 - 4x - 12y + 44 = 0$; $O'(2, 6)$.
19. $x^2 - 4x - 8y - 20 = 0$; $O'(2, -3)$.
20. $4x^2 + y^2 - 8x + 4y - 8 = 0$; $O'(1, -2)$.
21. $4x^2 + 9y^2 + 8x - 18y - 3 = 0$; $O'(-1, 1)$.
22. $9x^2 - 4y^2 - 24y - 72 = 0$; $O'(0, -3)$.
23. $x^2 - y^2 - 6x - 8y - 23 = 0$; $O'(3, -4)$.
24. $y^2 - 4x^2 + 8x + 4y - 16 = 0$; $O'(1, -2)$.
25. $(x - h)^2 + (y - k)^2 = r^2$; $O'(h, k)$.

17-7. Application of Translation of Axes to the Equations of the Conics. In the preceding sections the equations of the conics were found to be the following (the vertices of the parabolas and the centers of the other conics at the origin):

- (1) Parabola: $y^2 = 4px$, $y^2 = -4px$, $x^2 = 4py$, $x^2 = -4py$.
- (2) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.
- (3) Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

We shall consider the question: *What equations can be reduced by a translation of axes to the forms (1), (2), or (3)?*

For example, we want to know what equation can be reduced to

$$(4) \quad y'^2 = 4px'$$

by the substitution

$$(5) \quad x = x' + h, \quad y = y' + k.$$

The answer can be found by making in (4) the inverse substitution

$$(6) \quad x' = x - h, \quad y' = y - k.$$

This substitution gives the result

$$(7) \quad (y - k)^2 = 4p(x - h).$$

Thus, any equation of the form (7) can be reduced to the form (4) by the substitution (5). It follows that the locus of (7) is a parabola with vertex at the point (h, k) and axis parallel to the x -axis.

In like manner we can establish the following results.

The graph of each of the equations

$$(8) \quad (y - k)^2 = \pm 4p(x - h), \quad (x - h)^2 = \pm 4p(y - k)$$

is a parabola with vertex (h, k) and axis parallel to a coordinate axis.

The graph of each of the equations

$$(9) \quad \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad \frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

is an ellipse with center (h, k) and axes parallel to the coordinate axes.

The graph of each of the equations

$$(10) \quad \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \quad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

is an hyperbola with center (h, k) and axes parallel to the coordinate axes.

The converse of each statement also holds.

Equations 8, 9, and 10 are termed **standard forms**.

Thus, knowing the vertex of a parabola, or the center of an ellipse or hyperbola, we can make use of all the facts developed thus far in the chapter, provided the axes of the conics are parallel to the coordinate axes. An example will illustrate how this can be done.

Example. Find an equation of the ellipse whose foci are the points $(2, -2)$, $(2, 6)$ and the length of whose minor axis is 6.

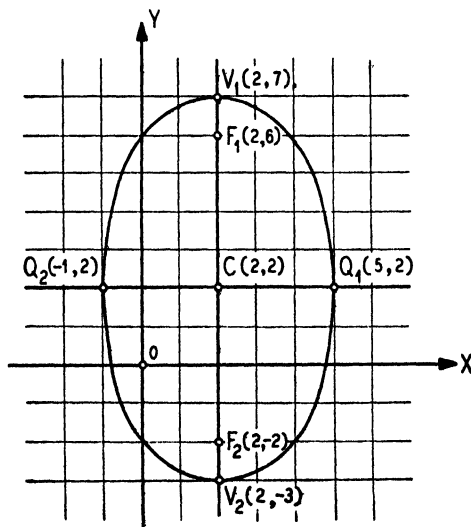


FIG. 17-18.

The center is midway between the foci, at $(2, 2)$; and hence in (9), $h = 2$, $k = 2$. The distance between foci is 8, whence $c = 4$. Also, $2b = 6$, or $b = 3$. Since $a^2 = b^2 + c^2$, $a = 5$. Hence the desired equation is

$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{25} = 1$$

or

$$25x^2 + 9y^2 - 100x - 36y = 89.$$

We plot the locus as shown in Fig. 17-18. We see that the major axis is parallel to the y -axis, the vertices are $(2, 7)$ and $(2, -3)$, the ends of the minor axis are $(-1, 2)$ and $(5, 2)$.

EXERCISES

Sketch the following parabolas. Find the coordinates of vertex, focus, ends of latus rectum, and length of latus rectum. Show this information on your figure.

1. $(y-2)^2 = 8(x-3)$.
2. $(x-5)^2 = -20(y+2)$.
3. $(y+3)^2 = 16(x+4)$.
4. $(x+4)^2 = -8(y-1)$.
5. $(x+1)^2 = 12(y+3)$.
6. $(x-10)^2 = 100(y+6)$.
7. $(x-3)^2 = 4y$.
8. $(x+5)^2 = -16(y+2)$.
9. $(y-3)^2 = 2(x+2)$.
10. $(y-3)^2 = -8(x+3)$.
11. $(y+6)^2 = -x$.
12. $(y+3)^2 = 16(x-3)$.

Sketch the following ellipses, showing on the figure the coordinates of the center, foci, vertices, and ends of minor axis.

13. $(x-2)^2 + 4(y+2)^2 = 16$.
14. $25(x-2)^2 + 36(y+1)^2 = 900$.
15. $4(x-2)^2 + (y+2)^2 = 16$.
16. $16(x+1)^2 + 9(y-1)^2 = 576$.
17. $9(x+1)^2 + 25(y-5)^2 = 225$.
18. $4(x+2)^2 + y^2 = 144$.
19. $9(x-1)^2 + 16(y+2)^2 = 144$.
20. $9(x-3)^2 + 16(y+4)^2 = 144$.
21. $25(x-4)^2 + 9(y+2)^2 = 225$.
22. $81(x+2)^2 + 100(y-3)^2 = 8100$.
23. $(x+5)^2 + 4(y-2)^2 = 36$.
24. $36(x-1)^2 + 25(y-2)^2 = 400$.

Sketch the following hyperbolas, showing on the figure the coordinates of the center, vertices, foci, ends of conjugate axis, and equations of asymptotes.

25. $9(x-1)^2 - 16(y+2)^2 = 144$.
26. $9(x-2)^2 - 4(y-4)^2 = 36$.
27. $4(x-3)^2 - 9(y-1)^2 = 36$.
28. $(x+2)^2 - (y-2)^2 = 64$.
29. $25(x+2)^2 - 144(y-3)^2 = 3600$.
30. $25(x-3)^2 - y^2 = 100$.
31. $16(y+1)^2 - x^2 = 16$.
32. $9(x+2)^2 - 4(y-1)^2 = -36$.
33. $y^2 - 4(x+4)^2 = 36$.
34. $x^2 - (y+4)^2 = -64$.
35. $(y+7)^2 - (x-3)^2 = 16$.
36. $4(y+1)^2 - 9x^2 = 25$.

Find an equation of the conic section determined by the following conditions, and sketch, supplying on the figure the standard data indicated in the directions for the preceding exercises.

37. Parabola, vertex $(2, 2)$, focus $(2, -6)$.
38. Parabola, vertex $(-2, 4)$, focus $(-6, 4)$.
39. Ellipse, major axis 10, foci $(3, -2)$ and $(-3, -2)$.
40. Ellipse, minor axis 6, focus $(2, 4)$, vertex $(2, 6)$.
41. Ellipse, major axis 10, foci $(3, 2)$ and $(1, 2)$.

42. Ellipse, minor axis 6, focus $(-2, -2)$, vertex $(0, -2)$.
43. Ellipse, minor axis 6, focus $(-3, -1)$, vertex $(-3, 1)$.
44. Hyperbola, center $(2, 1)$, focus $(7, 1)$, vertex $(5, 1)$.
45. Hyperbola, center $(-1, 4)$, vertex $(-1, 8)$, conjugate axis 6.
46. Hyperbola, center $(2, 3)$, vertex $(2, 7)$, conjugate axis 6.
47. Hyperbola, foci $(-3, -2)$ and $(7, -2)$, conjugate axis 6.
48. Hyperbola, foci $(0, -3)$ and $(10, -3)$, conjugate axis 8.

17-8. Reduction of Equations of Second Degree to Standard Forms.

If we carry out the multiplications indicated in the standard forms (8), (9), and (10) of Sec. 17-7, we see that all these equations have the general form

$$(1) \quad Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

From (8), (9), and (10) of Sec. 17-7 we see that

- (a) If (1) is a parabola, either $A = 0$ or $C = 0$;
- (b) If (1) is an ellipse, A and C have the same sign;
- (c) If (1) is an hyperbola, A and C have opposite signs.

The student may easily verify that in each of these three cases, (1) can be reduced to the corresponding standard form. A general proof may be formulated, following the lines of the examples below, but will not be presented here.

In this discussion we have ignored the fact that there may be no locus for (1), or that it may yield a special case of a conic. A situation of this kind has already been met in Sec. 16-2.

It is possible to show that the most general equation of second degree,

$$(2) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

can be reduced to the form (1) by a rotation of axes. Hence the locus of points satisfying (2) is a conic section, a special case of a conic section, or there may be no locus. However, a more detailed analysis of these matters is beyond the scope of this book.

Example 1. Reduce $x^2 + 6x + 8y = 7$ to standard form.

There is a squared term in x but not in y ; we therefore try to write the equation in one of the standard forms of the parabola. We obtain

$$x^2 + 6x = -8y + 7.$$

Adding 9 to both sides to complete the square in the left member, we obtain

$$\begin{aligned} x^2 + 6x + 9 &= -8y + 16, \\ (x + 3)^2 &= -8(y - 2), \end{aligned}$$

which is one of the standard forms (8) of Sec. 17-7. Thus the locus is a parabola with vertex at $(-3, 2)$ and axis parallel to the y -axis.

Example 2. Reduce $24x^2 + 49y^2 - 96x + 294y - 639 = 0$ to standard form.

There are squared terms in x and y and their coefficients have the same signs; we therefore try to reduce the equation to one of the forms of the ellipse. Thus,

$$24(x^2 - 4x) + 49(y^2 + 6y) = 639,$$

$$24(x^2 - 4x + 4) + 49(y^2 + 6y + 9) = 639 + 96 + 441,$$

$$24(x - 2)^2 + 49(y + 3)^2 = 1176,$$

which gives as the desired result:

$$\frac{(x - 2)^2}{49} + \frac{(y + 3)^2}{24} = 1.$$

Thus we have an ellipse with center at $(2, -3)$.

Example 3. Reduce $36x^2 - 25y^2 + 216x + 100y - 676 = 0$ to standard form.

There are squared terms in x and y , and their coefficients have opposite signs, so we reduce the equation to one of the forms of the hyperbola.

$$36(x^2 + 6x) - 25(y^2 - 4y) = 676,$$

$$36(x^2 + 6x + 9) - 25(y^2 - 4y + 4) = 676 + 324 - 100,$$

$$36(x + 3)^2 - 25(y - 2)^2 = 900,$$

giving the desired result

$$\frac{(x + 3)^2}{25} - \frac{(y - 2)^2}{36} = 1.$$

This is a hyperbola with center at $(-3, 2)$.

EXERCISES

Reduce each of the following equations to standard form and sketch, showing on the graph all the standard information as indicated in the directions for the exercises following Sec. 17-7.

1. $y^2 - 4x - 4y + 16 = 0$.
2. $x^2 + 12x + 12y + 12 = 0$.
3. $7x^2 + 16y^2 + 14x - 64y - 41 = 0$.
4. $3x - y^2 - 2y + 2 = 0$.
5. $9x^2 - 16y^2 - 108x + 96y + 36 = 0$.
6. $2y^2 + 8y + 6x = 0$.
7. $y^2 + 2x + 8y + 6 = 0$.
8. $y^2 + 6x - 5 = 0$.
9. $8x^2 + 4y^2 - 64x - 8y + 68 = 0$.
10. $3x^2 + 9x + 6y + 2 = 0$.
11. $16y^2 - x^2 - 6x - 80y + 75 = 0$.
12. $x^2 - 2x - 12y = 11$.
13. $4x^2 + 12x - 20y + 49 = 0$.
14. $y^2 - 12x + 12y = 12$.
15. $4x^2 + 9y^2 - 8x + 18y + 12 = 0$.
16. $x^2 + 8x + 8y = 0$.
17. $8x^2 - 28y^2 - 8x - 28y - 61 = 0$.
18. $x^2 + 4y^2 + 8x - 8y = 5$.
19. $2x^2 - 24x + 3y + 78 = 0$.
20. $4x^2 + 25y^2 - 8x + 50y = 35$.
21. $8x^2 + 9y^2 + 16x - 54y - 1 = 0$.
22. $9x^2 + y^2 - 18x - 8y = 56$.
23. $8x^2 - 9y^2 - 16x + 54y - 1 = 0$.
24. $16x^2 + y^2 + 16y = 105$.
25. $3y^2 + 15x - 12y + 20 = 0$.
26. $x^2 - y^2 + 20x = 0$.
27. $2x^2 - 18x + 15y - 21 = 0$.
28. $x^2 - 9y^2 + 4x + 18y = 30$.
29. $3y^2 - 4x^2 - 16x - 24y - 52 = 0$.
30. $9x^2 - y^2 - 18x - 10y = 0$.
31. $y = x^2 - 4x + 4$.
32. $16x^2 - y^2 + 32x - 20y + 60 = 0$.

PROGRESS REPORT

The present chapter was devoted to the study of the parabola, ellipse, and hyperbola; their equations were found and their properties were described.

In Chapter 15 it was shown that the general first-degree equation in two variables is the algebraic representation of a line. The conclusion of the present chapter is that the general second-degree equation in two variables is the algebraic representation of a conic.

CHAPTER 18

ELEMENTS OF SOLID ANALYTIC GEOMETRY

Just as plane analytic geometry is the application of the methods of algebra to the study of geometrical figures in the plane, so solid analytic geometry is the application of the methods of algebra to the study of figures in three-dimensional space. On the whole, engineers are concerned with structures in three-dimensional space. It is sometimes convenient to treat physical problems from a two-dimensional point of view, but such a simplification of real conditions is not always feasible. A solid body, whether it is a block of steel or the wing of a plane, can only be represented and analyzed as a three-dimensional configuration. Therefore, one of the purposes of this chapter is to train the student in the visualization of spatial relations.

18-1. Directed Lines. Consider any line and a point at position A on that line (Fig. 18-1). This point can move away from A and along the line in either of two directions. Suppose we move the point along

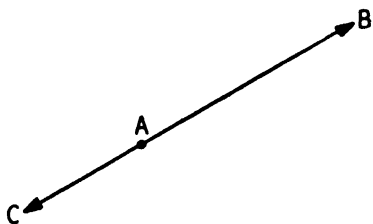


FIG. 18-1.

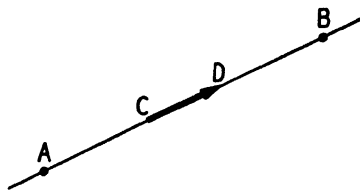


FIG. 18-2.

the line to position B . Then the line segment AB is called a **directed line segment** with **initial point** A and **terminal point** B . In Chapter 6 a directed line segment was called a **vector**, and we shall use the terms interchangeably in this chapter also.

The directed line segment or vector AC , obtained by moving a point from position A to position C , represents the other possibility for direction of motion along the line. The vectors AB and AC are said to have **opposite directions**.

A directed line segment which lies on a line may be thought of as giving its direction to the line. Thus *a line is said to be directed if there is given a vector or directed segment which lies on the line, the direction of*

the line being the same as the direction of the line segment. For example, in Fig. 18-2, the line passing through A and B has the direction left to right because the vector CD lying on it has that direction.

The direction of a number line is always chosen in the direction of increasing values of the numbers. Thus the usual arrows on the axes in a rectangular coordinate system indicate the directions of these lines.

18-2. Rectangular Coordinates. In Chapter 3 we adopted the convention of locating points in a plane by relating each point to a pair of real numbers called the coordinates of the point. A simple extension of this idea can be used to locate points in space.

Consider three lines in space which intersect at a point O and which are such that each line is perpendicular to the other two (Fig. 18-3).

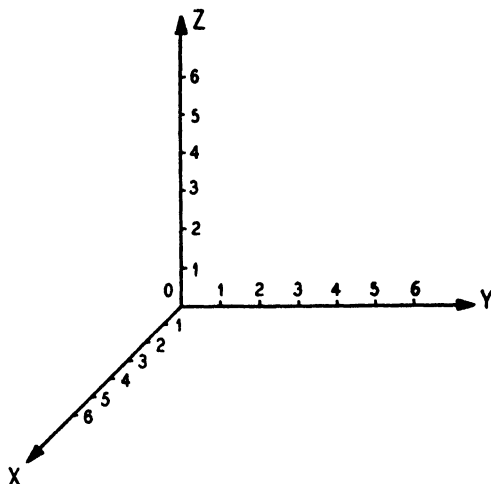


FIG. 18-3.

These three lines are called the x -axis, y -axis, and z -axis, and on each a unit length and a positive direction are chosen as shown in Fig. 18-3. Usually the same unit is chosen for all three axes. The point O at which the axes intersect is called the **origin**. The plane determined by the x -axis and the y -axis is called the xy -plane or the xy coordinate plane; the plane determined by the x -axis and the z -axis is the xz coordinate plane; the plane determined by the y -axis and the z -axis is the yz coordinate plane.

Now let P be any point in space (Fig. 18-4). Pass through P a plane parallel to the xy -plane, a second plane parallel to the xz -plane, and a third plane parallel to the yz -plane. These planes cut the x -axis, y -axis, and z -axis at A , B , and C , respectively (Fig. 18-4). If we denote

the number on the line OX at A by x , the number on the line OY at B by y , and the number on the line OZ by z , then the numbers (x, y, z) written in that order are called the **coordinates of the point**. Thus OA has length x , OB has length y , and OC has length z , as shown in Fig. 18-4. In this way we can associate a triple of numbers (x, y, z) with any point in space. It is obvious that for the point in Fig. 18-4 all three coordinates are positive numbers. For a point on the other side of the yz -plane, the x coordinate is negative; for a point on the other side of the xz -plane, the y coordinate would be negative, etc.

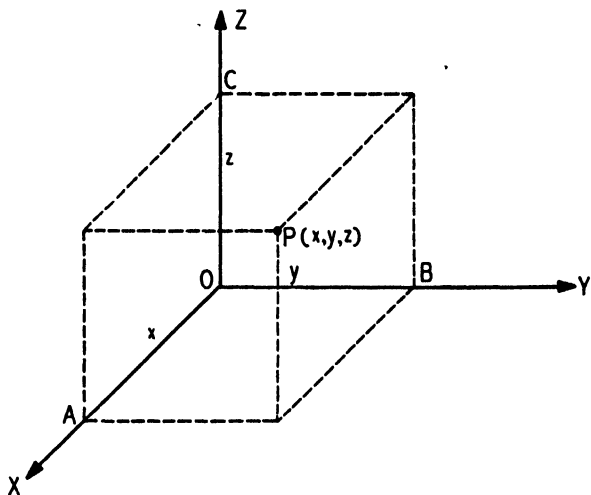


FIG. 18-4.

Each of the eight parts into which the coordinate planes divide space is called an **octant**. It is not customary to assign a numerical order to the octants. However, that octant in which all three coordinates are positive is usually called the **first octant**.

The point P with coordinates (x, y, z) can be located by reversing the above process. The point P can also be located as follows. Find the point A corresponding to x on the x -axis, the point B corresponding to y on the y -axis. Then find D , the intersection of the perpendicular to OX at A with the perpendicular to OY at B . Then erect a perpendicular to the xy -plane at D , and on this perpendicular measure the length z , up if z is positive, down if z is negative. Such a construction is carried out in Fig. 18-5.

That a point P has coordinates (x, y, z) is denoted by $P(x, y, z)$. Thus $P(0, 0, 0)$ is the origin.

In this way to every point P in space there corresponds one triple of numbers (x, y, z) , and to every triple of numbers (x, y, z) there corresponds one point.

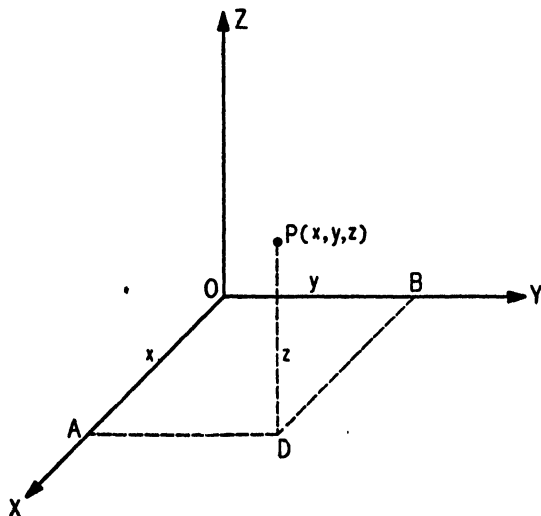


FIG. 18-5.

Example. Locate on a sketch the points $P_1(3, 5, 4)$, $P_2(-5, 7, 4)$, and $P_3(4, -4, -2)$.

These points are located in Fig. 18-6.

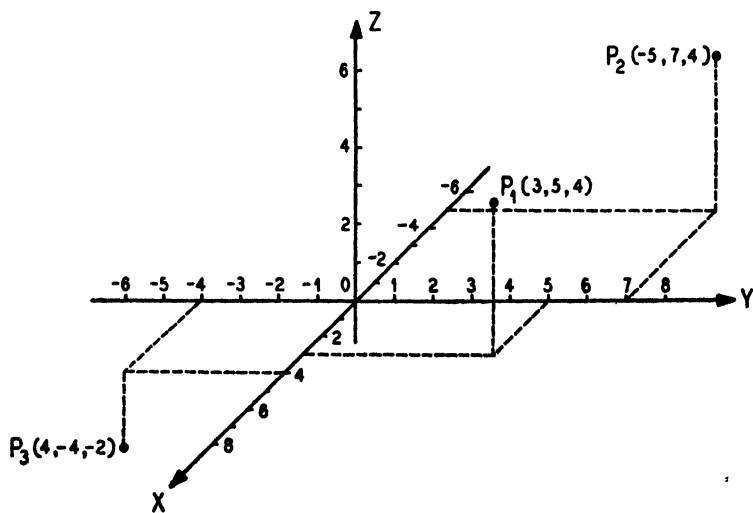


FIG. 18-6.

EXERCISES

Locate the following points on a sketch.

1. $P_1(3, 4, 5)$, $P_2(-3, 8, 5)$, $P_3(0, 0, -4)$.
2. $P_1(5, 4, -3)$, $P_2(-4, -6, 3)$, $P_3(5, -4, 3)$.
3. $P_1(5, 0, 0)$, $P_2(5, -5, 0)$, $P_3(0, 0, 5)$.
4. $P_1(3, 3, 3)$, $P_2(-3, 3, 3)$, $P_3(-3, -3, 3)$.
5. $P_1(0, 0, -4)$, $P_2(0, 0, 4)$, $P_3(4, -4, -4)$.
6. $P_1(-3, 0, 0)$, $P_2(0, -3, 0)$, $P_3(0, 0, -3)$.
7. $P_1(-5, -5, 0)$, $P_2(-5, -2, -4)$, $P_3(3, 3, -4)$.
8. $P_1(4, -2, -3)$, $P_2(4, 2, -3)$, $P_3(-4, -2, -3)$.

18-3. The Distance between Two Points. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two distinct points. Through each of the points

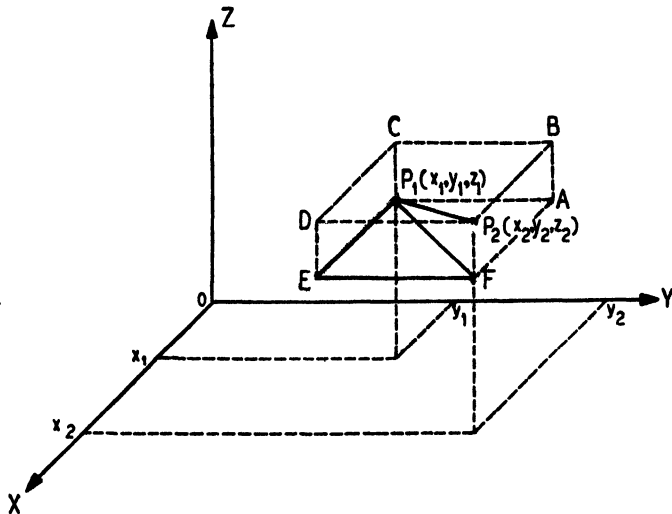


FIG. 18-7.

pass three planes, one parallel to each coordinate plane. These planes intersect and form a rectangular parallelepiped which has P_1P_2 as a diagonal (Fig. 18-7). It is easily seen that the dimensions of the rectangular parallelepiped are

$$(1) \quad P_1E = x_2 - x_1, \quad EF = y_2 - y_1, \quad FP_2 = z_2 - z_1.$$

Since the triangles P_1EF and P_1FP_2 are right triangles,

$$(P_1F)^2 = (P_1E)^2 + (EF)^2$$

and

$$(P_1P_2)^2 = (P_2F)^2 + (P_1F)^2,$$

whence

$$(2) \quad (P_1P_2)^2 = (P_2F)^2 + (P_1E)^2 + (EF)^2.$$

Combining the results of (1) and (2) we have that the distance d between P_1 and P_2 is given by

$$(3) \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Note that the order of subscripts does not matter. If P_2 is at the origin, (3) becomes

$$(4) \quad d = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

EXERCISES

Find the distance between the points of each pair.

- | | |
|---------------------------------|--------------------------------|
| 1. (5, 1, 2) and (3, 0, -3). | 2. (6, 1, 7) and (5, -2, 3). |
| 3. (5, 0, 2) and (-1, -3, -2). | 4. (3, 1, 5) and (-2, -3, 4). |
| 5. (-2, -3, 5) and (-3, -2, 1). | 6. (-3, -1, 2) and (4, 3, 5). |
| 7. (-8, 3, 4) and (-2, -1, -2). | 8. (-6, -3, -1) and (3, 4, 5). |
| 9. (6, -3, -2) and (5, -9, 3). | 10. (4, 1, 1) and (0, 0, 0). |

18-4. The Angle between Two Lines. Since two lines in space may not meet, we must define what we mean by the angle between them.

Let L_1 and L_2 be two lines in space, and assume that directions have been assigned to these lines. Choose a rectangular coordinate system in the space. Now parallel to L_1 , draw a directed segment l_1 which has its

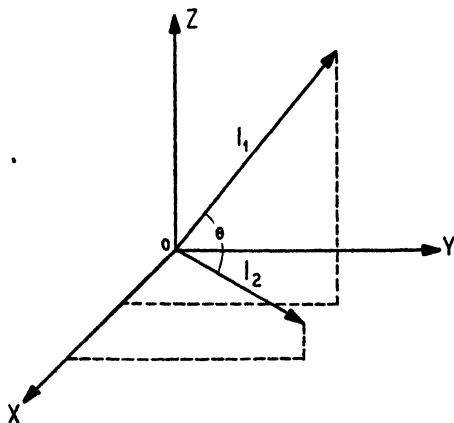


FIG. 18-8.

initial point at the origin and the same direction as L_1 ; parallel to L_2 draw a directed segment l_2 which has its initial point at the origin and the same direction as L_2 (Fig. 18-8). These two segments determine a plane and in this plane they determine many angles. We shall define the angle between L_1 and L_2 as the angle θ between 0° and 180° determined by l_1 and l_2 . Summarizing, we have this definition:

The angle θ between two directed lines L_1 and L_2 is the angle between 0° and 180° determined by the two directed segments l_1 and l_2 with initial points at the origin, which are parallel to, and have the same direction as, L_1 and L_2 respectively.

18-5. Direction Cosines. In plane analytic geometry the direction of a line with respect to the coordinate axes is given by the angle of

inclination, or its tangent, called the slope of the line. In solid analytic geometry we shall give the direction of a line with respect to the axes by three angles or by the cosines of these angles.

Let L be a directed line in space. As usual, we shall assume the coordinate axes to be directed as indicated by the arrows in Fig. 18-9. Let l be a directed segment with initial point at the origin parallel to L and with the same direction as L . Let α be the angle between l and the x -axis, β the angle between l and the y -axis, and γ the angle between l and the z -axis (Fig. 18-9). These are then by definition the angles

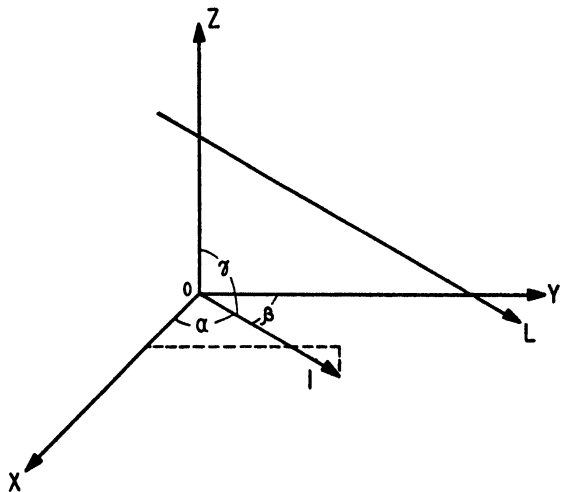


FIG. 18-9.

which L makes with the axes. The angles α , β , and γ are called the **direction angles** of L . The cosines of these angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of L .

Obviously, *two lines which are parallel and have the same direction have the same direction angles and the same direction cosines*. Let L_1 be a line parallel to L but with the opposite direction. Then the parallel directed segment l_1 with initial point at the origin has the direction opposite to that of l . Since l and l_1 are parallel, l_1 is then an extension of l through the origin. Then the direction angles α_1 , β_1 , and γ_1 of L_1 are

$$\alpha_1 = 180^\circ - \alpha, \quad \beta_1 = 180^\circ - \beta, \quad \gamma_1 = 180^\circ - \gamma.$$

Since $\cos (180^\circ - \theta) = -\cos \theta$, it follows that

$$\cos \alpha_1 = -\cos \alpha, \quad \cos \beta_1 = -\cos \beta, \quad \cos \gamma_1 = -\cos \gamma.$$

Thus the direction cosines of two parallel lines which have opposite directions are opposite in sign.

If the coordinates of two points on a line are given and its direction is known, the direction cosines of the line can be determined. Let L be a directed line on which the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given such that P_1 is the initial point of the directed segment P_1P_2 which has the same direction as L (Fig. 18-10). For convenience, let us suppose as in Fig. 18-10 that the direction angles α , β , and γ of L are all between 0° and 90° . The sides of the rectangle AP_1BD are

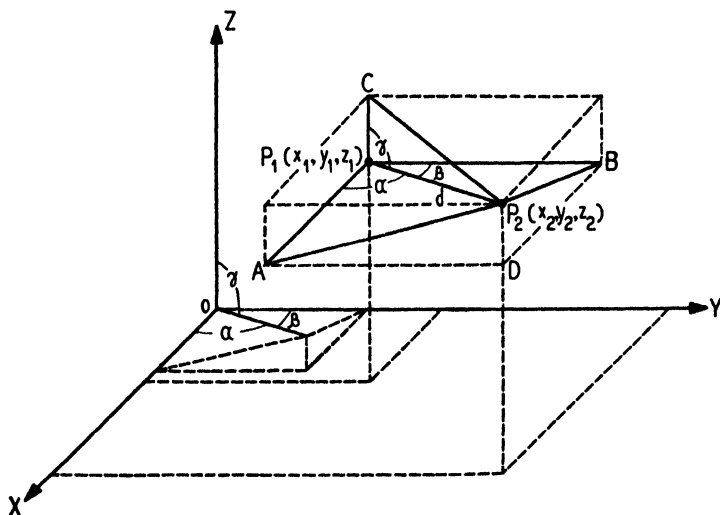


FIG. 18-10.

parallel to the x and y axes, and P_1C is parallel to the z -axis. It is evident that angle AP_1P_2 is equal to α , angle BP_1P_2 is equal to β , and angle CP_1P_2 is equal to γ , as shown. Now the triangles AP_1P_2 , BP_1P_2 , CP_1P_2 are right triangles. Since

$$d = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and

$$P_1A = x_2 - x_1, \quad P_1B = y_2 - y_1, \quad P_1C = z_2 - z_1,$$

it is obvious that

$$(1) \quad \cos \alpha = \frac{x_2 - x_1}{d}, \quad \cos \beta = \frac{y_2 - y_1}{d}, \quad \cos \gamma = \frac{z_2 - z_1}{d}.$$

If α is greater than 90° , then $\cos \alpha$ is negative. It is easily seen that in this case $|\cos \alpha| = \left| \frac{x_2 - x_1}{d} \right|$, but if $\alpha > 90^\circ$ then $x_2 < x_1$; $x_2 - x_1$ is negative; and $\cos \alpha = \frac{x_2 - x_1}{d}$. In this way it can be shown in general that:

If L is a directed line passing through the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and if the vector with initial point P_1 and terminal point P_2 has the same direction as L , then the direction cosines of L are

$$(2) \quad \frac{x_2 - x_1}{d}, \quad \frac{y_2 - y_1}{d}, \quad \frac{z_2 - z_1}{d},$$

where d is the length of the segment P_1P_2 .

Since the direction angles are by definition between 0° and 180° , the cosines (2) determine the direction angles uniquely (see Sec. 11-9).

If the direction of L is reversed, the new direction is that of the directed segment P_2P_1 with P_2 as initial point. By the rule the direction cosines are

$$\frac{x_1 - x_2}{d}, \quad \frac{y_1 - y_2}{d}, \quad \frac{z_1 - z_2}{d},$$

the negatives of those given in (2). This verifies the remark made previously that two parallel lines which have opposite directions have direction cosines that are opposite in sign.

From (2) we have that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{d^2} = 1.$$

Thus if α , β , and γ are the direction angles of a line,

$$(3) \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

From (3) we see that if any two direction angles or direction cosines are given, then the third direction cosine is determined except for sign.

Example. Find the direction cosines of the line passing through the points $P_1(1, 4, -2)$ and $P_2(-1, 0, 4)$, the direction of the line being given by the directed segment P_1P_2 which has P_1 as its initial point.

The distance d between the two points is

$$\begin{aligned} d &= \sqrt{(-1 - 1)^2 + (0 - 4)^2 + (4 - [-2])^2} \\ &= \sqrt{4 + 16 + 36} = \sqrt{56} = 2\sqrt{14}. \end{aligned}$$

Then by the rule the direction cosines are

$$\begin{aligned} \cos \alpha &= \frac{-1 - 1}{2\sqrt{14}} = -\frac{\sqrt{14}}{14} = -0.267, \\ \cos \beta &= \frac{0 - 4}{2\sqrt{14}} = -\frac{\sqrt{14}}{7} = -0.535, \\ \cos \gamma &= \frac{4 - (-2)}{2\sqrt{14}} = \frac{3\sqrt{14}}{14} = 0.802, \end{aligned}$$

and the direction angles are

$$\alpha = 105.5^\circ, \quad \beta = 122.3^\circ, \quad \gamma = 36.7^\circ.$$

As a check on the accuracy of the direction cosines we may verify that

$$\begin{aligned}\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= \left(-\frac{\sqrt{14}}{14}\right)^2 + \left(-\frac{\sqrt{14}}{7}\right)^2 + \left(\frac{3\sqrt{14}}{14}\right)^2 \\ &= \frac{14 + 56 + 126}{196} = \frac{196}{196} = 1.\end{aligned}$$

EXERCISES

Find the direction cosines of the line passing through the given points P_1 and P_2 , the direction of the line being given by the directed segment P_1P_2 which has P_1 as its initial point. Find the direction angles to the nearest tenth of a degree, and check the direction cosines by showing that the sum of their squares is 1.

- | | |
|------------------------------------|------------------------------------|
| 1. $P_1(1, 2, 3), P_2(4, 5, 8).$ | 2. $P_1(-1, 0, 5), P_2(5, 1, -2).$ |
| 3. $P_1(2, 1, -4), P_2(3, 1, -2).$ | 4. $P_1(4, 1, 2), P_2(3, 1, 0).$ |
| 5. $P_1(-5, -2, 7), P_2(4, 3, 1).$ | 6. $P_1(3, 1, 5), P_2(5, 1, 2).$ |
| 7. $P_1(1, 1, 2), P_2(1, 1, 5).$ | 8. $P_1(-4, 3, 1), P_2(0, 1, 3).$ |
| 9. $P_1(-5, 2, -7), P_2(0, 0, 6).$ | 10. $P_1(4, 1, 2), P_2(4, 5, 0).$ |
| 11. $P_1(0, 0, 5), P_2(5, 0, 0).$ | 12. $P_1(0, 0, 0), P_2(0, 5, 0).$ |
| 13. $P_1(0, 0, 2), P_2(0, 0, 0).$ | 14. $P_1(3, 0, 0), P_2(0, 0, -2).$ |
| 15. $P_1(0, -2, 0), P_2(0, 0, 0).$ | 16. $P_1(4, 0, 0), P_2(0, 0, 0).$ |

Using relation (3) of this section, find the direction cosines and direction angles not specified by the following data.

- | | |
|---|--|
| 17. $\alpha = 60^\circ, \beta = 90^\circ.$ | 18. $\alpha = 135^\circ, \beta = 90^\circ.$ |
| 19. $\beta = 30^\circ, \gamma = 90^\circ.$ | 20. $\alpha = 120^\circ, \gamma = 60^\circ.$ |
| 21. $\beta = 60^\circ, \gamma = 150^\circ.$ | 22. $\beta = 135^\circ, \gamma = 45^\circ.$ |
| 23. $\alpha = 45^\circ, \beta = 45^\circ.$ | 24. $\alpha = \beta = \gamma.$ |

25. What are the direction angles and direction cosines of the x -axis?

26. What are the direction angles and direction cosines of a line parallel to the y -axis, but with the opposite direction?

27. What are the direction angles and direction cosines of the z -axis?

18-6. A Formula for the Angle between Two Lines. Let L_1 and L_2 be two directed lines with direction angles $\alpha_1, \beta_1, \gamma_1$, and $\alpha_2, \beta_2, \gamma_2$, respectively, and let θ be the angle between L_1 and L_2 . Let l_1 and l_2 be directed segments with initial points at the origin, parallel to and with the same directions as L_1 and L_2 respectively (Fig. 18-11). Then by definition the angle θ between l_1 and l_2 is the angle between L_1 and L_2 . By definition, the direction angles α_1, β_1 , and γ_1 of L_1 are the direction angles of l_1 ; similarly α_2, β_2 , and γ_2 are the direction angles of l_2 . Let $P_1(x_1, y_1, z_1)$ be the terminal point of l_1 , $P_2(x_2, y_2, z_2)$ be the terminal point of l_2 . Denote the length of l_1 by d_1 , the length of l_2 by d_2 . Denote

the length of the segment P_1P_2 by d_3 . Applying the law of cosines (see Sec. 11-2) to the triangle P_1OP_2 we have that

$$(1) \quad \cos \theta = \frac{d_1^2 + d_2^2 - d_3^2}{2d_1d_2}.$$

Since

$$d_1^2 = x_1^2 + y_1^2 + z_1^2,$$

$$d_2^2 = x_2^2 + y_2^2 + z_2^2,$$

$$d_3^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

from (1) we have that

$$(2) \quad \cos \theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_1d_2}.$$

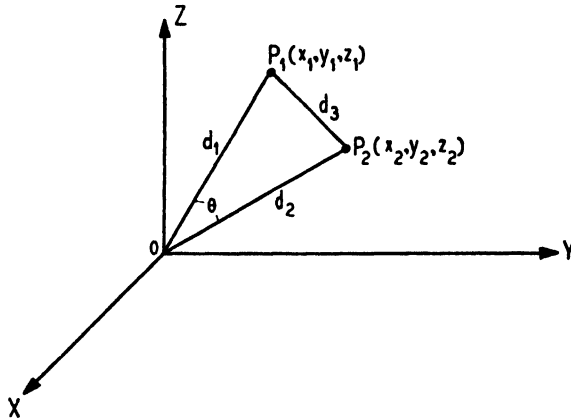


FIG. 18-11.

This expression can be written in the form

$$(3) \quad \cos \theta = \left(\frac{x_1}{d_1}\right)\left(\frac{x_2}{d_2}\right) + \left(\frac{y_1}{d_1}\right)\left(\frac{y_2}{d_2}\right) + \left(\frac{z_1}{d_1}\right)\left(\frac{z_2}{d_2}\right).$$

Now

$$\cos \alpha_1 = \frac{x_1}{d_1}, \quad \cos \alpha_2 = \frac{x_2}{d_2};$$

$$\cos \beta_1 = \frac{y_1}{d_1}, \quad \cos \beta_2 = \frac{y_2}{d_2};$$

$$\cos \gamma_1 = \frac{z_1}{d_1}, \quad \cos \gamma_2 = \frac{z_2}{d_2}.$$

Using these formulas in (3) we have the result:

If the direction angles of L_1 are $\alpha_1, \beta_1, \gamma_1$ and the direction angles of L_2 are $\alpha_2, \beta_2, \gamma_2$, then the angle θ between L_1 and L_2 is given by

$$(4) \quad \cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

Since θ by definition lies between 0° and 180° , (4) determines θ uniquely (see Sec. 11-9).

If L_1 and L_2 are parallel, $\theta = 0^\circ$ and

$$(5) \quad \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 1.$$

If L_1 and L_2 are perpendicular, $\theta = 90^\circ$ and

$$(6) \quad \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0.$$

18-7. Direction Numbers. Any three numbers a, b, c which are proportional to the direction cosines of a directed line, and also have their respective signs, are called a set of **direction numbers** of the line. Thus, if L is a line with direction angles α, β, γ , then any three numbers a, b, c such that

$$(1) \quad a = k \cos \alpha, \quad b = k \cos \beta, \quad c = k \cos \gamma,$$

where k is positive, form a set of direction numbers for L .

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points on L , and L is directed from P_1 to P_2 , then from (1) of Sec. 18-5 we have that

$$x_2 - x_1 = d \cos \alpha, \quad y_2 - y_1 = d \cos \beta, \quad z_2 - z_1 = d \cos \gamma$$

where d is positive. Thus the differences $x_2 - x_1, y_2 - y_1, z_2 - z_1$ form a set of direction numbers of L .

Example 1. For the example of Sec. 18-5, the numbers $-2, -4, 6$ form a set of direction numbers, for they are $2\sqrt{14}$ times the direction cosines. Obviously $-4, -8, 12$ and $-1, -2, 3$ are also sets of direction numbers for this line. On the other hand $2, 4, -6$ and $8, 16, -24$ are sets of direction numbers for a parallel but oppositely directed line.

If a set of direction numbers a, b, c are given, it is possible to find the direction cosines. From (1) and the identity (3) of Sec. 18-5,

$$a^2 + b^2 + c^2 = k^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = k^2.$$

Since by definition k is positive, we have

$$k = \sqrt{a^2 + b^2 + c^2}.$$

Thus finally we have

$$(2) \quad \cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Obviously, *two lines are parallel if their direction numbers a_1, b_1, c_1 and a_2, b_2, c_2 are proportional*; that is, if

$$(3) \quad a_1 = ka_2, \quad b_1 = kb_2, \quad c_1 = kc_2,$$

where k is a positive or negative constant. If none of the direction numbers are zero, (3) can be written

$$(4) \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

If L_1 and L_2 are two lines with direction numbers a_1, b_1, c_1 and a_2, b_2, c_2 , respectively, by Sec. 18-6 and (2) above, *the angle θ between L_1 and L_2 is given by*

$$(5) \quad \cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

The two lines are perpendicular if $\cos \theta = 0$, *whence the lines are perpendicular if*

$$(6) \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0,$$

and conversely.

Example 2. Given a line L_1 with direction numbers 1, -2, 2 and a line L_2 with direction numbers 3, 1, -2. Find the angle between the lines.

By (5)

$$\begin{aligned} \cos \theta &= \frac{(1 \cdot 3) - (2 \cdot 1) - (2 \cdot 2)}{\sqrt{1^2 + (-2)^2 + 2^2} \sqrt{3^2 + 1^2 + (-2)^2}} \\ &= \frac{-3}{\sqrt{9} \sqrt{14}} = -\frac{\sqrt{14}}{14} = -0.267, \end{aligned}$$

whence

$$\theta = 105.5^\circ.$$

EXERCISES

Find the direction cosines of the lines with the following direction numbers. Find the direction angles to the nearest tenth of a degree.

- | | |
|---------------|--------------|
| 1. 1, 2, -2. | 2. 3, 4, 0. |
| 3. 5, 0, -12. | 4. 1, 1, 1. |
| 5. 3, 0, 0. | 6. 1, 2, -3. |
| 7. -2, -5, 3. | 8. -1, 4, 8. |
| 9. -8, 1, -4. | 10. 2, 3, 4. |

Find to the nearest tenth of a degree the angle between the lines of each pair, if the direction numbers are as follows.

- | | |
|---|---|
| 11. L_1 : 0, 0, 2; L_2 : 5, 12, 0. | 12. L_1 : 1, 1, 1; L_2 : 1, -2, 2. |
| 13. L_1 : 0, 1, 1; L_2 : 2, 0, 0. | 14. L_1 : 1, -2, 3; L_2 : 1, 3, 2. |
| 15. L_1 : 4, 1, $-2\sqrt{2}$; L_2 : 0, 0, 5. | 16. L_1 : 2, 3, 4; L_2 : -1, -2, 3. |

17. Given six lines by two points through which each passes as follows:

$$\begin{aligned} L_1: (1, 2, -2), (0, 0, 0); & L_2: (0, 1, -4), (4, 7, 4); \\ L_3: (5, 1, 2), (4, -1, 4); & L_4: (2, 3, 4), (0, 0, 0); \\ L_5: (2, 1, -5), (0, -2, -9); & L_6: (4, -2, 3), (1, -8, 9). \end{aligned}$$

Find the groups of parallel lines in this set.

18. Find all mutually perpendicular pairs of lines among the six lines of Exercise 17.
 19. Given six lines by two points through which each passes as follows:

$$\begin{aligned} L_1: (3, 1, -1), (0, 0, 0); & L_2: (5, 1, 4), (3, 4, 1); \\ L_3: (3, 4, 5), (3, 2, 3); & L_4: (-1, 5, 4), (-1, 2, 1); \\ L_5: (0, 0, 0), (0, 1, 1); & L_6: (2, -3, 3), (0, 0, 0). \end{aligned}$$

Find the groups of parallel lines in this set.

20. Find all groups of mutually perpendicular lines among the six lines of Exercise 19.

21. Given six lines whose direction numbers are:

$$\begin{aligned} L_1: 1, 2, -1; & L_4: -7, 4, 1; \\ L_2: 1, 1, 3; & L_5: \frac{1}{2}, -\frac{2}{7}, -\frac{1}{14}; \\ L_3: -2, -4, 2; & L_6: 1, 1, 2. \end{aligned}$$

Find all groups of mutually perpendicular lines.

22. Among the six lines of Exercise 21, find all groups of parallel lines.

23. Show that the points (3, -1, -1), (1, 2, -2), and (5, 5, -1) are the vertices of a right triangle. What is the perimeter of the triangle? Find the two acute angles.

24. Show that the quadrilateral with vertices at (2, 3, 4), (3, 7, 3), (5, 0, 1), (6, 4, 0) is a parallelogram. Find the angles of the parallelogram.

25. Show that the quadrilateral with vertices (0, 0, 0), (0, 3, 3), ($3\sqrt{2}$, 3, 3), ($3\sqrt{2}$, 0, 0) is a square.

26. Show that the points (0, 1, 3), (-1, -1, -1), (-2, 2, -1) and (-1, 4, 3) form the vertices of a parallelogram.

27. Show that the points (2, 3, 7), (1, 3, 2), and (0, 3, -3) lie on a straight line.

28. Find the area of the triangle whose vertices are (1, 2, -5), (3, 1, 4), and (5, -6, -3).

18-8. Simple Loci in Space : Surfaces. The locus of a point moving in space so as to satisfy a single condition is a surface. For example, the locus of a point moving at a constant distance r from a fixed point is a sphere with center at the fixed point and radius r . To find the equation of such a locus, the geometrical condition must be expressed as an equation involving the variable coordinates x, y, z of the moving point. The procedure is what we would expect from our knowledge of plane analytic geometry.

Example 1. Find the equation of the locus of a point moving 5 units from the origin.

If $P(x, y, z)$ is the point, then $OP = \sqrt{x^2 + y^2 + z^2}$, and since $OP = 5$ the equation of the sphere is

$$x^2 + y^2 + z^2 = 25.$$

Example 2. Find the equation of the locus of a point moving so that it is 4 times as far from the xy -plane as from the yz -plane.

Let the point P have coordinates (x, y, z) . The distance from the xy -plane is $|z|$, and the distance from the yz -plane is $|x|$. Hence the requirement is that $|z| = 4|x|$, whence the equations of the locus are

$$z = 4x, \quad \text{or} \quad z = -4x,$$

which can be written together as $z^2 = 16x^2$.

In some simple cases the locus of an equation is easily described. For example, $x = 5$ is the equation of the plane perpendicular to the x -axis at the point $x = 5$; $x^2 + y^2 = 25$ is the equation of a right circular cylinder whose axis is the z -axis. The sections of this chapter which follow will show how the loci of first degree equations and some second degree equations can be described.

EXERCISES

Find the locus in space of the point moving so as to satisfy the following condition:

1. The point is the same distance from the xy -plane as from the xz -plane.
2. The point is the same distance from the yz -plane as from the xz -plane.
3. The point is twice as far from the xy -plane as from the yz -plane.
4. The point is three times as far from the xz -plane as from the xy -plane.
5. The point is 5 units from the xy -plane.
6. The point is 2 units from the yz -plane.
7. The point is 4 units from the xz -plane.
8. The point is 10 units from the origin.
9. The point is 4 units from the origin.
10. The point is 3 units from the x -axis.
11. The point is 5 units from the z -axis.
12. The point is 8 units from the y -axis.
13. The point is equidistant from the origin and the point $(0, 0, 8)$.

14. The point is equidistant from $(4, 4, 0)$ and $(-4, -4, 0)$.
15. The point is 5 units from $(0, 0, 5)$.
16. The point is 4 units from $(1, 2, 3)$.
17. The point is equidistant from $(0, 5, 2)$ and $(3, -4, -2)$.
18. The point is equidistant from $(2, 1, 4)$ and $(-3, 4, 8)$.

Describe the surface which is the locus of each of the following equations.

- | | | |
|-----------------------|-----------------|------------------------|
| 19. $x = 0$. | 20. $y = 0$. | 21. $z = 0$. |
| 22. $x = y$. | 23. $y = -z$. | 24. $y = 5$. |
| 25. $z = 6$. | 26. $z = 3x$. | 27. $x^2 + y^2 = 25$. |
| 28. $y^2 + z^2 = 4$. | 29. $y^2 = 4$. | 30. $x^2 = 9$. |

18-9. The Plane. A line which is perpendicular to a plane is called a **normal** to the plane. Since all normals to a plane are parallel, their direction numbers are proportional (Sec. 18-7).

Let RS (Fig. 18-12) be any plane, and let $P_0(x_0, y_0, z_0)$ be a fixed point of the plane. Let L be a directed normal to the plane at P_0 , and let A, B, C be a set of direction numbers for L . Now the plane may be regarded as the locus of points $P(x, y, z)$ such that P_0P is perpendicular to L . The numbers $x - x_0, y - y_0$, and $z - z_0$ form a set of direction numbers for P_0P (Sec. 18-7). By (6) of Sec. 18-7, P_0P is perpendicular to L if and only if

$$(1) \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

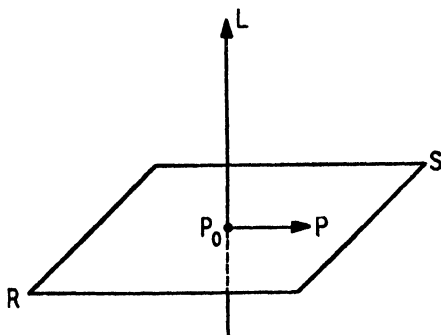


FIG. 18-12.

Hence the equation of a plane passing through a point $P_0(x_0, y_0, z_0)$ and perpendicular to a line with direction numbers A, B , and C is given by (1). The direction numbers A, B, C are called the **direction numbers for the plane**.

Since any plane can be specified in this way, we see from (1) that **every plane is represented by an equation of first degree**.

We shall now prove the converse of this theorem: **Every equation of the first degree represents a plane**.

The general equation of first degree is

$$(2) \quad Ax + By + Cz + D = 0,$$

where not all three of the constants A , B , and C are zero. Let x_0 , y_0 , z_0 be a set of values satisfying (2). Then

$$(3) \quad Ax_0 + By_0 + Cz_0 + D = 0.$$

Subtracting (3) from (2) we obtain

$$(4) \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which is the equation of a plane containing the point (x_0, y_0, z_0) and normal to a line with direction numbers A, B, C .

As a consequence of these results, (2) is called the **general equation of the plane**, and (1) is called the **general equation of the plane containing (x_0, y_0, z_0)** .

Since two planes are parallel if and only if they have parallel normals, *two planes are parallel if and only if the coefficients of x , y , and z in their equations are proportional*.

Similarly, *two planes*

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if and only if

$$(5) \quad A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

Example 1. Find the equation of the plane which passes through $(2, -1, \frac{3}{2})$ and whose normal has the direction numbers $\frac{1}{2}, -2, 3$.

Using (1) we have

$$\frac{1}{2}(x - 2) - 2(y + 1) + 3(z - \frac{3}{2}) = 0,$$

which becomes upon simplifying

$$x - 4y + 6z - 15 = 0.$$

Example 2. Show that the plane $4x + y = 0$ is perpendicular to the plane of Example 1.

Since $4, 1, 0$ is a set of direction numbers for this plane, and $1, -4, 6$ is a set for the plane of Example 1, by (5),

$$(4 \cdot 1) + (1 \cdot [-4]) + (0 \cdot 6) = 0$$

shows that the planes are perpendicular.

18-10. The Plane Determined by Three Points. The following example illustrates how to find the equation of a plane determined by three points.

Example. Find the equation of the plane which passes through the points $P_1(7, 1, 2)$, $P_2(-3, -3, 1)$, and $P_3(5, -1, 1)$.

Since P_1 lies in the plane, the equation of the plane has the form

$$(1) \quad A(x - 7) + B(y - 1) + C(z - 2) = 0.$$

Substituting the coordinates of P_2 in (1) we have

$$(2) \quad -10A - 4B - C = 0,$$

and substituting the coordinates of P_3 in (1) we have

$$(3) \quad -2A - 2B - C = 0.$$

Solving (2) and (3) simultaneously for A and B in terms of C (Sec. 13-8), we obtain $A = \frac{1}{6}C$ and $B = -\frac{2}{3}C$, whence (1) becomes

$$\frac{1}{6}C(x - 7) - \frac{2}{3}C(y - 1) + C(z - 2) = 0.$$

This reduces to

$$x - 4y + 6z - 15 = 0,$$

the required equation.

18-11. Sketching a Plane. The x -coordinate of the point at which a plane crosses the x -axis is called the **x -intercept** of the plane; the

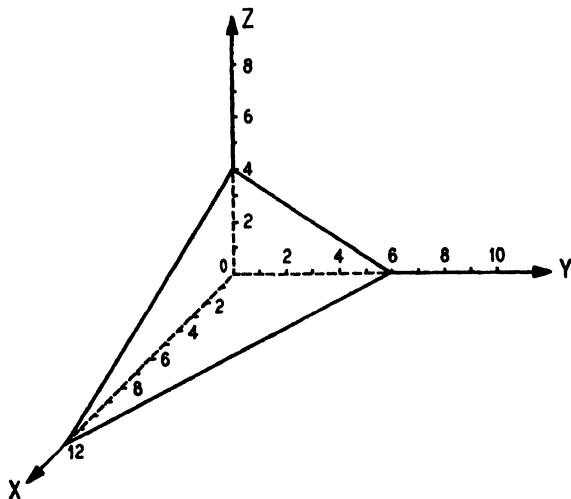


FIG. 18-13.

y -intercept and **z -intercept** are defined similarly. To find the x -intercept of a plane find the value of x when $y = z = 0$ in the equation of the plane; to find the y -intercept of a plane find the value of y when $x = z = 0$ in the equation of the plane, etc.

When a plane is not parallel to any of the coordinate axes it can be sketched conveniently by finding its intercepts on the axes as indicated

in the following example. A plane parallel to an axis can be sketched easily on the basis of what has been done in Sec. 18-8.

Example. Sketch the plane $x + 2y + 3z - 12 = 0$.

The x -intercept is 12, the y -intercept is 6, and the z -intercept is 4. Since the intersection of two planes is a straight line, the given plane and the xy -plane intersect in a straight line passing through the x -intercept and y -intercept of the given plane, called the *trace* of the given plane in the xy -plane. It is convenient to draw the portion of this trace between the intercepts; similarly in the other coordinate planes. Then the triangular section as shown in Fig. 18-13 makes a convenient sketch of the plane.

EXERCISES

Find the equation of the plane which passes through the given point and whose normal has the given direction numbers.

1. $(1, 2, -1)$; direction numbers $1, \frac{1}{2}, -1$.
2. $(-1, 1, \frac{1}{2})$; direction numbers $2, \frac{1}{3}, -2$.
3. $(2, -1, 0)$; direction numbers $-2, 1, -\frac{1}{2}$.
4. $(0, 0, 0)$; direction numbers $0, 1, 0$.
5. $(4, -6, 5)$; direction numbers $-3, 2, -2$.
6. $(1, -2, 3)$; direction numbers $1, 2, 3$.
7. $(3, -2, 1)$; direction numbers $0, 1, 1$.
8. $(-1, 0, 2)$; direction numbers $2, -1, 3$.

For each of the following planes, (a) find the intercepts, (b) find a set of direction numbers for the normal to the plane, (c) sketch the plane.

- | | |
|-------------------------------|------------------------------|
| 9. $x + 2y + 3z = 6$. | 10. $2x + 3y - 4z = 12$. |
| 11. $5x + 2y + 3z = 10$. | 12. $3x - 2y + z - 12 = 0$. |
| 13. $6x - 4y - 5z - 24 = 0$. | 14. $4y - 3x - z - 12 = 0$. |
| 15. $z = 5y - 2x - 13$. | 16. $7x - 2y + 5z = 14$. |

Determine whether the planes are parallel, perpendicular, or neither.

17. $x + 2y + 3z = 6$, $3x + 6y + 9z - 15 = 0$.
18. $x + 2y + 3z = 6$, $12x - 3y - 2z - 12 = 0$.
19. $2x + y + 4y = 15$, $3x - 2y - z - 6 = 0$.
20. $3x - 2y + z = 7$, $9x - 6y + 3z = 17$.
21. $6x + 4y + 2z = 15$, $9x + 6y + 3z = 19$.
22. $3x - y - 2z = 6$, $3y - x - 3z = 10$.
23. $4y - 3x - z - 12 = 0$, $7x - 2y + 5z = 14$.
24. $5x + 2y + 3z = 10$, $3x - 2y + z - 12 = 0$.

Determine the equation of the plane passing through the given three points.

25. $(0, 1, 2)$, $(1, -1, 2)$, $(2, 4, -5)$.
26. $(1, 0, -2)$, $(2, 3, -3)$, $(-5, -4, -1)$.
27. $(4, 1, 4)$, $(0, 4, -1)$, $(0, 0, 0)$.
28. $(1, 2, \frac{1}{2})$, $(3, -1, 2)$, $(-2, 2, -\frac{3}{2})$.
29. $(1.2, -2.3, 4.0)$, $(-3.1, -2.7, 0.9)$, $(1.8, -1.2, -0.7)$.
30. $(3.2, -1, -2)$, $(5, -1.2, 3.7)$, $(5.6, -7, 4.3)$.

31. Find the equation of the plane passing through the point $(1, 2, 5)$ which is parallel to the plane $z = 0$.

32. Find the equation of the plane passing through the point $(1, 1, 1)$ which is parallel to the plane $2x - 3y + 4z - 1 = 0$.

33. Find the equation of the plane passing through $(2, 1, -1)$ which is parallel to the plane $x - 6y + 4z - 5 = 0$.

34. Find the equation of the plane through $(2, 1, 3)$ perpendicular to the segment joining this point and $(-1, -2, 6)$.

35. Find the equation of the plane through $(3, -2, 4)$ perpendicular to the segment joining this point and $(0, 4, -2)$.

36. Find the equation of the plane which passes through $(5, 1, 2)$ and is parallel to the plane $3x - 2y + z - 12 = 0$.

37. Find the equation of the plane which passes through $(2, 0, -5)$ and is parallel to the plane $x + 2y + 3z = 6$.

38. Find the equation of the plane which passes through $(3, 1, -2)$ and is parallel to $5x - y + z = 15$.

39. Find the equation of the plane which passes through $(1, 2, -1)$ and $(2, 4, -2)$ and is perpendicular to $2x + 3y - z + 4 = 0$.

40. Find the equation of the plane which passes through $(4, -1, -2)$ and $(7, 3, 1)$ and is perpendicular to $x - 3y + 4z - 2 = 0$.

41. Find the equation of the plane which passes through $(1, 2, 3)$ and is perpendicular to each of the planes $x - y + 2z = 3$ and $2x - y - 3z = 0$.

42. Find the equation of the plane which passes through $(1, 0, 2)$ and is perpendicular to each of the two planes $2x - y + 4z = 0$ and $3x + 2y - z = 10$.

18-12. Sketching a Surface. The surfaces corresponding to the equations discussed in the previous sections were easily identified. Frequently, however, the surface corresponding to an equation is not so easily identified but can be discovered by a simple analysis. We shall show how such an analysis can be carried out in a few simple cases.

A surface cannot be plotted satisfactorily by plotting points and joining them, as can a plane curve. However, a study of the traces, intercepts, and plane sections is sufficient to reveal the nature of the surfaces we shall consider.

1. *Intercepts.* To find the intercepts of the surface on a given axis, let the variables measured along the other two axes be zero, and then solve the equation of the surface for the third variable.

2. *Traces.* The trace of a surface on a coordinate plane is the curve of intersection of the surface and the plane. The equation of a trace of a surface on a coordinate plane is found by substituting zero for the variable measured perpendicular to that plane.

3. *Plane Sections.* The curve in which a plane cuts a surface is called a plane section of the surface. Thus the traces of a surface in the coordinate planes are plane sections. If $x = a$ is substituted in the equation of the surface, the resulting equation in the variables y and z is the equation of the plane section of the curve lying in the plane $x = a$.

Similar statements hold for planes parallel to the other two coordinate planes. By sketching a few plane sections the shape of a surface can be indicated very easily.

Example. Sketch the portion of the surface

$$\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{9} = 1$$

which lies in the first octant.

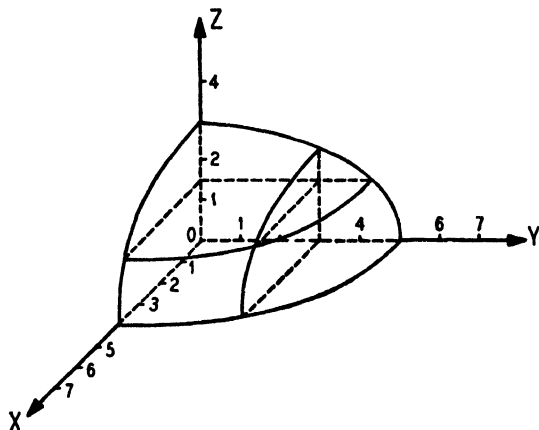


FIG. 18-14.

Setting y and z equal to zero, we find the x -intercepts to be $+4$ and -4 . Similarly the y -intercepts are $+5$ and -5 , and the z -intercepts are $+3$ and -3 . The traces in the xy -plane, xz -plane, and yz -plane are, respectively,

$$\frac{x^2}{16} + \frac{y^2}{25} = 1, \quad \frac{x^2}{16} + \frac{z^2}{9} = 1, \quad \frac{y^2}{25} + \frac{z^2}{9} = 1,$$

which are ellipses and which can be sketched very easily, as shown in Fig. 18-14. The plane section in the plane $z = \frac{3}{2}$ is

$$\frac{x^2}{12} + \frac{y^2}{\left(\frac{75}{4}\right)} = 1,$$

an ellipse as shown in the figure. The plane section in the plane $y = 3$ is

$$\frac{x^2}{\left(\frac{256}{25}\right)} + \frac{z^2}{\left(\frac{144}{25}\right)} = 1$$

as shown.

4

18-13. Curves in Space. The locus of a point satisfying two conditions is a curve. Each condition separately defines a surface, and the curve is then the intersection of these two surfaces. Thus a curve in

space can be represented by giving the equations of two surfaces which intersect to give the line. Thus a straight line may be specified by giving the equations of two planes which intersect in the line.

Example. Sketch the intersection in the first octant of the surfaces $y^2 + z^2 = 4$ and $x^2 = -8(z - 2)$.

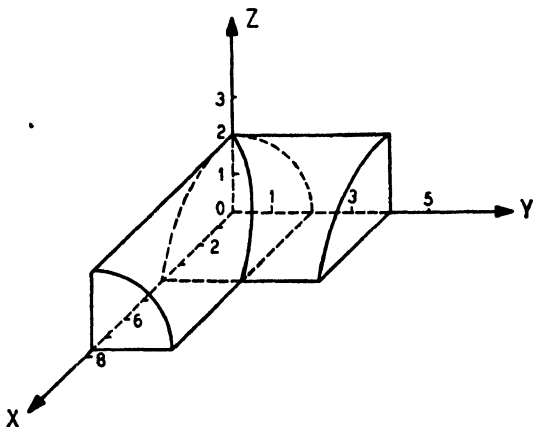


FIG. 18-15.

The surface $y^2 + z^2 = 4$ is a right circular cylinder of radius 2 whose axis is the x -axis. The surface $x^2 = -8(z - 2)$ is a cylinder such that any plane section parallel to the xz -plane is a parabola. The surfaces and curve are shown in Fig. 18-15.

EXERCISES

Sketch the portion of each of the following surfaces which lies in the first octant.

1. $4z = 16 - x^2$.
2. $4x = 16 - z^2$.
3. $3y = 9 - x^2$.
4. $4x + y^2 = 12$.
5. $z^2 + y^2 = 16$.
6. $x^2 + y^2 = 25$.
7. $y^2 = 9x$.
8. $x^2 = 4z$.
9. $4x^2 + 9y^2 = 36$.
10. $25x^2 + 9z^2 = 225$.
11. $x^2 + y^2 + z^2 = 4$.
12. $x^2 + y^2 + z^2 = 25$.
13. $x^2 + y^2 = z^2$.
14. $x^2 + y^2 = \frac{1}{4}z^2$.
15. $x^2 + y^2 + z^2 = 36$.
16. $x^2 + y^2 = 4z$.
17. $y^2 + 2z^2 = 8x$.
18. $x^2 + 3z^2 = 12y$.
19. $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.
20. $\frac{x^2}{4} + \frac{y^2}{36} + \frac{z^2}{16} = 1$.
21. $\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{36} = 1$.
22. $\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{16} = 1$.
23. $\frac{x^2}{25} + \frac{y^2}{9} - \frac{z^2}{36} = 1$.
24. $\frac{x^2}{4} - \frac{y^2}{36} + \frac{z^2}{16} = 1$.
25. $x^2 - 4y^2 = 16z$.
26. $4y^2 - x^2 = 16z$.

Sketch the curve of intersection in the first octant of the following pairs of surfaces.

27. $x^2 + y^2 = 16, x + z = 4.$

28. $x^2 + z^2 = 25, x + y = 5.$

29. $x^2 + z^2 = 25, x + y = 3.$

30. $x^2 + y^2 = 16, x^2 + z^2 = 16.$

31. $x^2 + y^2 = 25, y^2 + z^2 = 25.$

32. $x^2 + y^2 + z^2 = 36, x^2 + y^2 = 9.$

33. $x^2 + y^2 + z^2 = 25, x^2 + z^2 = 4.$

34. $x + y = 5, y + z = 5.$

35. $x + y = 4, x + y + z = 8.$

36. $\frac{x^2}{4} + \frac{y^2}{36} + \frac{z^2}{16} = 1, x + z = 2.$

37. $\frac{x^2}{4} + \frac{y^2}{36} + \frac{z^2}{16} = 1, x + y = 3.$

38. $\frac{x^2}{25} + \frac{y^2}{36} + \frac{z^2}{16} = 1, x = y.$

39. $\frac{x^2}{25} + \frac{y^2}{36} + \frac{z^2}{16} = 1, x^2 + z^2 = 16.$

40. $\frac{x^2}{25} + \frac{y^2}{36} + \frac{z^2}{16} = 1, x^2 = -2(z - 6).$

PROGRESS REPORT

In this chapter we studied the equation of the plane and learned to sketch surfaces and the intersections of two surfaces. The method of analysis was similar to that in plane analytic geometry: we merely added another dimension.

CHAPTER 19

THE ELEMENTS OF DIFFERENTIAL CALCULUS

The concept of a function was introduced in Chapter 3 and was used throughout the book. In almost every chapter, new functions were examined, and new methods were developed which permitted us to investigate properties of these functions and to solve problems in which they were involved. There still remains one extremely important type of problem for which no adequate method of solution has been developed so far. These problems can be exemplified by the following questions. What is the best way of describing the speed of a car or the cooling of a hot object? How does the change of the plate current of a tube depend on the change of the grid voltage? These are instances of innumerable problems in which the rate of change of different quantities has to be investigated. For such investigations new ideas have been introduced and new methods have been developed. The system of all these concepts and procedures, called **differential calculus**, is the main topic of this chapter.

19-1. Increments and Δ -Notation. The plate current I_p of a triode tube is a function of the grid potential E_g . The following table gives the results of an experiment in which I_p was measured for different values of E_g (I_p measured in milliamperes, E_g in volts).

E_g	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
I_p	0.2	0.5	1.1	1.8	2.7	3.8	5.2	6.9	8.9	11.1	13.5

Assume that the experiment starts with a grid potential of -10 volts and that this potential is then successively changed. Each increase or decrease of the potential is called an **increment** and is denoted by the symbol ΔE_g , read: **increment of E_g** or simply **delta- E_g** . Thus the increment of E_g is 2 volts if E_g is changed from -10 to -8 volts; and, if the potential is changed from -10 volts to -14 , the increment is -4 volts. In the first case we write $\Delta E_g = 2$ volts, in the second case $\Delta E_g = -4$ volts.

If E_g changes from -10 to -8 , the current changes from 5.2 to 8.9 milliamperes. The change of the current is then 3.7 milliamperes and is called *the increment of the current which corresponds to the increment of*

2 volts of the grid potential. The symbol ΔI_p is used to denote the increment of the current. The experiment has shown that I_p increases 3.7 milliamperes if the grid voltage is increased by 2 volts from -10 to -8 volts. These two values $\Delta E_g = 2$ volts and $\Delta I_p = 3.7$ milliamperes are called *corresponding increments for the interval from $E_g = -10$ volts to $E_g = -8$ volts.*

Example 1. What are the corresponding increments in the above experiment if E_g changes from -7 volts to -14 volts?

In this case $\Delta E_g = -14 - (-7) = -7$ volts and, from the table, the corresponding $\Delta I_p = 1.1 - 11.1 = -10$ milliamperes.

Example 2. From a table of logarithms find the increment Δy of $y = \log x$ corresponding to $\Delta x = 0.3$ at the point $x = 2$.

From the table of logarithms, $\log 2 = 0.3010$ and $\log 2.3 = 0.3617$, hence

$$\Delta x = 0.3, \quad \Delta y = \log 2.3 - \log 2 = 0.3617 - 0.3010 = 0.0607.$$

The functions used in the preceding examples were given by tables. But a function can be given also by means of a formula or by a graph. The concept of **corresponding increments** refers to a function given in any form and can be explained generally in the following way.

Assume that a function $y = f(x)$ is given. In order to compute the increment Δy corresponding to an increment Δx , compute the value $f(x + \Delta x)$. The increment of the function is the difference between its second and the first value. Symbolically

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x).$$

Since $y = f(x)$, it follows that

$$f(x + \Delta x) = y + \Delta y.$$

Example 3. Compute the corresponding increments of x and the function

$$y = f(x) = 3x^2 - 5$$

if x changes from 2 to 2.5.

Following the above explanation we have

$$x = 2, \quad \Delta x = 2.5 - 2 = 0.5,$$

$$f(x) = f(2) = 3 \cdot 4 - 5 = 7,$$

$$f(x + \Delta x) = f(2.5) = 3 \cdot 6.25 - 5 = 13.75,$$

$$\Delta x = f(x + \Delta x) - f(x) = 13.75 - 7 = 6.75.$$

Example 4. From the formula $I = \frac{110}{R}$ compute the corresponding increments of I and R if R changes from 50 to 60.

In this case we have

$$R = 50, \quad \Delta R = 10,$$

$$I = f(R) = f(50) = \frac{110}{50} = 2.20,$$

$$f(R + \Delta R) = f(60) = \frac{110}{60} = 1.83,$$

$$\Delta I = f(R + \Delta R) - f(R) = 1.83 - 2.20 = -0.37.$$

Corresponding increments are found very easily if the function $y = f(x)$ is given by a graph. Let P and Q be two points on the graph (Fig. 19-1), x the abscissa of P , and Δx the increment of x if we proceed

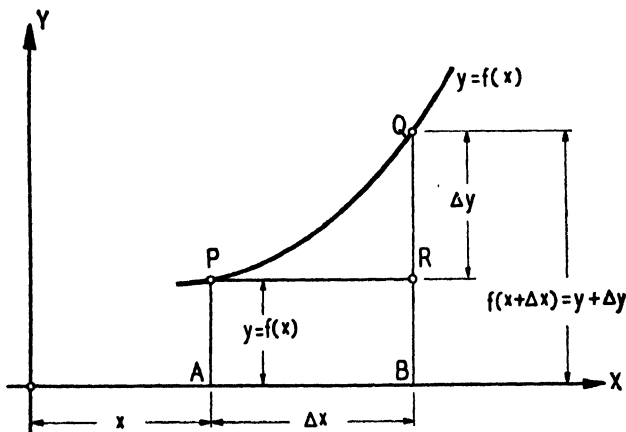


FIG. 19-1.

from P to Q . From the figure it can be seen that $\Delta x = PR$ is the difference of the abscissas of P and Q . The corresponding increment of $f(x)$ is, as seen from the figure:

$$\Delta y = RQ = BQ - BR = f(x + \Delta x) - f(x).$$

Example 5. The degree of magnetization or magnetic flux density in a piece of iron depends on the magnetizing force. For a certain kind of steel, the flux density B , measured in gausses, is plotted in Fig. 19-2 against the magnetizing force H , measured in oersteds. What is the increment ΔB if H increases from 0.4 to 0.8, and what is the value of ΔB if H increases from 1.0 to 1.4?

In both cases $\Delta H = 0.4$. The graph shows that the increment of B in the first case is

$$\Delta B = R_1Q_1 = 4000,$$

and that the increment in the second case is

$$\Delta B = R_2Q_2 = 1000$$

approximately.

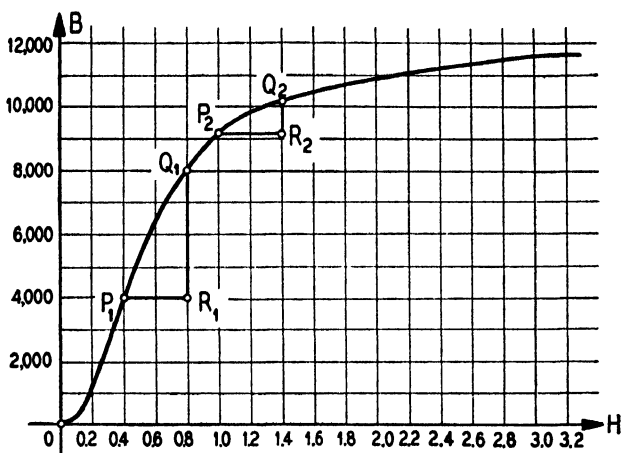


FIG. 19-2.

EXERCISES

Compute $\Delta \log x$ for $\Delta x = 10, 5, 1, 0.5, 0.1$ at a point where

1. $x = 45$.

2. $x = 170$.

3. $x = 60$.

Compute $\Delta \sin \theta$ for $\Delta \theta = 2^\circ, 1^\circ, 0.5^\circ, 0.1^\circ$, if

4. $\theta = 5^\circ$.

5. $\theta = 20^\circ$.

6. $\theta = 80^\circ$.

Compute $\Delta \tan \theta$ for $\Delta \theta = 2^\circ, 1^\circ, 0.5^\circ, 0.1^\circ$, if

7. $\theta = 10^\circ$.

8. $\theta = 50^\circ$.

9. $\theta = 85^\circ$.

Using a table of squares and square roots, compute the increments indicated in the following exercises for $\Delta x = 5, 1, 0.1$.

10. $\Delta(x^2)$ at $x = 10, x = 50, x = 90$.

11. $\Delta(\sqrt{x})$ at $x = 20, x = 65, x = 80$.

12. $\Delta(\frac{1}{2}x^2 + 10\sqrt{x})$ at $x = 10, x = 20, x = 30$.

Compute the increments of the functions given in Exercises 13-18.

13. $y = 2x + 5$ at $x = 2$ for $\Delta x = -1, 1, 0.5, 0.1$.

14. $y = \frac{1}{x}$ at $x = 1$ for $\Delta x = 0.5, 0.3, -0.1$.

15. $y = \frac{1}{x}$ at $x = 10$ for $\Delta x = 0.5, -0.3, 0.1$.

16. $I = \frac{110}{R}$ at $R = 50$ for $\Delta R = -5, 2, 1$.

17. $I = \frac{500}{\sqrt{R^2 + 5000}}$ at $R = 100$ for $\Delta R = 10, 5, 1$.

18. $P = I^2R$ at $R = 5$, $I = 10$ for $\Delta I = 2, 1, 0.5$.

19. The volume of a sphere is given by the formula $V = \frac{4\pi}{3}r^3$. Compute ΔV if r increases from 10 to 11.

20. The current through a certain tube is $I_p = 0.0021(E_p + \mu E_g)^{\frac{3}{2}}$ milliamperes where $\mu = 13.8$. It is operated at a plate potential $E_p = 200$ and a grid potential of $E_g = -12$. Compute ΔI_p if $\Delta E_g = -2, 1, 2, 5$ volt.

21. The inductance L of a single-layer coil of radius r and length l is given by the formula

$$L = \frac{r^2 N^2}{9r + 10l},$$

where N is the number of turns. Compute ΔL if $r = 3$ in., $l = 2$ in., $N = 100$, $\Delta N = 120$.

22. The power in an alternating-current circuit is $P = EI \cos \theta$. Compute ΔP if $E = 110$ volts, $I = 5$ amperes, and if θ changes from 40° to 50° .

23. The magnetization B of a certain kind of steel as a function of the magnetizing force is given in Fig. 19-2. Find ΔB for $\Delta H = 0.2$ at $H = 0.2$, at $H = 0.4$, and at $H = 2$.

24. Using Fig. 19-2, find ΔB for $\Delta H = 0.4$ at $H = 0$, at $H = 0.5$, and at $H = 2.4$.

19-2. The Average Rate of Change. It is obvious that the increment of a function depends on the corresponding increment of the independent variable and on the point where the increment begins. The table in Sec. 19-1 for corresponding values of plate current and grid voltage, which is repeated here, illustrates this fact very clearly.

E_g	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
I_p	0.2	0.5	1.1	1.8	2.7	3.8	5.2	6.9	8.9	11.1	13.5

If the increments of I_p corresponding to $\Delta E_g = 1, 2, 3$ are computed, first starting with $E_g = -15$, and then with $E_g = -9$, the following corresponding increments are obtained:

$$E_g = -15, \quad \Delta E_g = 1, \quad \Delta I_p = 0.6$$

$$\Delta E_g = 2, \quad \Delta I_p = 1.3$$

$$\Delta E_g = 3, \quad \Delta I_p = 2.2$$

$$E_g = -9, \quad \Delta E_g = 1, \quad \Delta I_p = 2.0$$

$$\Delta E_g = 2, \quad \Delta I_p = 4.2$$

$$\Delta E_g = 3, \quad \Delta I_p = 6.6$$

The increment ΔI_p increases in both cases if the corresponding ΔE_g increases, but the increase of ΔI_p is much greater at the point $E_g = -9$ than at the point $E_g = -15$. When E_g increases from $E_g = -15$ to $E_g = -13$ by $\Delta E_g = 2$, the corresponding $\Delta I_p = 1.3$; when E_g increases

from $E_g = -9$ to $E_g = -7$, we have again $\Delta E_g = 2$, but the corresponding $\Delta I_p = 4.2$. We say then that I_p increases faster in the interval from $E_g = -9$ to -7 than in the interval from $E_g = -15$ to -13 .

In order to compare the behavior of I_p as a function of E_g in both intervals, the ratio $\frac{\Delta I_p}{\Delta E_g}$ of corresponding increments can be used. The value of this ratio for the interval $E_g = -15$ to $E_g = -13$ is

$$\frac{\Delta I_p}{\Delta E_g} = \frac{1.3}{2} = 0.65;$$

the corresponding ratio computed for the interval $E_g = -9$ to $E_g = -7$ is

$$\frac{\Delta I_p}{\Delta E_g} = \frac{4.2}{2} = 2.1.$$

The ratio $\frac{\Delta I_p}{\Delta E_g}$ is called **average rate of change** of I_p in the interval from E_g to $E_g + \Delta E_g$. Thus 2.1 is the average rate of change of I_p in the interval from $E_g = -9$ to $E_g = -7$, and 0.65 is the average rate of change of I_p if E_g changes from -15 to -13 .

The idea of an average rate of change is used very frequently. A few examples will illustrate the application of this concept.

Example 1. A car starts at noon, and at 2 P.M. is at a distance of 50 miles, and at 5 P.M. a distance of 140 miles, from the starting point. Within the three hours from 2 P.M. to 5 P.M. the car covered a distance of $140 - 50 = 90$ miles at an average rate of $\frac{90}{3} = 30$ miles per hour.

Example 2. On a certain day, the temperature at 7 A.M. was 30° , and at 1 P.M. it was 60° . The increment of the temperature was 30° during 6 hours, and the temperature changed at an average rate of $\frac{30^\circ}{6} = 5^\circ$ per hour.

Example 3. A city had a population of 80,000 people in 1920 and 120,000 people in 1940. The population increased by 40,000 people in 20 years. The average rate of increase was $\frac{40,000}{20} = 2000$ people per year.

In the above examples, it has not been stated that the observed quantities—distance, temperature, population—change uniformly. There is no reason why the population of the city in Example 3 should have increased by exactly 2000 every year. The meaning of the result obtained in Example 3 is that the change from 80,000 to 120,000 could have been produced by a uniform increase of 2000 people per year.

The average rate of change in the above examples was computed as the ratio of the increment of an observed quantity and the corresponding increment of time. The observed quantity was regarded as a function

of the time. This concept of an average rate of change can be defined generally in the following way.

The average rate of change of a function $y = f(x)$ in the interval from x to $x + \Delta x$ is defined as the ratio of the corresponding increments,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

It is easy to see that the average rates in the above examples were computed in this way. Consider for instance Example 1. The distance s from the starting point may be regarded as a function of the time t , denoted by the symbol $s = f(t)$. We have then that $f(2) = 50$ and $f(5) = 140$. The increment of t is $\Delta t = 5 - 2 = 3$, and the increment of s is $\Delta s = f(2 + \Delta t) - f(2) = f(5) - f(2) = 140 - 50 = 90$. Hence $\frac{\Delta s}{\Delta t} = \frac{90}{3} = 30$ miles per hour.

A very striking geometrical interpretation of the average rate of change can be given if the graph of the function is plotted. The function $y = f(x)$ may be represented by the graph of Fig. 19-3. The increment

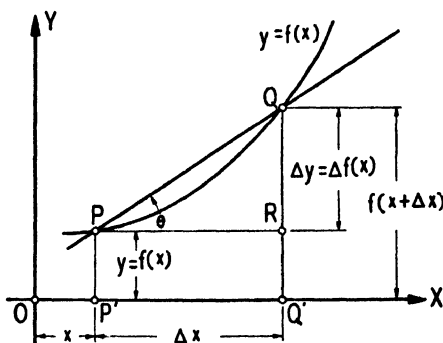


FIG. 19-3.

Δx is given by the segment $P'Q' = PR$. The corresponding increment $\Delta y = \Delta f(x)$ is given by the segment RQ . Hence it follows that the average rate of change in the interval $P'Q'$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{RQ}{PR} = \tan \theta$$

where θ is the angle between the line PQ and the direction of the x -axis.

Thus, $\frac{\Delta y}{\Delta x}$ is the slope of the line PQ , and we may state that the *average rate of change of the function $y = f(x)$ in the interval from x to $x + \Delta x$ is*

equal to the slope of the straight line PQ connecting the points on the graph which correspond to the values x and $x + \Delta x$. Such a straight line connecting two points of a curve is called a **secant** of the curve.

The average rate of change is computed very easily when the function is given by a formula. The following examples will show the procedure.

Example 4. Compute the average rate of change of the function $y = f(x) = 3x^2 - 2$ if x changes from 1 to 3.

Following the explanations given above we have:

$$\Delta x = 3 - 1 = 2,$$

$$f(x) = f(1) = 1,$$

$$f(x + \Delta x) = f(1 + 2) = f(3) = 25,$$

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x) = 25 - 1 = 24,$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{24}{2} = 12.$$

Example 5. The plate current in milliamperes of a particular triode tube is a function of the grid potential, in volts, approximately given by the formula

$$I_p = f(E_g) = 0.4(16 + E_g)^{\frac{3}{2}}.$$

Compute the average rate of change of the current when the grid potential changes from -12 to -8 volts.

We have in this case:

$$\Delta E_g = 4 \text{ volts,}$$

$$I_p = f(-12) = 0.4(16 - 12)^{\frac{3}{2}} = 3.20 \text{ milliamperes,}$$

$$I_p + \Delta I_p = f(-8) = 0.4(16 - 8)^{\frac{3}{2}} = 9.05 \text{ milliamperes,}$$

$$\Delta I_p = f(-8) - f(-12) = 5.85 \text{ milliamperes,}$$

$$\frac{\Delta I_p}{\Delta E_g} = \frac{5.85}{4} = 1.46 \text{ milliamperes per volt.}$$

EXERCISES

Compute the average rate of change of each of the following functions for the indicated intervals.

1. $y = \log x$; (a) from $x = 10$ to $x = 12$, (b) from $x = 3$ to $x = 4$, (c) from $x = 3$ to $x = 3.1$, (d) from $x = 3$ to $x = 3.01$, (e) from $x = 40$ to $x = 50$, (f) from $x = 300$ to $x = 305$.

2. $y = \sqrt{x}$; (a) from $x = 1$ to $x = 1.5$, (b) from $x = 1$ to $x = 1.1$, (c) from $x = 5$ to $x = 6$, (d) from $x = 10$ to $x = 11$, (e) from $x = 10$ to $x = 12$, (f) from $x = 50$ to $x = 52$.

3. $s = 16t^2$; (a) from $t = 0$ to $t = 1$, (b) from $t = 0$ to $t = 10$, (c) from $t = 3$ to $t = 4$, (d) from $t = 4$ to $t = 6$, (e) from $t = 3$ to $t = 7$, (f) from $t = 2$ to $t = 5$.

4. $I = 10 \sin 120\pi t$; (a) from $t = 0$ to $t = 0.001$, (b) from $t = 0$ to $t = \frac{1}{120}$, (c) from $t = 1$ to $t = 1.002$, (d) from $t = 1$ to $t = 1.008$.

5. The plate current I_p in amperes of a certain cathode tube is given by the formula $I_p = 4.5 \cdot 10^{-6} E_p^{\frac{3}{2}}$ where E_p is the plate potential in volts. Compute the average change of the plate current if E_p changes (a) from 0 to 50 volts, (b) from 0 to 200 volts, (c) from 50 to 100 volts.

19-3. The Instantaneous Rate of Change. There are many rate problems in practical life which are not satisfactorily solved by merely computing an average rate of change of a function. If an automobile accident happens, the driver cannot shake off his responsibility by proving that he drove at an average rate of 20 miles per hour during the last two hours. The important questions are: What was his rate at the moment of the accident? Did he drive at this very moment at a rate of 15 or 60 miles per hour? The interest is concentrated on his **instantaneous rate** or his speed at the instant of the accident.

The precise meaning of this concept of an instantaneous rate needs to be investigated. It was easy to define an average rate of change of a function, but difficulties arise if one tries to use the same approach to the idea of an instantaneous rate. There is no increment of time for the moment of the accident and no corresponding distance covered by the automobile. The actual experiments which have been designed in order to find the speed of a car, or of a bullet, measure the average rate during a very short time. It is assumed that during this short time the motion of the object is uniform so that the average rate during a short time interval can be used in order to characterize the motion at any instant during this interval. This is a somewhat vague description of how to arrive at the idea of an instantaneous rate. The problem of this section is to clarify and to give a correct definition of this concept.

Consider the rate of a car which moves with a constant speed of 30 miles per hour. The average rate for two hours is $\frac{60}{2} = 30$ miles per hour, the rate for 0.5 hour is $\frac{15}{0.5} = 30$ miles per hour, the average rate

for one minute is $\frac{\frac{1}{2}}{\frac{1}{60}} = 30$ miles per hour. It is obvious that the rate

can be computed for any interval of time and that the value 30 miles per hour will always result. If the time interval is small, the distance covered by the car during this interval is small, but the ratio of this distance Δs and the time Δt is the same. For example, if $\Delta t = 1$ sec. =

$\frac{1}{3600}$ of an hour, the corresponding $\Delta s = \frac{30}{3600} = \frac{1}{120}$ mile and $\frac{\Delta s}{\Delta t} = \frac{3600}{120} = 30$ miles per hour.

In more complicated cases, the rates are not constant, but observations show that in most cases they are nearly constant if small intervals are investigated. This experience has been encountered so often that it is used, almost instinctively, in many applications. Assume, for instance, that an experiment shows an increment of $\Delta I_p = 2$ milliamperes of the plate current of a tube if the grid voltage changes from $E_g = -9$ to $E_g = -8$ volts. The average rate of change in this interval is $\frac{\Delta I_p}{\Delta E_g} = \frac{2}{1} = 2$ milliamperes per volt. It can therefore be assumed that the increment of the current within this interval is approximately proportional to the increment of E_g , so that $\Delta I_p = 1, \frac{1}{2}, \frac{1}{5}$ milliamperes, when $\Delta E_g = \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$ volt, respectively, and that $\frac{\Delta I_p}{\Delta E_g} = 2$ for all corresponding increments of I_p and E_g within the observed interval.

The whole situation can more easily be examined, using a function for which many values are given. The function $y = \log x$ may be selected for this purpose, and the average rates of change will be computed for different intervals starting with $x = 1.5$. The following table contains the required data.

x	$\log x$	$\Delta x = x - 1.5$	$\Delta y = \log x - \log 1.5$	$\frac{\Delta y}{\Delta x}$
1.3	0.1139	-0.2	-0.0622	0.311
1.4	0.1461	-0.1	-0.0300	0.300
1.5	0.1761	0	0	
1.6	0.2041	0.1	0.0280	0.280
1.7	0.2304	0.2	0.0543	0.272

It can be observed that the average rates of change of $y = \log x$ for segments to the left of $x = 1.5$ are greater than the average rates for segments to the right of this point, and that the values of these rates are, for the investigated segments, rather close to each other. We may expect that for still smaller intervals, starting with the same point $x = 1.5$, the rates of change will be between 0.300 and 0.280. A four-place table of logarithms is not sufficient to study the behavior of these rates for smaller intervals and, therefore, a seven-place logarithmic table has been used to obtain the following values.

x	$\log x$	$\Delta x = x - 1.500$	$\Delta y = \log x - \log 1.500$	$\frac{\Delta y}{\Delta x}$
1.498	0.1755118	-0.002	-0.0005795	0.2898
1.499	0.1758016	-0.001	-0.0002897	0.2897
1.500	0.1760913	0	0	
1.501	0.1763807	0.001	0.0002894	0.2894
1.502	0.1766699	0.002	0.0005786	0.2893

It may be observed that the values of the average rates of change for these small intervals are very close to each other, and it may be expected that, if still smaller intervals of both sides of $x = 1.5$ are investigated, the corresponding average rates of change will be between 0.2897 and 0.2894. We say that the behavior of $y = \log x$ in the neighborhood of $x = 1.500$ is characterized by a number between 0.2894 and 0.2897, which could be found with any precision if logarithmic tables with enough decimals were available. This number is called the **instantaneous rate of change** or the **derivative** of the function $y = \log x$ with respect to x at the point $x = 1.5$. Its correct value with seven decimals is 0.2895297. This number means that the average rate of change of the function $y = \log x$ for intervals adjacent to $x = 1.5$ approaches this value if smaller and smaller intervals are investigated.

A good approximation for this instantaneous rate of change or derivative at a given point may be found by computing the average rates of change for two small segments of equal length adjacent to the given point at both sides of it and by taking half the sum of the two values. In this way we obtain, from the first of the two tables given above,

$$\frac{\Delta \log x}{\Delta x} = 0.300 \text{ in the interval from } x = 1.4 \text{ to } x = 1.5,$$

$$\frac{\Delta \log x}{\Delta x} = 0.280 \text{ in the interval from } x = 1.5 \text{ to } x = 1.6.$$

Taking half the sum of these two values, we get

$$\frac{1}{2}(0.300 + 0.280) = 0.290$$

as the value of the instantaneous rate of change, correct to three decimals.

The following examples will illustrate this new concept of an instantaneous rate of change.

Example 1. If the angle θ is expressed in radians, compute approximately the derivative of the function $y = \sin x$ at the point $x = 1.2$.

From Table 5 in the Appendix we have the following table of values.

x	$\sin x$	$\Delta x = x - 1.2$	$\Delta y = \sin x - \sin 1.2$	$\frac{\Delta y}{\Delta x}$
1.10	0.8912	-0.10	-0.0408	0.41
1.19	0.9284	-0.01	-0.0036	0.36
1.20	0.9320			
1.21	0.9356	0.01	0.0036	0.36
1.30	0.9636	0.10	0.0316	0.32

Hence the derivative or instantaneous rate of change of $y = \sin x$, at $x = 1.2$, is with two decimals, 0.36.

Example 2. Using the table of Sec. 19-2, compute an approximate value for the instantaneous rate of change of I_p with respect to E_g at the point $E_g = -10$ volts.

From the given table we obtain the following values.

E_g	I_p	ΔE_g	ΔI_p	$\frac{\Delta I_p}{\Delta E_g}$
-11	3.8	-1	-1.4	1.4
-10	5.2	0	0	
-9	6.9	+1	1.7	1.7

An approximate value for the instantaneous rate of change of I_p with respect to E_g at the point $E_g = -10$, is given by

$$\frac{1}{2}(1.4 + 1.7) = 1.6 \text{ milliamperes per volt.}$$

In radio engineering, this value is called **mutual conductance** or **grid-plate transconductance** of the tube for the grid voltage $E_g = -10$ and the plate voltage which has been used in collecting the data of the table.

EXERCISES

Compute with the precision attainable by the tables in the Appendix, the derivatives of the function in Exercises 1-6 at the indicated points:

1. $y = \log x$ at (a) $x = 2$, (b) $x = 7$, (c) $x = 0.3$, (d) $x = 4.2$.
2. $y = \sqrt{x}$ at (a) $x = 1$, (b) $x = 0.5$, (c) $x = 4$, (d) $x = 1.3$.
3. $y = \sin x$ (x in radians) at (a) $x = 0.5$, (b) $x = 0.8$, (c) $x = 1.5$, (d) $x = 0$.
4. $y = \cos x$ (x in radians) at (a) $x = 0.1$, (b) $x = 0.3$, (c) $x = 1$, (d) $x = \frac{\pi}{2}$.
5. $y = \log \sin x$ (x in radians) at (a) $x = 0.5$, (b) $x = 0.8$, (c) $x = 1.5$, (d) $x = 1.2$.
6. $y = \log \cot x$ (x in radians) at (a) $x = 0.1$, (b) $x = 0.3$, (c) $x = 1.2$, (d) $x = 1.5$.
7. Compute, using the table of Exercise 25 of Sec. 3-9, the instantaneous rate of change of I_p with respect to E_p for (a) $E_p = 140$ volts, (b) 170 volts, (c) 200 volts.
8. Compute, using the table of Exercise 27 of Sec. 3-9, the instantaneous rate of change of I_p with respect to E_g for (a) $E_g = -6.0$ volts, (b) $E_g = -2.5$ volts, (c) $E_g = -1.5$ volts.
9. Compute, using the table of Exercise 31 of Sec. 3-9, the instantaneous rate of change of E_o with respect to I_o at (a) $I_o = 39.5$ milliamperes, (b) $I_o = 24$ milliamperes, (c) $I_o = 14$ milliamperes.
10. Compute, using the table of Exercise 32 of Sec. 3-9, the instantaneous rate of change of I with respect to E at (a) $E = 40$ volts, (b) $E = 70$ volts, (c) $E = 90$ volts.
11. Compute, using the table of Exercise 40 of Sec. 3-9, the instantaneous rate of change of B with respect to H at (a) $B = 4000$ gaussess, (b) $B = 8000$ gaussess, (c) $B = 12,000$ gaussess.
12. Compute, using the table of Exercise 48 of Sec. 3-9, the instantaneous rate of change of L with respect to θ at (a) $\theta = 30^\circ$, (b) $\theta = 60^\circ$, (c) $\theta = 90^\circ$.
13. Compute, using the table of Exercise 50 of Sec. 3-9, the instantaneous rate of change of T with respect to t at (a) $t = 3$ minutes, (b) $t = 6$ minutes, (c) $t = 8$ minutes.
14. Compute, using the table of Exercise 51 of Sec. 3-9, the instantaneous rate of change of V with respect to D at (a) $D = 3$ miles, (b) $D = 5$ miles, (c) $D = 7$ miles.

19-4. Limits. The explanation of the instantaneous rate of change and the method of computing an approximate value of this rate were based on the observation that the average rate of change approaches a certain value if the interval for which it is computed becomes very small. Situations of this kind, where a quantity approaches a certain value, occur in different branches of mathematics and its applications and are especially important for the subject of this chapter.

Consider, as an example, the sequence of numbers

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$$

which can be continued indefinitely. The numbers obviously approach one. In order to state this fact, we say that 1 is the **limit** of the numbers of the given sequence. Every number of the given sequence can be found by the formula $a_n = \frac{n-1}{n}$, for when n is replaced by particular values we then obtain,

$$a_1 = \frac{1-1}{1} = 0, a_2 = \frac{2-1}{2} = \frac{1}{2}, a_3 = \frac{3-1}{3} = \frac{2}{3}, a_4 = \frac{4-1}{4} = \frac{3}{4}, \text{etc.}$$

From the expression $a_n = \frac{n-1}{n} = 1 - \frac{1}{n}$ it can be inferred that a_n approaches the value 1 when n increases indefinitely, for $\frac{1}{n}$ becomes as small as we want if n increases. In order to state the fact that the numbers of the above sequence have the limit 1, the following notation is used.

$$\lim a_n = \lim \frac{n-1}{n} = 1, \quad \text{if } n \rightarrow \infty,$$

or more briefly

$$\lim_{n \rightarrow \infty} \frac{n-1}{n} = 1$$

which reads as follows: *the limit of $\frac{n-1}{n}$ is equal to 1 if n increases indefinitely, or if n becomes indefinitely great, or if n approaches infinity.*

An illustration of the concept of limit is given by the following engineering application.

Example 1. The combined resistance of two resistances R_1 and R_2 , connected in parallel, is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}.$$

What is the value of the limit of R as R_2 approaches infinity? The general method for finding the limit, in this and similar cases, is to divide numerator and denominator of R by R_2 , yielding

$$R = \frac{R_1}{\frac{R_1}{R_2} + 1}.$$

The fraction $\frac{R_1}{R_2}$ approaches zero, when R_2 increases indefinitely, and, therefore, R approaches the value $\frac{R_1}{1} = R_1$; so that we have finally,

$$\lim R = R_1, \text{ if } R_2 \rightarrow \infty.$$

The concept of limit refers to many similar situations. Two more examples will illustrate this.

Example 2. Examine the values of the function

$$y = \frac{\log x}{x - 1}$$

when x assumes values which approach 1.

First observe that no value of y is obtained when $x = 1$ because the numerator and the denominator of the given function become zero when the value $x = 1$ is substituted. We shall therefore investigate the value of y in the neighborhood of $x = 1$. This is done in the following table, where the values of y are computed for different values of x approaching 1.

x	$\log x$	$x - 1$	$\frac{\log x}{x - 1}$
0.50	$0.6990 - 1 = -0.3010$	-0.50	0.602
0.80	$0.9031 - 1 = -0.0969$	-0.20	0.484
0.90	$0.9542 - 1 = -0.0458$	-0.10	0.458
0.99	$0.9956 - 1 = -0.0044$	-0.01	0.44
1	0	0	
1.01	0.0043	0.01	0.43
1.10	0.0414	0.10	0.414
1.20	0.0792	0.20	0.396
1.50	0.1761	0.50	0.352

From this table it is seen that when x approaches 1, the function $\frac{\log x}{x - 1}$ approaches the value M , which lies between 0.43 and 0.44. Using logarithmic tables with more decimals this value M could be found with greater precision. Thus we could get $M = 0.434$ correct to three significant digits.

Using the notation which has been introduced before, we write

$$\lim_{x \rightarrow 1} \frac{\log x}{x - 1} = 0.434,$$

which reads: *the limit of $\frac{\log x}{x - 1}$ is 0.434 if x approaches 1.*

The situation appearing in the above example occurs very often and can be described as follows. *The limit of a fraction has to be computed if numerator and denominator become very small.* Indeed, the concept of instantaneous rate is based on such a situation, as will be shown in the following example.

Example 3. Compute the instantaneous rate of change of the function $y = \sqrt{x}$ at the point $x = 1$.

When $x = 1$, then $y = 1$; and when $x = 1 + \Delta x$, then $y = \sqrt{1 + \Delta x}$. Hence the average rate of change of \sqrt{x} in the interval from 1 to $1 + \Delta x$ is given by the expression

$$\frac{\sqrt{1 + \Delta x} - 1}{\Delta x},$$

and the instantaneous rate of change is the limit of this expression as Δx approaches zero. But when $\Delta x = 0$ both the numerator and the denominator of the last expression are equal to zero, and we must investigate its value in the neighborhood of $\Delta x = 0$. This is done in the following table.

Δx	$\sqrt{1 + \Delta x} - 1$	$\frac{\sqrt{1 + \Delta x} - 1}{\Delta x}$
-0.1	-0.05132	0.5132
-0.01	-0.005013	0.5013
-0.001	-0.0005001	0.5001
0	0	
0.001	0.0004999	0.4999
0.01	0.004988	0.4988
0.1	0.04881	0.4881

We can infer, from this table, that the value sought lies between 0.5001 and 0.4999 and that it is 0.5. Hence

$$\lim_{\Delta x \rightarrow 0} \frac{\sqrt{1 + \Delta x} - 1}{\Delta x} = 0.5, \text{ if } \Delta x \rightarrow 0,$$

and 0.5 is the value of the instantaneous rate of change, or the derivative, of the function \sqrt{x} at the point $x = 1$.

EXERCISES

Compute the following limits.

1. $\lim \left(4 + \frac{1}{n} \right), n \rightarrow \infty.$

2. $\lim \left(3 - \frac{2}{n} \right), n \rightarrow \infty.$

3. $\lim \left(\frac{4}{n^2} + 1 \right), n \rightarrow \infty.$

4. $\lim \frac{2n + 1}{n + 1}, n \rightarrow \infty. \quad (\text{Hint. Divide numerator and denominator by } n.)$

5. $\lim \frac{3n - 2}{5n - 7}, n \rightarrow \infty.$

6. $\lim \frac{n^2 + 1}{n^2 - 1}, n \rightarrow \infty.$

The angle θ in the following exercises is expressed in radians. Find the results with the precision which can be attained if the tables in the Appendix are used.

7. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}, \theta \rightarrow 0.$

8. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}, \theta \rightarrow 0.$

9. $\lim_{\theta \rightarrow 0} \theta \cot \theta, \theta \rightarrow 0.$

10. $\lim_{t \rightarrow \infty} 2^{-t}, t \rightarrow \infty.$

11. $\lim_{t \rightarrow \infty} (10 + 2 \cdot 2.7^{-60t}), t \rightarrow \infty.$

12. $\lim_{t \rightarrow \infty} (25 - 30 \cdot 10^{-5t}), t \rightarrow \infty.$

In the following problems assume E, R, L, C to be positive quantities, $e = 2.718$, and compute the limits if $t \rightarrow \infty$.

13. $\lim_{t \rightarrow \infty} \frac{E}{R} (1 - e^{-\frac{Rt}{L}}).$

14. $\lim_{t \rightarrow \infty} \frac{E^2}{R} (e^{-\frac{t}{RC}} - e^{-\frac{2t}{RC}}).$

15. $\lim_{t \rightarrow \infty} \frac{E}{10L} (e^{-10t} - e^{-30t}).$

16. $\lim_{t \rightarrow \infty} e^{-10t} \cdot \sin 60t.$

17. $\lim_{t \rightarrow \infty} Ee^{-5t} (2 \sin 360t + 5 \cos 360t).$

18. $\lim_{t \rightarrow \infty} (5 - 10e^{-2t} \cos 150t).$

19-5. Geometrical Investigation of the Instantaneous Rate of Change.

A geometrical interpretation of the average rate of change as the slope of a straight line was studied in Sec. 19-2. This will be used in this section in discussing the instantaneous rate of change.

Assume that a function is given by the graph of Fig. 19-4. The average rate of change of the function for the segment QQ_1 is equal to the

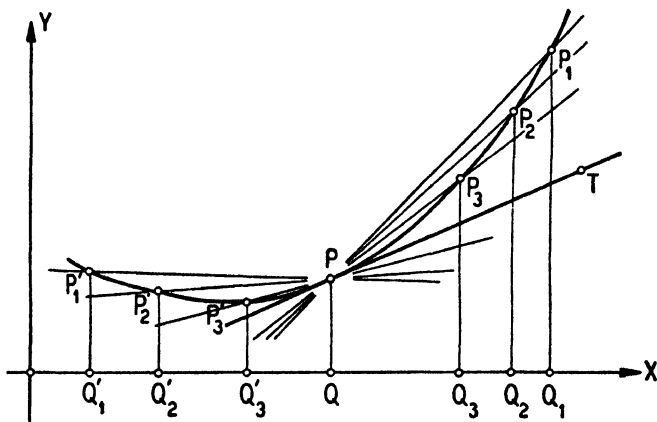


FIG. 19-4.

slope of the secant PP_1 . If different points, $P_1, P_2, P_3 \dots$ to the right of P are chosen, different secants $PP_1, PP_2, PP_3 \dots$ are obtained. Similarly, different points $P_1', P_2', P_3' \dots$ to the left of P may be used in order to construct the secants $PP_1', PP_2', PP_3' \dots$. From the figure it is seen that there is a line PT which separates the secants meeting the curve to the right of P , from the secants meeting the curve to the

left of P . This line is called the **tangent of the curve at P** . The secants PP_1, PP_2, PP_3, \dots approach the tangent if the points P_1, P_2, P_3, \dots approach P , or, which is the same, if the lengths of the segments QQ_1, QQ_2, QQ_3, \dots approach zero.

The slopes of the secants PP_1, PP_2, PP_3, \dots approach the slope of the tangent PT when the points P_1, P_2, P_3, \dots approach P . We may say that *the slope of the tangent at P is the limit of the slopes of the secants through P whose second point of intersection with the graph approaches P* .

The slopes of the secants PP_1, PP_2, PP_3, \dots are the average rates of change of the function for the intervals QQ_1, QQ_2, QQ_3, \dots respectively. These average rates approach the instantaneous rate of change at P . But we have seen before that the slopes of the secants approach the slope of the tangent. We can state, therefore, that *the instantaneous rate of change at P is equal to the slope of the tangent at P* .

In order to find the slope of the tangent, consider an arbitrary point T on the tangent (Fig. 19-5). The increments of the coordinates, if one

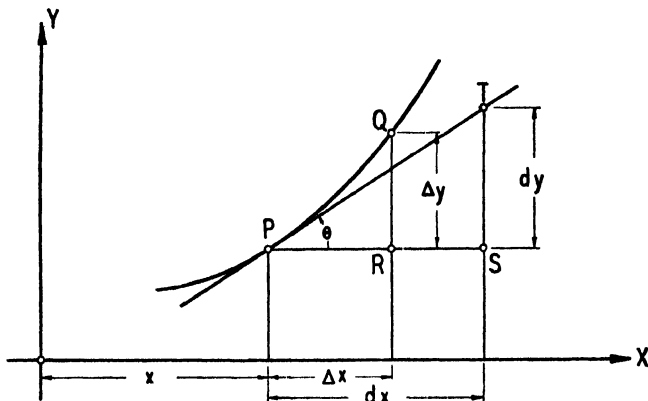


FIG. 19-5.

passes from P to T , may be denoted by dx and dy . We have, then, for the slope of the tangent PT ,

$$\tan \theta = \frac{dy}{dx}.$$

But it was stated above that the instantaneous rate of change at the point P , which may be denoted by m , is equal to the slope of the tangent PT and therefore

$$m = \frac{dy}{dx}.$$

Since the instantaneous rate of change was defined as the limit of the average rate of change for the interval Δx , as Δx approaches zero, we have then

$$m = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}, \quad \Delta x \rightarrow 0.$$

Summarizing the contents of this and the preceding section, *the derivative, or instantaneous rate of change, of a function $y = f(x)$ at a particular point P is the value of $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ as Δx approaches zero. This value is equal to the slope of the tangent at P . If dx and dy are the increments of x and y from P to an arbitrary point on the tangent, then the derivative at P is given by*

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

19-6. Derivative of a Function Given by a Formula. It has been explained, in the preceding sections of this chapter, how to find the **derivative** (this will be used in preference to **instantaneous rate of change**) of a function which is defined by a table or by a graph. The next step is to show how to find the derivative of a function which is defined by a formula.

Consider, for example, the function

$$(1) \quad y = f(x) = 3x^2 - 5.$$

The problem is to find the derivative of this function at some particular point, for example, $x = 2$. According to the discussion in the preceding sections, our first step is to find the average rate of change of the function in the interval from $x = 2$ to $x = 2 + \Delta x$. We thus obtain the following:

$$f(2) = 3 \cdot 4 - 5 = 7,$$

$$f(2 + \Delta x) = 3(2 + \Delta x)^2 - 5 = 7 + 12 \cdot \Delta x + 3 \cdot (\Delta x)^2,$$

$$\Delta y = f(2 + \Delta x) - f(2) = 12 \cdot \Delta x + 3 \cdot (\Delta x)^2,$$

$$\frac{\Delta y}{\Delta x} = 12 + 3 \cdot \Delta x.$$

The next step is to find the limit of the average rate of change $\frac{\Delta y}{\Delta x}$ as $\Delta x \rightarrow 0$. We have then

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (12 + 3 \cdot \Delta x) = 12, \quad \text{as } \Delta x \rightarrow 0.$$

Hence 12 is the value of the derivative of the given function $y = 3x^2 - 5$ at the point $x = 2$.

The same method can be used if the derivative of this function (1) is wanted for other values of x . In order to avoid the repetition of the same steps, we shall derive a formula which gives the derivative of (1) for any desired value of x . This can be accomplished if in the preceding computations the number 2 is replaced by the symbol x . We thus obtain

$$f(x) = 3x^2 - 5,$$

$$f(x + \Delta x) = 3(x + \Delta x)^2 - 5 = 3x^2 + 6x \cdot \Delta x + 3 \cdot (\Delta x)^2 - 5.$$

Hence $\Delta f(x)$ which is the increment of $f(x)$ corresponding to the increment Δx of x , is given by

$$\Delta f(x) = f(x + \Delta x) - f(x) = 6x \cdot \Delta x + 3 \cdot (\Delta x)^2.$$

Thus the average rate of change of $f(x)$ for the interval from x to $x + \Delta x$ is given by

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{6x \cdot \Delta x + 3 \cdot (\Delta x)^2}{\Delta x} = 6x + 3 \cdot \Delta x,$$

whence

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \lim (6x + 3 \cdot \Delta x) = 6x, \text{ as } \Delta x \rightarrow 0.$$

The result $6x$ gives the derivative of the function $y = 3x^2 - 5$ at any point x . Substituting $x = 2$ yields the value 12, which is the same as the answer previously obtained.

It is necessary to have a convenient notation for derivatives. In the preceding section it has been shown that the derivative is equal to the ratio dy/dx , where dx and dy are the corresponding increments if one passes from the point $P(x, y)$ to an arbitrary point on the tangent at P . This ratio dy/dx will be used systematically in denoting the derivative. Thus the following are the notations used to denote the derivative of the function (1) at an arbitrary point x :

$$\text{Derivative} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \frac{dy}{dx} = \frac{df(x)}{dx} = \frac{d(3x^2 - 5)}{dx} = \frac{d}{dx} (3x^2 - 5),$$

and we have

$$\frac{d(3x^2 - 5)}{dx} = 6x,$$

which is read: *the derivative of $3x^2 - 5$ with respect to x is equal to $6x$ or simply $d(3x^2 - 5)$ over dx is $6x$.*

Sometimes it is convenient to have a still shorter notation for the derivative of $y = f(x)$, and we write

$$\frac{dy}{dx} = y' = f'(x),$$

which is read: *y-prime*, or *f-prime of x*.

The derivative of a function, when computed for an arbitrary value x is another function of this variable x . The operation which consists in finding the derivative is called **differentiation**. The following examples will illustrate the concepts of this section.

Example 1. The electric current in amperes produced by a battery of 6 volts is given by $I = \frac{6}{R}$, where R is the resistance in ohms. Compute the derivative $\frac{dI}{dR}$.

The current I is here a function $f(R)$ of the independent variable R . If R increases by ΔR , the corresponding increment of I is given by

$$\Delta I = f(R + \Delta R) - f(R) = \frac{6}{R + \Delta R} - \frac{6}{R} = \frac{-6 \cdot \Delta R}{(R + \Delta R)R},$$

and hence the average rate of change is

$$\frac{\Delta I}{\Delta R} = -\frac{6}{(R + \Delta R)R}.$$

The derivative is the limit of this expression as $\Delta R \rightarrow 0$, and hence we get

$$\frac{dI}{dR} = \lim_{\Delta R \rightarrow 0} \left(\frac{-6}{(R + \Delta R)R} \right) = -\frac{6}{R^2} \text{ amperes per ohm.}$$

If it is given that $R = 3$ ohms, we then obtain

$$I = \frac{6}{R} = 2 \text{ amperes, } \frac{dI}{dR} = -\frac{6}{R^2} = -\frac{6}{9} = -\frac{2}{3} \text{ amperes per ohm.}$$

Example 2. The speed of a moving object is equal to the instantaneous rate of change with respect to the time of the distance between the object and its starting point. If an object is thrown vertically upwards with an initial speed of 100 ft. per sec., its distance from the starting point after t seconds is given by the expression

$$s = 100t - 16t^2.$$

What is its speed after t seconds? After 3 sec.?

Proceeding as before, we have the following

$$s = f(t) = 100t - 16t^2,$$

$$\begin{aligned} s + \Delta s &= f(t + \Delta t) = 100(t + \Delta t) - 16(t + \Delta t)^2 \\ &= 100t + 100 \cdot \Delta t - 16t^2 - 32t \cdot \Delta t - 16 \cdot (\Delta t)^2, \end{aligned}$$

$$\Delta s = f(t + \Delta t) - f(t) = 100 \cdot \Delta t - 32t \cdot \Delta t - 16 \cdot (\Delta t)^2.$$

Hence the average speed during the time interval Δt is given by

$$\frac{\Delta s}{\Delta t} = \frac{100 \cdot \Delta t - 32t \cdot \Delta t - 16 \cdot (\Delta t)^2}{\Delta t} = 100 - 32t - 16 \cdot \Delta t,$$

whence,

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} (100 - 32t - 16 \cdot \Delta t) = 100 - 32t.$$

We have, finally, that the speed of the object after t seconds is $(100 - 32t)$ feet per second. After 3 sec. the speed is $100 - 32 \cdot 3 = 4$ ft. per sec.

EXERCISES

Compute the derivatives of the following functions at the indicated points.

1. $f(x) = 9x^2$ at (a) $x = 5$, (b) $x = -3$, (c) $x = 0$, (d) $x = 1.5$.
2. $f(x) = \frac{1}{3}x^2 - 3x + 6$ at (a) $x = 2$, (b) $x = 3$, (c) $x = 6$, (d) $x = 0$.
3. $f(x) = x^3 + x$ at (a) $x = 1$, (b) $x = -1$, (c) $x = 0$, (d) $x = 1.2$.

Compute the derivatives of the following functions for arbitrary values of the independent variable:

- | | |
|-----------------------------|----------------------------------|
| 4. $s = 20t + 4.9t^2$. | 5. $s = 16t^2 + 30t$. |
| 6. $s = 2t^3 - t^2$. | 7. $s = 7 + 2t - 4t^2 - 12t^3$. |
| 8. $I = \frac{20}{5 + R}$. | 9. $X_c = \frac{1}{3000C}$. |
| 10. $P = 30I^2$. | 11. $I = \frac{10}{2 + R}$. |

19-7. Differential Calculus. The computation of derivatives as described in the preceding section is troublesome, especially if derivatives of more complicated functions are needed. *The differential calculus is a system of formulas and rules which permits one to find, in a comparatively simple way, the derivatives of practically all functions used in the various applications of mathematics.* The method for computing derivatives is easily understood if one observes that functions defined by complicated formulas are a combination of a small number of simple functions which are connected in a more or less complicated way. Thus, for example, the computation of the function

$$y = \sqrt{x^3 + 5x^2}$$

can be resolved into the following steps:

1. Compute the value of x^2 .
2. Compute the value of $5u$ where $u = x^2$.
3. Compute the value of x^3 .
4. Add the results of step 2 and 3.
5. Compute the function \sqrt{v} where $v = x^3 + 5x^2$.

It is obvious that the given function is a combination of the simple functions $y = x^2$, $y = x^3$, $y = \sqrt{x}$, which are special cases of $y = x^n$.

The fundamental result of the differential calculus is that the derivatives of all such functions which are built up from simpler functions can be computed if

1. The derivatives of a small number of elementary functions are known.

2. Rules are known which permit one to compute the derivative of a function which is a combination of functions whose derivatives are already known.

The purpose of this chapter is to serve as an introduction to differential calculus. Therefore, not all the formulas and rules will be given here, but only those which are needed to obtain a general idea of the calculus and to solve simple problems.

19-8. The Derivatives of x^n , $\sin x$, and $\cos x$. It will be sufficient for our needs to know the derivatives of the function x^n , $\sin x$, and $\cos x$. These derivatives are given without proof by the following formulas:

$$(1) \quad \frac{dx^n}{dx} = nx^{n-1},$$

$$(2) \quad \frac{d \sin x}{dx} = \cos x,$$

$$(3) \quad \frac{d \cos x}{dx} = -\sin x.$$

The formula given in (1) is true for all possible values of n . The following are a few important special cases of (1):

$$n = 0, \quad \frac{dx^0}{dx} = \frac{d1}{dx} = 0,$$

$$n = 1, \quad \frac{dx}{dx} = 1,$$

$$n = 2, \quad \frac{dx^2}{dx} = 2x,$$

$$n = \frac{1}{2}, \quad \frac{dx^{\frac{1}{2}}}{dx} = \frac{d\sqrt{x}}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},$$

$$n = -1, \quad \frac{dx^{-1}}{dx} = \frac{d\left(\frac{1}{x}\right)}{dx} = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

It must be pointed out that formulas (2) and (3) are true only when x is measured in radians. When the degree measure of x is used these formulas have to be replaced by more complicated ones. This is the reason why the radian measure of angles is important in many problems.

19-9. Rules for Computing Derivatives. The second group of fundamental theorems in calculus consists of a set of rules which permits one to find the derivatives of functions formed by means of simpler functions with known derivatives. These rules are given without proof in this section.

Rule 1. The derivative of the product of a constant and a function

$$y = af(x)$$

is given by the formula

$$\frac{dy}{dx} = \frac{d[af(x)]}{dx} = a \frac{df(x)}{dx}.$$

Example 1. Find the derivative of $y = 7x^3$.

Using the above rule we have

$$\frac{d(7x^3)}{dx} = 7 \frac{dx^3}{dx} = 7 \cdot 3x^2 = 21x^2.$$

Example 2. Find the derivative of a constant function $y = C$.

Using Rule 1 and (1) of Sec. 19-8, we obtain

$$\frac{dC}{dx} = \frac{d(C \cdot 1)}{dx} = C \frac{d1}{dx} = 0.$$

Hence the derivative of a constant is zero.

Rule 2. The derivative of the sum of two functions

$$y = f(x) + g(x)$$

is given by the formula

$$\frac{dy}{dx} = \frac{d[f(x) + g(x)]}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}.$$

Example 3. $\frac{d(x^3 + x^5)}{dx} = \frac{dx^3}{dx} + \frac{dx^5}{dx} = 3x^2 + 5x^4$.

Example 4. Find the derivative of $y = 3x^3 - 2x + 5 - 3\sqrt{x} + \frac{1}{2\sqrt{x}}$.

Since Rule 2 can be extended to a sum consisting of any number of terms, we have

$$\begin{aligned} \frac{dy}{dx} &= 3 \frac{dx^3}{dx} - 2 \frac{dx}{dx} + \frac{d5}{dx} - 3 \frac{dx^{\frac{1}{2}}}{dx} + \frac{1}{2} \frac{dx^{-\frac{1}{2}}}{dx} \\ &= 3 \cdot 3x^2 - 2 + 0 - 3 \cdot \frac{1}{2} x^{\frac{1}{2}-1} + \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} \\ &= 9x^2 - 2 - \frac{3}{2\sqrt{x}} - \frac{1}{4x\sqrt{x}}. \end{aligned}$$

Before differentiating an expression, radicals have to be replaced by fractional exponents, as shown in the last example, in order to apply formula (1) of Sec. 19-8.

Example 5. Find the derivative of $E = 10 \sin t + 5 \cos t$.

Using the various formulas and rules, we obtain,

$$\frac{dE}{dt} = \frac{d(10 \sin t)}{dt} + \frac{d(5 \cos t)}{dt} = 10 \frac{d \sin t}{dt} + 5 \frac{d \cos t}{dt} = 10 \cos t - 5 \sin t.$$

Rule 3. The derivative of the product of two functions

$$y = f(x) g(x)$$

is given by the formula

$$\frac{dy}{dx} = \frac{d[f(x)g(x)]}{dx} = g(x) \frac{df(x)}{dx} + f(x) \frac{dg(x)}{dx}.$$

Example 6. Find the derivative of the function $y = (3 + 5x)\sqrt[3]{x}$.

Using Rule 3 we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d[(3 + 5x)\sqrt[3]{x}]}{dx} = \sqrt[3]{x} \frac{d(3 + 5x)}{dx} + (3 + 5x) \frac{dx^{\frac{1}{3}}}{dx} \\ &= \sqrt[3]{x}(0 + 5 \cdot 1) + (3 + 5x) \cdot \frac{1}{3} x^{\frac{1}{3}-1} \\ &= 5\sqrt[3]{x} + \frac{1}{3}(3 + 5x) \frac{1}{\sqrt[3]{x^2}}. \end{aligned}$$

The result can be simplified by rationalizing the second term:

$$\begin{aligned} \frac{dy}{dx} &= 5\sqrt[3]{x} + \frac{1}{3}(3 + 5x) \frac{\sqrt[3]{x}}{x} = 5\sqrt[3]{x} + \frac{\sqrt[3]{x}}{x} + \frac{5}{3} \sqrt[3]{x} \\ &= \frac{20}{3} \sqrt[3]{x} + \frac{\sqrt[3]{x}}{x} = \left(\frac{20}{3} + \frac{1}{x} \right) \sqrt[3]{x}. \end{aligned}$$

Example 7. Find the derivative of $r = 5\theta \sin \theta$.

We obtain

$$\begin{aligned} \frac{dr}{d\theta} &= 5 \frac{d(\theta \sin \theta)}{d\theta} = 5 \sin \theta \frac{d\theta}{d\theta} + 5\theta \frac{d \sin \theta}{d\theta} \\ &= 5 \sin \theta + 5\theta \cos \theta = 5(\sin \theta + \theta \cos \theta). \end{aligned}$$

Rule 4. The derivative of the quotient of two functions

$$y = \frac{f(x)}{g(x)}$$

is given by the formula

$$\frac{dy}{dx} = \frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x) \frac{df(x)}{dx} - f(x) \frac{dg(x)}{dx}}{[g(x)]^2}.$$

Example 8. Find the derivative of $y = \frac{2+3x}{1+x^2}$.

Using the last rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d \frac{2+3x}{1+x^2}}{dx} = \frac{(1+x^2) \frac{d(2+3x)}{dx} - (2+3x) \frac{d(1+x^2)}{dx}}{(1+x^2)^2} \\ &= \frac{(1+x^2)(0+3 \cdot 1) - (2+3x)(0+2x)}{(1+x^2)^2} \\ &= \frac{3+3x^2-4x-6x^2}{(1+x^2)^2} = \frac{3-4x-3x^2}{(1+x^2)^2}.\end{aligned}$$

Example 9. Find the derivative of $\tan \theta$.

Using the relation $\tan \theta = \frac{\sin \theta}{\cos \theta}$, we have

$$\begin{aligned}\frac{d \tan \theta}{d\theta} &= \frac{d \left(\frac{\sin \theta}{\cos \theta} \right)}{d\theta} = \frac{\cos \theta \frac{d \sin \theta}{d\theta} - \sin \theta \frac{d \cos \theta}{d\theta}}{\cos^2 \theta} \\ &= \frac{\cos \theta \cdot \cos \theta - \sin \theta \cdot (-\sin \theta)}{\cos^2 \theta} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta.\end{aligned}$$

Rule 5. If u is a function of x , then the formulas of Sec. 19-8 are replaced by the following:

$$\begin{aligned}\frac{du^n}{dx} &= nu^{n-1} \frac{du}{dx}, \\ \frac{d \sin u}{dx} &= \cos u \cdot \frac{du}{dx}, \\ \frac{d \cos u}{dx} &= -\sin u \cdot \frac{du}{dx}.\end{aligned}$$

Example 10. Compute $\frac{d\sqrt{1+x^2}}{dx}$.

Substituting $u = 1+x^2$ we obtain,

$$\frac{d\sqrt{1+x^2}}{dx} = \frac{du^{\frac{1}{2}}}{dx} = \frac{1}{2} u^{\frac{1}{2}-1} \frac{du}{dx} = \frac{1}{2\sqrt{1+x^2}} \frac{d(1+x^2)}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}.$$

Example 11. Compute $\frac{d \cos 5t}{dt}$.

Substituting $u = 5t$ we obtain,

$$\frac{d \cos 5t}{dt} = \frac{d \cos u}{dt} = -\sin u \cdot \frac{du}{dt} = -\sin u \cdot 5 = -5 \sin 5t.$$

Example 12. Compute $\frac{dI}{dt}$ where $I = I_o \sin(\omega t + \theta)$, I_o , ω , and θ being constants.

According to the rules developed in this section we obtain

$$\begin{aligned}\frac{dI}{dt} &= I_o \frac{d \sin(\omega t + \theta)}{dt} \\ &= I_o \frac{d \sin u}{dt},\end{aligned}$$

where $u = \omega t + \theta$. Hence

$$\frac{dI}{dt} = I_o \cos u \cdot \frac{du}{dt} = I_o \cos(\omega t + \theta) \cdot (\omega + 0) = \omega I_o \cos(\omega t + \theta).$$

EXERCISES

Find the derivatives of the following functions.

1. $y = 5x^2 - 3$.
2. $y = 3x^2 + 2x - 1$.
3. $y = x^3 - 2x + 5$.
4. $u = 3 + \frac{5}{t} - \frac{6}{t^2}$.
5. $q = \frac{2 \cdot 7}{t^2} - \frac{1 \cdot 3}{t}$.
6. $R = \frac{20}{d^2}$.
7. $y = 3\sqrt{x} - 5\sqrt[3]{x}$.
8. $y = \sqrt{3x} + 2\sqrt[3]{5x}$.
9. $y = 3\sqrt[4]{x^4} + 4x\sqrt[3]{x}$.
10. $s = 3 \sin t - 5 \cos t$.
11. $u = 5t^2 - 6 \sin t$.
12. $r = 2 \cos \theta + 3\theta$.
13. $y = 4(1 + x)^5$.
14. $s = \sqrt{10 - t^2}$.
15. $r = \frac{1}{3 + \theta}$.
16. $r = 10 \cos 2\theta$.
17. $E = 110 \sin 377t$.
18. $I = 10 \sin(150t + 12)$.
19. $r = 2 \sin^2 \theta$.
20. $s = 10 \cos^2 t$.
21. $P = 3 \sin^2 5t + 8 \sin^2 10t$.
22. $y = 10x \sin 3x$.
23. $s = t^2 \cos 2t$.
24. $r = 10\theta^3 \sin(2\theta + 5)$.
25. $y = (1 + x)^4 \sqrt{x}$.
26. $s = (1 + t^2)\sqrt{t^5}$.
27. $u = 10(1 + v)^2(5 + 7v)^3$.
28. $y = (3x^2 + 5)^3$.
29. $y = \sqrt{3 + 2x^2}$.
30. $u = \sqrt[3]{1 + x}$.
31. $y = \frac{1}{\sqrt{1 + x}}$.
32. $y = \frac{1}{\sqrt{1 + x^2}}$.
33. $y = \frac{1}{\sqrt{2 + 5x}}$.
34. $y = \frac{2 + x}{3 + x}$.
35. $y = \frac{3x - 5}{2x + 3}$.
36. $s = \frac{3t}{t^2 - 2}$.
37. $y = \frac{1 - x^2}{1 + x^2}$.
38. $u = \frac{2t^2 - 1}{t + 1}$.
39. $v = \frac{u^4 + 1}{u^2 + 1}$.
40. $y = \frac{\sin x}{x}$.
41. $y = \frac{\cos x}{1 + x}$.
42. $y = \frac{\sin x}{\cos 2x}$.
43. $u = \sqrt{1 + \theta \cdot 8 \sin^2 \theta}$.
44. $z = \sqrt{25 + 4\omega^2}$.
45. $z = \sqrt{4 + \frac{1}{3\omega^2}}$.

Compute the derivatives of the following functions of t , assuming that all the other symbols denote constants.

46. $s = \frac{a + bt}{c + dt}.$

47. $u = \sqrt{1 + R^2 \sin^2 t}.$

48. $E = E_o \cos (\omega t + \alpha).$

49. $E = A \sin \omega t + B \cos \omega t.$

50. $W = A \sin^2 \omega t.$

51. $I = I_o + I_1 \sin (\omega t + \alpha_1) + I_2 \sin (2\omega t + \alpha_2) + I_3 \sin (3\omega t + \alpha_3).$

52. $u = a_0 + a_1 \cos t + a_2 \cos^2 t + a_3 \cos^3 t.$

19-10. Applications of the Differential Calculus. There are many problems in geometry and engineering which can be solved by the methods of the preceding sections. A few of them will be discussed in this section.

Example 1. Compute the slope of the tangent of the curve given by the equation $y = \frac{1}{6}(x^2 + 4x - 11)$ at the point P with the abscissa $x = 1$ (Fig. 19-6).

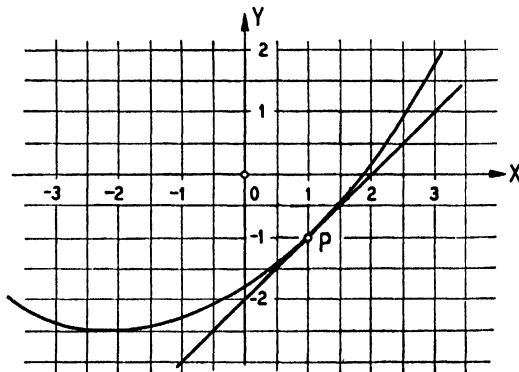


FIG. 19-6.

The required slope is, according to Sec. 19-5, equal to the derivative of y at the point $x = 1$. This derivative is found to be

$$\frac{dy}{dx} = \frac{d\frac{1}{6}(x^2 + 4x - 11)}{dx} = \frac{1}{3}x + \frac{2}{3},$$

and its value for $x = 1$ is 1.

The equation of the tangent line to the given curve at the point $x = 1$, $y = \frac{1}{6}(1 + 4 - 11) = -1$, can now be found, for this is the equation of a straight line passing through the point $(1, -1)$ and having the slope 1. Hence, using the point slope form of the equation of a straight line (Sec. 15-8), the equation of the tangent is found to be

$$y + 1 = x - 1,$$

$$y = x - 2.$$

Example 2. The motion of a particle P along a straight line is completely described if the distance s between a point of reference O on this line and the particle (Fig. 19-7) is given as a function $s = f(t)$ of the time t .

The speed v of the particle at the time t is the instantaneous rate of change of s with respect to t and, therefore, can be found by differentiating the function $f(t)$ with respect to t ,

$$v = \frac{ds}{dt} = \frac{df(t)}{dt}.$$

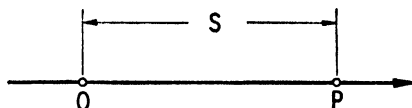


FIG. 19-7.

In Sec. 5-16 we discussed the *harmonic* motion of a point described by the equation $s = A \sin \omega t$. The speed of the moving point at the time t is given by

$$v = \frac{ds}{dt} = \frac{d(A \sin \omega t)}{dt} = A \omega \cos \omega t.$$

Now the speed of the moving particle is itself a function of the time. The rate of change of this function with respect to time is called **acceleration**, often denoted by the letter a . For the harmonic motion we have

$$a = \frac{dv}{dt} = \frac{d(A \omega \cos \omega t)}{dt} = -A \omega^2 \sin \omega t.$$

The acceleration is found by computing the derivative $\frac{ds}{dt} = \frac{df(t)}{dt} = v$ and by differentiating this result another time, so that

$$a = \frac{dv}{dt} = \frac{d\left(\frac{df(t)}{dt}\right)}{dt}.$$

This value which has been obtained by differentiating the result of a first differentiation is called the **second derivative** of $s = f(t)$ with respect to t . It is generally denoted by $\frac{d^2s}{dt^2} = \frac{d^2f(t)}{dt^2}$ or simply $s'' = f''(t)$ (read: s two prime, or s double prime, f two prime of t or f double prime of t). In this connection, $f'(t) = \frac{df(t)}{dt}$ is often called the **first derivative** of $f(t)$.

The result of the preceding discussion is summarized in the following statement: *The speed of a particle moving along a straight line is the first derivative of the distance s with respect to time and the acceleration is the second derivative of the distance s with respect to time, where s is measured from a fixed point of reference to the moving particle.*

Example 3. The electromotive force E , across a circuit of resistance R ohms and inductance L henries, when a variable current I amperes is flowing, is given by

$$E = RI + L \frac{dI}{dt}.$$

This equation can be used to compute E when I is given. In an alternating-current circuit I may be given by the equation

$$I = I_0 \sin \omega t.$$

The voltage across this circuit is then obtained as follows.

$$E = RI + L \frac{dI}{dt} = RI_0 \sin \omega t + L \frac{d(I_0 \sin \omega t)}{dt} = RI_0 \sin \omega t + I_0 L \omega \cos \omega t.$$

EXERCISES

Find the equations of the tangents of the following curves at the indicated points.

1. $y = \frac{1}{2}x^2 - 1$, (a) $x = 2$, (b) $x = -3$, (c) $x = 0$.

2. $y = \sqrt{1 - x^2}$, (a) $x = \frac{1}{2}$, (b) $x = -0.6$, (c) $x = 0$.

3. $y = \sin x$, (a) $x = 0$, (b) $x = \frac{\pi}{6}$, (c) $x = \frac{\pi}{2}$.

4. $y = 2 \cos x - 3 \cos 2x$, (a) $x = 0$, (b) $x = \frac{\pi}{6}$, (c) $x = \frac{\pi}{2}$.

Find the speed and the acceleration at the time t of a particle moving according to the following equations.

5. $s = 5t^3$.

6. $s = 6t^4 - 10t^2$.

7. $s = \frac{1}{t^2 + 1}$.

8. $s = v_0 t + \frac{g}{2} t^2$.

9. $s = A_1 \sin \omega t + A_2 \sin 2\omega t$.

10. $s = 5t \sin 2t$.

A circuit has a resistance of 40 ohms and an inductance of 0.2 henry. Find the electromotive force across this circuit if the current is

11. $I = 5$ amperes.

12. $I = (0.1t + 2)$ amperes.

13. $I = 10 \sin (377t + 0.1)$ amperes.

14. $I = \cos (0.5t + 3)$ amperes.

19-11. Maxima and Minima. There are many practical problems where it is necessary to obtain a quantity as great or as small as possible. Examples of such problems are given in the following statements: Find the sides of the greatest rectangular lot which can be fenced with 200 ft. of wire. Find the dimensions of a tin can of given volume if as little sheet metal as possible should be used for its construction. Find the value of the resistance which must be in series with a source of electricity if the power output should be as large as possible. The differential calculus permits us to develop a method for dealing with these and similar problems. All problems of this kind, when expressed in mathematical language, can be stated in the following way: *Find the particular values of x for which a given function $y = f(x)$ assumes its greatest or smallest value, and compute this value.* The first of the problems mentioned above will be used to illustrate the subject of the present section.

Our problem is to find the sides of the greatest rectangular lot which can be fenced with 200 ft. of wire. Let x and x_1 denote the unknown sides of the lot. The sum of the four sides (perimeter) of this lot is given by

$$x + x + x_1 + x_1,$$

and hence we have the equation

$$2x + 2x_1 = 200,$$

or

$$x_1 = (100 - x) \text{ feet.}$$

Now the area of this lot is given by

$$y = xx_1 = x(100 - x) \text{ square feet.}$$

We have to find a value of x for which y has the greatest value. For this purpose the following table of corresponding values of x and y is constructed:

x	$x_1 = 100 - x$	$y = x(100 - x)$
0	100	0
10	90	900
20	80	1600
30	70	2100
40	60	2400
50	50	2500
60	40	2400
70	30	2100
80	20	1600
90	10	900
100	0	0

From this table and from the corresponding graph (Fig. 19-8), it can be seen that the greatest value or **maximum** of the area is 2500 sq. ft. and that the lot having this area is a square whose sides are 50 ft. in length.

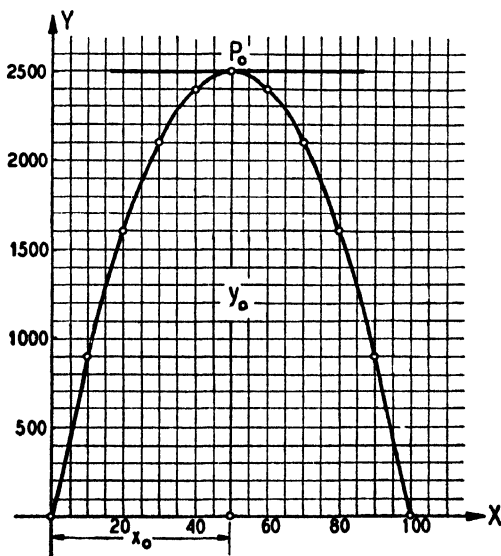


FIG. 19-8.

The next step is to solve the above problem without constructing a table or plotting a graph. From Fig. 19-8 it is obvious that the tangent at P_0 , the highest point of the graph, is parallel to the x -axis and,

therefore, has slope zero. From Sec. 19-5 it is known that the slope of the tangent is equal to the derivative of the function of the graph. Hence it can be stated that the derivative of this function, computed at the point P_0 , is zero. Now the equation of the graph of Fig. 19-8 is given by

$$y = x(100 - x) = 100x - x^2,$$

and hence its derivative is

$$\frac{dy}{dx} = 100 - 2x.$$

This derivative is zero when computed for $x = x_0$, the abscissa of P_0 . Hence the following equation for x_0 is obtained:

$$100 - 2x_0 = 0,$$

and therefore

$$x_0 = 50,$$

$$y_0 = x_0(100 - x_0) = 2500.$$

These values are the same as those previously obtained by constructing a table of corresponding values of the sides and the area of the lot.

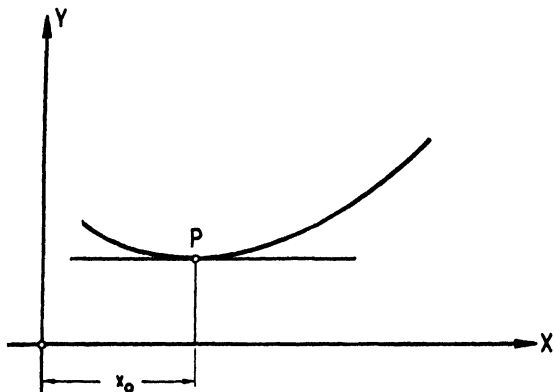


FIG. 19-9.

The same method could be used to find the smallest value or **minimum** of a function given by $y = f(x)$. Assume that the graph of this function is given in Fig. 19-9 and P is the lowest point of the graph. Again it can be stated that $f'(x) = 0$ for $x = x_0$, because the tangent at P has slope zero.

The foregoing discussion shows that the problem of finding the maximum or minimum of a function $y = f(x)$ may be reduced to the problem of solving the equation

$$\frac{df(x)}{dx} = 0.$$

Assume that the graph of some function $y = f(x)$ is plotted in Fig. 19-10. The roots of the equation

$$\frac{df(x)}{dx} = 0$$

are the abscissas of the points P_1, P_2, P_3 where the tangent is parallel to the x -axis. The figure shows that $Q_1P_1 = f(x_1)$ is the maximum of $y = f(x)$, but that $Q_2P_2 = f(x_2)$ is not the *smallest* value of y . The ordinate Q_2P_2 is only *smaller* than the ordinates of the other points in the neighborhood of P_2 , but there are points whose ordinates are smaller

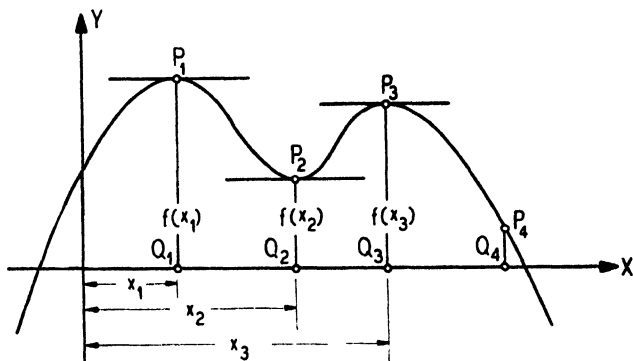


FIG. 19-10.

than Q_2P_2 . For example, P_4 is such a point since Q_4P_4 is smaller than Q_2P_2 . At points like P_2 the given function is said to have a **relative minimum**. Similarly the ordinate Q_3P_3 is a **relative maximum** of the function. There are many problems in which it is important to find such a relative minimum or a relative maximum. *In the following discussion, the expressions maximum and minimum are used to denote a relative maximum or relative minimum.* The value $f(x_1) = Q_1P_1$ is then called an **absolute maximum**.

The solution of the equation $\frac{df(x)}{dx} = 0$ yields the abscissas of the points P_1, P_2, P_3 , but it does not settle the question whether, at any one of these points, the function has a maximum or a minimum. This question can be answered by plotting the graph although it is sometimes clear from the nature of the problem whether we have a maximum or minimum. A better method for determining whether a point is a maximum or a minimum is given by the following rule:

The function $y = f(x)$ has a maximum or a minimum for $x = x_0$ if, at this point, the first derivative $f'(x_0) = 0$ and the second derivative $f''(x_0)$ is respectively negative or positive.

The results of this section can now be summarized in the following statement.

In order to find the maxima or minima of a function $y = f(x)$, compute the first derivative $f'(x)$ and the second derivative $f''(x)$. Then solve the equation $f'(x) = 0$. If $x = x_0$ is a particular root of this equation, the value $f(x_0)$ is a maximum if $f''(x_0) < 0$, and a minimum if $f''(x_0) > 0$.

A few examples will illustrate the application of the material developed in this section.

Example 1. Find the maxima and minima of the function

$$y = f(x) = \frac{x^3}{4} - \frac{3x^2}{4} + 2.$$

According to the rule given above we find the first and second derivatives,

$$f'(x) = \frac{3x^2}{4} - \frac{3x}{2},$$

$$f''(x) = \frac{3}{2}x - \frac{3}{2}.$$

Solving the equation

$$f'(x) = \frac{3x^2}{4} - \frac{3x}{2} = 0,$$

we obtain

$$\frac{3}{4}x(x - 2) = 0,$$

$$x_1 = 0, \quad x_2 = 2.$$

Computing $f''(x)$ for these values we get

$$f''(0) = -\frac{3}{2} < 0, \quad f''(2) = \frac{3}{2} > 0.$$

It can be stated, therefore, that the value $f(0) = 2$ is a maximum and the value $f(2) = 1$ is a minimum of the given function. These results are illustrated by the graph of Fig. 19-11.

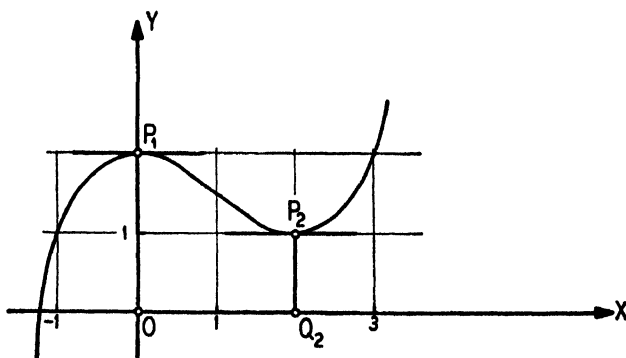


FIG. 19-11.

Example 2. What is the resistance R of a load connected in series with a battery of electromotive force E and internal resistance R_i if the power output is to be as great as possible?

The power output P is given by the formula

$$P = I^2 R.$$

By Ohm's law we have

$$I = \frac{E}{R_i + R}.$$

We obtain, therefore,

$$P = \frac{E^2 R}{(R_i + R)^2}.$$

In order to find the value of R for which P is a maximum, we have to compute $\frac{dP}{dR}$ and $\frac{d^2 P}{dR^2}$ from the last formula. This computation is troublesome in this case and can be simplified by observing that $\frac{1}{P}$ has a minimum when P has a maximum.

Hence if we substitute $\frac{1}{P} = u$ in the expression for P , we obtain

$$u = \frac{(R_i + R)^2}{E^2 R} = \frac{R_i^2 + 2R_i R + R^2}{E^2 R} = \frac{1}{E^2} \left(\frac{R_i^2}{R} + 2R_i + R \right).$$

Finding the first and second derivative we have,

$$\begin{aligned} \frac{du}{dR} &= \frac{1}{E^2} \left(-\frac{R_i^2}{R^2} + 1 \right), \\ \frac{d^2 u}{dR^2} &= \frac{1}{E^2} \left(\frac{2R_i^2}{R^3} \right) = \frac{2R_i^2}{E^2 R^3}. \end{aligned}$$

Equating the first derivative $\frac{du}{dR}$ to zero, we obtain

$$\frac{1}{E^2} \left(-\frac{R_i^2}{R^2} + 1 \right) = 0,$$

and hence

$$R = R_i \quad \text{or} \quad R = -R_i.$$

Since the negative value $R = -R_i$ has no physical meaning in our problem, we discard it and use the value $R = R_i$. When this value $R = R_i$ is substituted in the

second derivative $\frac{d^2 u}{dR^2} = \frac{2R_i^2}{E^2 R^3}$, the result is positive, so that the corresponding

value of u is a minimum and hence the value $P = \frac{1}{u}$ is a maximum. The power output, therefore, is a maximum if the load resistance R is equal to the internal resistance R_i . Substituting $R = R_i$ in the expression for P , we get that the maximum power output is given by

$$P = \frac{E^2 R_i}{4R_i^2} = \frac{E^2}{4R_i}.$$

EXERCISES

Find the maxima and minima of the following functions.

1. $y = 2x^2 - 3x + 7.$

2. $s = 100t - 16t^2.$

3. $m = 0.23N^2 + 0.18N - 1.2.$

4. $y = x^3 - 6x^2 - 15x - 6.$

5. $y = 4x^3 + 3x^2 - 90x + 144.$

6. $y = 3x^3 - x.$

7. $x = t + \frac{1}{t}.$

8. $y = \frac{x^2}{3} + 2 + \frac{27}{x^2}.$

9. $y = \frac{1}{1 + x^2}.$

10. $s = \sin t + \cos t.$

11. $s = \cos t + \cos 2t.$

12. $u = 8 \sin t - \tan t.$

13. An open box is formed from a square sheet of tin, the side of which is 18 in., by cutting off equal squares at each corner and turning up the sides. What size of squares should be cut out in order to obtain a box of maximum content?

14. Solve the problem of Exercise 13 if, instead of the square piece of tin, a rectangular sheet 15 by 24 in. is given.

15. Solve the problem of Exercise 13 for a rectangular sheet whose sides are a and b inches.

16. A closed cylindrical can is to be made so that its volume is 54 cu. in. Find its dimensions if the total surface is to be a minimum.

17. A cylindrical can open at one end is to be made so that its volume is 64 cu. in. and its surface as small as possible. Find the diameter and the height of the can.

18. Find the side and height of a parallelepiped with a square base 27 cu. in. in volume if the surface is to be a minimum.

✓ 19. A water tank is to be constructed having a square base, an open top, and vertical walls; it is to contain 256 cu. yd. Find the side of the base and the height if the sum of the area of the bottom and the walls is to be a minimum.

✓ 20. The time, in seconds, needed for an object to slide down an inclined plane, if friction and air resistance are neglected, is given by the formula

$$t = \frac{1}{4} \sqrt{\frac{2L}{\sin 2\theta}},$$

where θ is the angle between the inclined plane and its base and L is the length of the base measured in feet. Find the value of θ which makes t a minimum.

✓ 21. The power output P of a battery of electromotive force E and internal resistance R_i is given by

$$P = EI - RI^2,$$

where I is the current delivered by the battery. Find the current for which the power output is a maximum.

✓ 22. The power developed in watts in an alternating-current circuit is given by

$$P = \frac{E^2 R}{R^2 + X^2},$$

where E is the impressed voltage, R the resistance in ohms, and X the reactance in ohms. If E and X have constant values, compute the value of R which makes P a maximum.

✓23. If, in an electric transformer, the primary current is an alternating current of impressed voltage E and frequency f and if both circuits are properly tuned, then the secondary current is given by the formula,

$$I = \frac{E\omega M}{R_1 R_2 + \omega^2 M},$$

where $\omega = 2\pi f$, M the mutual inductance, and R_1 and R_2 the resistances of the two circuits. Find the value of M for which I is as large as possible and compute this maximum.

19-12. Increments and Differentials. The computation of an increment of a function $y = f(x)$ according to the formula (Sec. 19-1)

$$\Delta y = f(x + \Delta x) - f(x)$$

is often troublesome. In many cases the computation can be greatly simplified if the increment Δx is small and only an approximate value of Δy is required, conditions which are satisfied in many practical problems.

Assume that P and P_1 are the two points on the graph of $y = f(x)$, Fig. 19-12, which correspond to the abscissas x and $x + \Delta x$. The increment Δy of the ordinate is given by the segment $P_1 Q$. Now construct

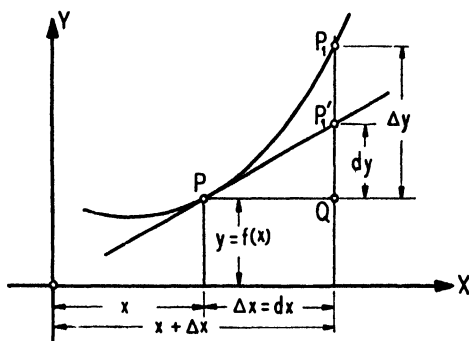


FIG. 19-12.

the tangent to the curve at P and let P_1' be the point on the tangent which has the same abscissa as P_1 . In Sec. 19-5, the term differential was used for each of the increments connected with a point on the tangent, and the notations dx and dy were adopted. Accordingly, $dx = PQ$, $dy = QP_1'$, and $\Delta x = dx$.

The differential dy is easier to compute than the increment Δy because, according to Sec. 19-5, $\frac{dy}{dx}$ is the derivative $f'(x)$ computed at the point P , and therefore

$$dy = f'(x)dx.$$

From Fig. 19-12 it is obvious that dy is an approximation for Δy when $\Delta x = dx$ is small, because the points P_1 and P_1' approach one another if Q approaches P . This discussion permits us to state the following rule:

The increment of $y = f(x)$ for a small increment of x , denoted by $\Delta x = dx$, is given approximately by the formula for the differential

$$dy = f'(x)dx.$$

A few examples will illustrate the use of the differential dy as an approximation to the increment Δy .

Example 1. Compute the differential of $u = \sqrt{1+v^2}$.

Since $u = f(v)$, and

$$\frac{du}{dv} = f'(v) = \frac{v}{\sqrt{1+v^2}},$$

it follows that

$$du = \frac{v}{\sqrt{1+v^2}} dv.$$

Example 2. What is the increase of the volume V of a sphere if its radius r is increased from 24 in. to 24.2 in.? The volume of a sphere is given by the formula

$$V = \frac{4\pi r^3}{3}.$$

Using the differential dV as an approximation to ΔV in order to simplify the computation, we obtain

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r^2 \\ dV &= 4\pi r^2 dr.\end{aligned}$$

Since $r = 24$ and $dr = 0.2$, we obtain

$$dV = 4\pi(24)^2(0.2) = 1450 \text{ cu. in.},$$

the computation being carried out by the slide rule.

In order to investigate how close this value is to the correct value ΔV , it is necessary to compute V for $r = 24$ in. and $r = 24.2$ in. and to subtract the first value from the second. The results obtained by use of the slide rule are not precise enough for this purpose. Use of logarithms gives the following results.

$$V = \frac{4\pi}{3} (24)^3 = 57,900 \text{ cu. in.},$$

$$V + \Delta V = \frac{4\pi}{3} (24.2)^3 = 59,360 \text{ cu. in.},$$

$$\Delta V = 1460 \text{ cu. in.}$$

The error made by using $dV = 1450$ instead of $\Delta V = 1460$ is less than 1 per cent.

✓ *Example 3.* The frequency of the electric oscillations which can be produced by a circuit of inductance of L henries and capacitance of C farads is

$$f = \frac{1}{2\pi\sqrt{LC}} \text{ cycles per second.}$$

What is the approximate change of f if C is increased by 1 per cent?

In this problem, f is regarded as a function of C , and the value of df is to be computed for $dC = 0.01C$. We compute first the derivative

$$\frac{df}{dC} = \frac{d}{dC} \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{L}} \frac{dC^{-\frac{1}{2}}}{dC} = \frac{1}{2\pi\sqrt{L}} \left(-\frac{1}{2} \right) C^{-\frac{3}{2}} = -\frac{1}{4\pi C\sqrt{LC}} = -\frac{f}{2C}.$$

Then

$$df = -\frac{f}{2C} \cdot dC = -\frac{f}{2} \cdot \frac{dC}{C} = -\frac{0.01}{2} f = -0.005f.$$

This result shows that the frequency decreases approximately $\frac{1}{2}$ per cent if the capacitance is increased by 1 per cent.

✓ *Example 4.* Compute approximately $\sin 30.1^\circ$, using the known value $\sin 30^\circ = 0.5$.

Since

$$\Delta \sin x = \sin(x + \Delta x) - \sin x$$

or

$$(1) \quad \sin(x + \Delta x) = \sin x + \Delta \sin x$$

we can compute $\sin 30.1^\circ$ by computing (1) for $x = 30^\circ$ and $\Delta x = 0.1^\circ$.

As an approximation to the term $\Delta \sin x$ we shall use the differential

$$(2) \quad d \sin x = \cos x \, dx.$$

We stressed in Sec. 19-8 that in the formulas for the differentiation of trigonometric functions it is assumed that x is measured in radians. Hence

$$dx = 0.1^\circ = (0.1) \frac{\pi}{180} \text{ radians} = 0.00175 \text{ radians,}$$

and therefore, if $x = 30^\circ = \frac{\pi}{6}$ radians, we have by (2)

$$d \sin x = \left(\cos \frac{\pi}{6} \right) (0.00175) = (0.8660)(0.00175) = 0.0015.$$

By (1) we have that

$$\begin{aligned} \sin 30.1^\circ &= \sin 30^\circ + 0.0015 \\ &= 0.5000 + 0.0015 \\ &= 0.5015. \end{aligned}$$

The approximation in this case is so good that all four digits of the result are correct as the student may verify from the tables of trigonometric functions.

EXERCISES

Compute the differentials of the following functions.

1. $y = 7x^3 - 5x + 1.$
2. $y = 3x^4 + 7.$
3. $y = 1 - 2x^2.$
4. $y = \frac{1}{x}.$
5. $y = \sqrt{x} + \frac{1}{\sqrt{x}}.$
6. $y = \frac{1}{\sqrt{1-x^2}}.$
7. $y = 10 \sin x$
8. $y = 5 \sin x - 3 \cos x.$
9. $y = \sin 2x.$

10. Compute approximately the volume of a hollow sphere of internal radius $r = 4$ in. and thickness $a = 0.1$ in.

11. How much does the volume of a cube with 5-ft. sides increase if each side is increased by 1 in.?

12. The diameter and height of a cylindrical tin can are 4 in. and 7 in., respectively. By how much will the volume of the can be increased if the diameter is 4.1 in., the height remaining unchanged?

13. The impedance of an alternating-current circuit, measured in ohms, is

$$Z = \sqrt{R^2 + 4\pi^2 f^2 L^2},$$

where R is the resistance in ohms, f the frequency in cycles per second, and L the inductance in henries. Compute approximately the increase of Z if

(a) $R = 10$ ohms, $L = 0.5$ henry, $f = 60$ cycles per second, and f increases to 60.5 cycles per second.

(b) $R = 120$ ohms, $L = 4 \cdot 10^{-3}$ henry, $f = 5 \cdot 10^5$ cycles per second, and f increases to $5.05 \cdot 10^5$ cycles per second.

(c) $R = 180$ ohms, $L = 0.20$ henry, $f = 1000$ cycles per second, and L increases to 0.21 henry.

Compute, approximately, the following quantities.

14. $\sqrt{16.2}$, starting from the value $\sqrt{16} = 4.$

15. $\sqrt[5]{32.3}$, using the value $\sqrt[5]{32} = 2.$

16. $\sqrt[3]{1.05}$, using $\sqrt[3]{1} = 1.$

17. $\sin 60.2^\circ.$

18. $\cos 60.2^\circ.$

19. $\tan 45.3^\circ.$

PROGRESS REPORT

In order to study how a function varies when the independent variable changes, the concepts of corresponding increments, average rate of change, and instantaneous rate of change were introduced. The new idea of an instantaneous rate, or derivative, was carefully investigated by means of the concept of limit. The computation of derivatives and their use in solving problems in geometry and engineering were then shown.

CHAPTER 20

THE ELEMENTS OF INTEGRAL CALCULUS

Newton and Leibnitz made one of the greatest discoveries of mathematics when they found that there is a close connection between computing an area and finding a function whose derivative is given, and that the first problem can be reduced to the second. The consequences of this discovery together with the solution of the second problem are of the utmost importance and possess innumerable applications in engineering and the physical sciences.

20-1. The Approximate Computation of an Area. The problem of computing an area is often presented in the following way. The graph

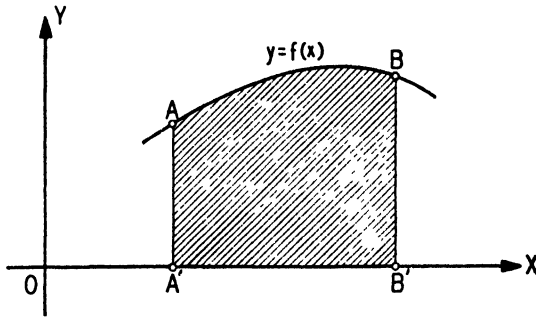


FIG. 20-1.

of a function $y = f(x)$ is plotted (Fig. 20-1) and the shaded area bounded by the graph AB and the segments AA' , BB' , and $A'B'$ is to be found. In order to obtain an approximate value for this area, the segment $A'B'$ is divided into a number of equal parts. For example, in Fig. 20-2 the segment $A'B'$ is divided into 5 equal parts. The length of each of these parts is denoted by Δx . Now in each of these parts select a point arbitrarily. In this way we select P_1, P_2, \dots , whose abscissas are denoted by x_1, x_2, \dots . The points on the curve with these abscissas are denoted by Q_1, Q_2, \dots , whence $P_1Q_1 = f(x_1)$, $P_2Q_2 = f(x_2)$, \dots . Now, as shown in the figure, rectangles are constructed whose bases are the segments of length Δx on the x -axis and whose altitudes are the

segments P_1Q_1, P_2Q_2, \dots . The area of a rectangle is the product of the length of its base and its altitude and therefore,

$$\text{Area of rectangle I} = (P_1Q_1) \Delta x = f(x_1) \cdot \Delta x$$

$$\text{Area of rectangle II} = (P_2Q_2) \Delta x = f(x_2) \cdot \Delta x$$

...

The sum of these areas

$$\text{I} + \text{II} + \dots = (P_1Q_1) \Delta x + (P_2Q_2) \Delta x + \dots$$

$$= f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \dots$$

is obviously an approximation for the area $ABB'A'$. The error made by using the sum of the rectangles instead of the correct, but unknown

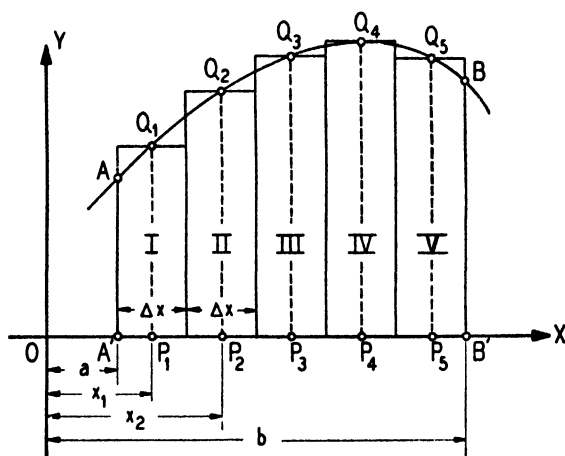


FIG. 20-2.

value of the area $ABB'A'$ becomes smaller if the number of segments into which $A'B'$ is divided is increased. Thus the area can be found with any desired degree of precision if the number of segments is large enough or, what means the same thing, if the bases Δx are small enough.

The sum

$$f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \dots$$

which is used above can be denoted symbolically by

$$Sf(x) \cdot \Delta x,$$

where the symbol S denotes that a series of similar terms $f(x_1) \cdot \Delta x, f(x_2) \cdot \Delta x, \dots$ are to be added. In order to indicate that the area can be found as precisely as we desire by taking Δx small enough and com-

putting these sums, the symbol dx is used instead of Δx , and the sum is written with the symbol \int instead of S :

$$\text{Area } ABB'A' = \int f(x) dx.$$

In order to indicate that the area bounded by the lines $x = a$ and $x = b$ is to be computed, we write

$$\text{Area} = \int_a^b f(x) dx.$$

The right member of this equation is called **the definite integral of $f(x) dx$ between the limits a and b or from a to b** . This discussion has shown that the definite integral is an area; the symbol \int (read "the integral of") indicates the method of computing the area.

For practical computation it is advantageous to replace the rectangles of Fig. 20-2 by trapezoids as shown in Fig. 20-3. The area of a trapezoid

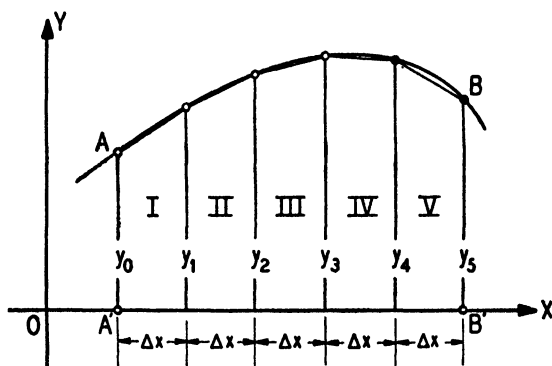


FIG. 20-3.

is the product of one-half the sum of the two parallel sides and the distance between them. If the notations of Fig. 20-3 are used, these areas are

$$\text{I} = \frac{1}{2}(y_0 + y_1) \Delta x,$$

$$\text{II} = \frac{1}{2}(y_1 + y_2) \Delta x,$$

$$\text{III} = \frac{1}{2}(y_2 + y_3) \Delta x,$$

$$\text{IV} = \frac{1}{2}(y_3 + y_4) \Delta x,$$

$$\text{V} = \frac{1}{2}(y_4 + y_5) \Delta x,$$

whence

$$I + II + III + IV + V$$

$$\begin{aligned} &= \frac{\Delta x}{2} [(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + (y_4 + y_5)] \\ &= \frac{\Delta x}{2} (y_0 + y_5 + 2y_1 + 2y_2 + 2y_3 + 2y_4) \\ &= \Delta x \left(\frac{y_0 + y_5}{2} + y_1 + y_2 + y_3 + y_4 \right). \end{aligned}$$

When the interval $A'B'$ is divided into any number of equal parts, a similar formula can be found. The general result of this discussion is:

In order to find an approximate value for the area $ABB'A'$ bounded by a curve, the axis of the abscissas, and two ordinates, divide the segment $A'B'$ into n equal parts, the length of each part being denoted by Δx . Then find the ordinates $y_0, y_1, y_2, \dots, y_n$ corresponding to the division points of $A'B'$. The formula

$$\Delta x \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right)$$

gives an approximation for the area $ABB'A'$. This formula is known as the trapezoidal formula.

A few examples will show how this formula is applied.

Example 1. Compute the area S bounded by the graph of $y^2 = 4x$, the ordinates corresponding to $x = 4$ and $x = 9$, and the x -axis, as shown in Fig. 20-4.

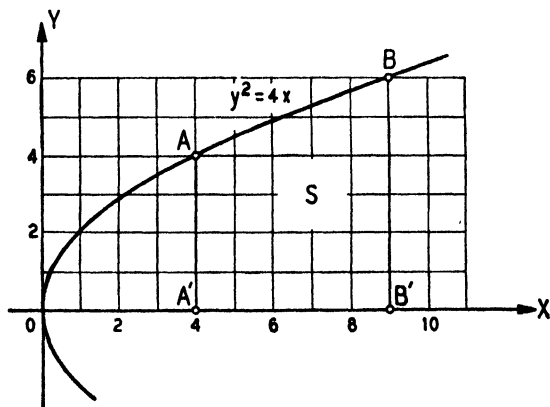


FIG. 20-4.

If the segment $A'B'$ is divided into five equal parts each of length $\Delta x = 1$, the following table can be constructed giving corresponding values of x and $y = \sqrt{4x}$, the values of x being the points of division of the segment $A'B'$.

x	y
4	4.000
5	4.472
6	4.899
7	5.292
8	5.657
9	6.000

The trapezoidal formula gives as an approximate value of the area

$$S = \int_4^9 \sqrt{4x} \, dx = 1 \left(\frac{4.000 + 6.000}{2} + 4.472 + 4.899 + 5.292 + 5.657 \right) \\ = 5.000 + 20.320 = 25.320.$$

Since the correct value is 25.333, the value obtained by this computation is correct to about four significant figures, a very good approximation.

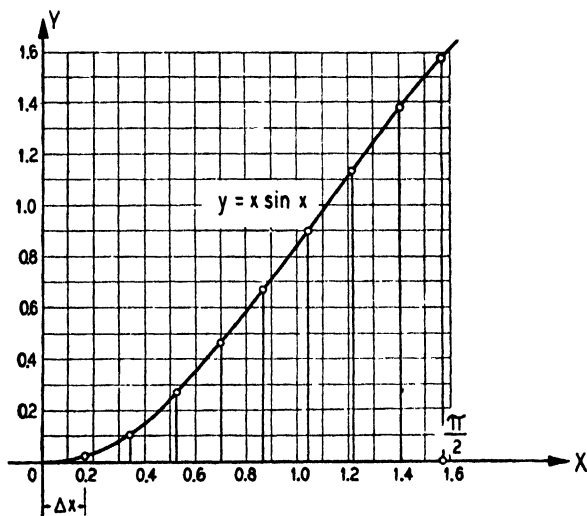


FIG. 20-5.

Example 2. Compute the area denoted by the integral $\int_0^{\pi/2} x \sin x \, dx$. The graph of the curve $y = x \sin x$ is plotted in Fig. 20-5. The area to be computed is included by the graph, the x -axis from 0 to $\frac{\pi}{2}$, and the line $x = \frac{\pi}{2}$. The interval from 0 to $\frac{\pi}{2}$ is divided in 9 equal segments of the length $\Delta x = \frac{\pi}{18} = 0.1745$, corre-

sponding to 10° . In order to apply the trapezoidal formula, the following table is computed, the computations being performed with the slide rule.

NUMBER	x	$\sin x$	$y = x \sin x$
0	0°	0.0000	0.000
1	$10^\circ = \frac{\pi}{18} = 0.174$	0.1736	0.030
2	$20^\circ = 2 \frac{\pi}{18} = 0.349$	0.3420	0.119
3	$30^\circ = 3 \frac{\pi}{18} = 0.524$	0.5000	0.262
4	$40^\circ = 4 \frac{\pi}{18} = 0.698$	0.6428	0.449
5	$50^\circ = 5 \frac{\pi}{18} = 0.873$	0.7660	0.668
6	$60^\circ = 6 \frac{\pi}{18} = 1.047$	0.8660	0.907
7	$70^\circ = 7 \frac{\pi}{18} = 1.222$	0.9397	1.148
8	$80^\circ = 8 \frac{\pi}{18} = 1.396$	0.9848	1.375
9	$90^\circ = \frac{\pi}{2} = 1.571$	1.0000	1.571

By the trapezoidal formula, we add the average of the first and last y -values to the sum of the others.

$$\frac{y_0 + y_9}{2} + y_1 + y_2 + \cdots + y_8 = 0.786 + 4.958 = 5.744.$$

The area is this sum multiplied by $\Delta x = 0.1745$. The result is

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = S = 0.1745 \cdot 5.744 = 1.000.$$

EXERCISES

Using the trapezoidal rule, find to three significant figures the area bounded by the given curve, the given two vertical lines, and the x -axis. Use the given value of Δx .

- $y = x^2, x = 0, x = 3; \Delta x = 1.$
- $y = x^3, x = 1, x = 4; \Delta x = 1.$
- $y = \sqrt{x}, x = 0, x = 8; \Delta x = 2.$
- $y = \sqrt[3]{x}, x = 0, x = 8; \Delta x = 2.$
- $y = \sin x, x = 0, x = \frac{\pi}{2}; \Delta x = \frac{\pi}{8}.$
- $y = \cos x, x = 0, x = \frac{\pi}{2}; \Delta x = \frac{\pi}{10}.$
- $y = \tan x, x = 0, x = \frac{\pi}{3}; \Delta x = \frac{\pi}{12}.$
- $y = \frac{1}{x}, x = 1, x = 4; \Delta x = \frac{1}{2}.$
- $y = \log x, x = 1, x = 5; \Delta x = \frac{1}{2}.$
- $y = 2^x, x = 0, x = 5; \Delta x = \frac{1}{2}.$

Compute the area denoted by the given integral to three significant figures, using the given value of Δx .

$$11. \int_0^4 x^2 dx, \Delta x = 1.$$

$$12. \int_0^3 3^x dx, \Delta x = \frac{1}{2}.$$

$$13. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx, \Delta x = \frac{\pi}{6}.$$

$$14. \int_0^4 x^{\frac{3}{2}} dx, \Delta x = \frac{1}{2}.$$

$$15. \int_1^4 x^{-\frac{1}{2}} dx, \Delta x = \frac{1}{2}.$$

$$16. \int_0^{\frac{\pi}{2}} x \cos x dx, \Delta x = \frac{\pi}{12}.$$

$$17. \int_0^4 \sqrt{16 - x^2} dx, \Delta x = 1.$$

$$18. \int_0^3 \sqrt{9 - x^2} dx, \Delta x = \frac{1}{2}.$$

$$19. \int_0^{\frac{\pi}{2}} x^2 \sin x dx, \Delta x = \frac{\pi}{12}.$$

$$20. \int_0^{\pi} x^2 \sin 2x dx, \Delta x = \frac{\pi}{12}.$$

21. Compute the value of $\int_0^5 x^2 dx$ using the trapezoidal rule and $\Delta x = 1$. Then compute the same integral using $\Delta x = \frac{1}{2}$. If the correct value of the integral is $\frac{125}{3}$, by how much does each approximation differ from the correct value and what is the percent of error in each case?

22. The graph of $y = \sqrt{4 - x^2}$ is a semi-circle with center at the origin and lying above the x -axis. Compute the area of the semi-circle by the trapezoidal rule, using $\Delta x = \frac{1}{2}$, and find the amount by which this approximation differs from the correct result.

23. The effective value of a sinusoidal alternating current with a peak value of I_0 amperes is kI_0 amperes, where

$$k = \sqrt{\frac{1}{\pi} \int_0^{\pi} \sin^2 x dx}.$$

Compute the value of k correct to three significant figures.

24. The average value of a sinusoidal alternating current with a peak value of I_0 amperes is mI_0 , where

$$m = \frac{1}{\pi} \int_0^{\pi} \sin x dx.$$

Compute the value of m correct to three significant figures.

20-2. The Definite Integral and the Inverse Operation of Differentiation. The method given in the preceding section for computing approximate values of an area made use of a table of values of the function corresponding to equally spaced values of the independent variable, but did not make use of any further properties of the function defining the area. When a function is defined by a mathematical formula, it is desirable to find the area under the curve precisely, without the use of the approximation method of Sec. 20-1. That such a precise result can be achieved by using the properties of the function defining the area is one of the most important discoveries of mathematics. How this precise result can be obtained will be discussed in this section.

The problem considered in Section 20-1 was: Given a function $y = f(x)$, compute the area bounded by an arc of the graph of $y = f(x)$, the two lines $x = a$ and $x = b$, and the x -axis. Now we shall investigate how the area S changes if the value of b changes, or using functional language we shall investigate the area S as a function of b . If we write x instead of b we may state the problem as follows. Investigate the area S under the curve $y = f(x)$ and above the x -axis between the abscissas a and x (see Fig. 20-6). Since this area depends on x , it is a function of x , which we shall denote by $S(x)$.

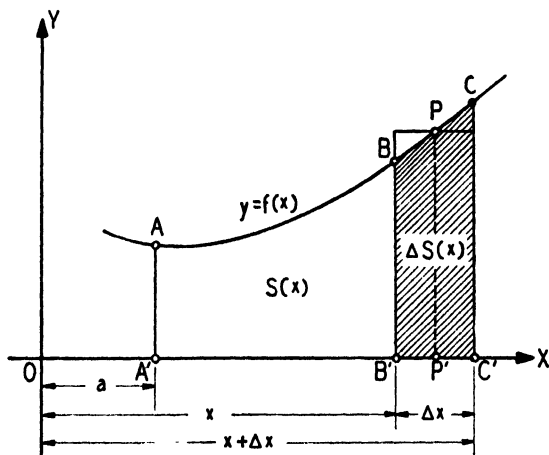


FIG. 20-6.

We shall now compute the derivative of $S(x)$ with respect to x . The increase $\Delta S(x)$ corresponding to the increase Δx of x is the shaded area $BCC'B'$. This area is equal to the area of a rectangle with base $B'C'$ and altitude PP' , where P is a point on the curve between B and C . Thus

$$\Delta S(x) = \text{area } BCC'B' = (PP')(B'C') = (PP') \cdot \Delta x.$$

The average rate of change of $S(x)$ as x increases from x to $x + \Delta x$ is

$$\frac{\Delta S(x)}{\Delta x} = PP'.$$

The derivative $\frac{dS(x)}{dx}$ is the limit of this expression as Δx approaches zero.

From Fig. 20-6 it can be inferred that PP' approaches BB' as Δx approaches zero, and therefore C approaches B . We have, then, that

$$\frac{dS(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta S(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} PP' = BB'.$$

The length of BB' is the value of the function $y = f(x)$ for the value of x indicated in the figure. Hence we have the fundamental formula

$$\frac{dS(x)}{dx} = f(x).$$

The meaning of this formula can be stated as follows.

If the area bounded by the curve $y = f(x)$, the x -axis, and the ordinates corresponding to the abscissas a and x is denoted by $S(x)$, then the derivative of $S(x)$ with respect to x is the given function $f(x)$.

The problem of finding a formula for $S(x)$ is thus the problem of finding a function whose derivative is the given function $f(x)$. This statement shows that the problem of finding an area is closely connected with the problem of *finding a function whose derivative is a given function of the independent variable*.

The operation of finding a function whose derivative is known is called **integration**. Integration is the inverse operation of differentiation, just as subtraction is the inverse operation of addition and division is the inverse operation of multiplication.

The operation of differentiation applied to a given function yields a completely determined result, the derivative of the given function. The inverse operation does not yield a unique result, because it can be shown that there are infinitely many functions having the same derivative. For example, the functions $2x^2$, $2x^2 + 10$, $2x^2 - 5$ have the same derivative $4x$, because the constant terms $+10$ and -5 have the derivative zero. All the functions which are obtained by adding an arbitrary constant C to a function $F(x)$ have the same derivative as $F(x)$ because

$$\frac{d[F(x) + C]}{dx} = \frac{dF(x)}{dx}.$$

Conversely, it can be shown that *all functions which have the same derivative as $F(x)$ can be obtained from the formula $F(x) + C$* . This fact can be expressed as follows.

Theorem A. *If $F(x)$ is a function with $\frac{dF(x)}{dx} = f(x)$, then every function of x having the same derivative $f(x)$ can be written as $F(x) + C$, where C is a constant.*

The result of this theorem will now be applied to the area problem. Consider the shaded area $S(x)$ under the curve $y = f(x)$ as shown in Fig. 20-7. We have already seen that

$$\frac{dS(x)}{dx} = f(x).$$

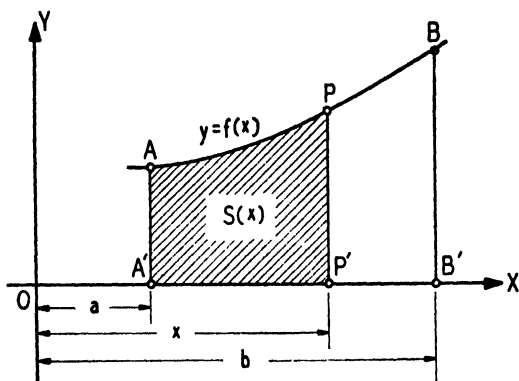


FIG. 20-7.

Suppose that we succeed in finding a function $F(x)$ such that $\frac{dF(x)}{dx} = f(x)$. Then according to Theorem A, there is a constant C such that

$$S(x) = F(x) + C.$$

When $x = a$ the area is obviously zero, so that $S(a) = 0$. Therefore,

$$S(a) = F(a) + C = 0$$

or

$$C = -F(a).$$

Hence the formula for $S(x)$ becomes

$$S(x) = F(x) - F(a).$$

This formula permits us to find the area for any value of x after $F(x)$ has been found. Substituting $x = b$, we obtain, in particular, the area $ABB'A'$, which was previously denoted by

$$\int_a^b f(x) dx.$$

Therefore

$$\int_a^b f(x) dx = S(b) = F(b) - F(a).$$

We have established then this important result:

Theorem B. If $f(x)$ is given and $F(x)$ is a function such that

$$\frac{dF(x)}{dx} = f(x)$$

then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example 1. In Example 1 of Sec. 20-1, the area under the curve $y = \sqrt{4x}$ and above the x -axis lying between $x = 4$ and $x = 9$ was computed. This area can be denoted by

$$\int_4^9 \sqrt{4x} \, dx.$$

According to the rule just discussed, this definite integral can be computed if a function $F(x)$ can be found such that $\frac{dF(x)}{dx} = \sqrt{4x} = 2\sqrt{x}$. The function

$$F(x) = \frac{4}{3}x^{\frac{3}{2}}$$

is such a function, for

$$\frac{d(\frac{4}{3}x^{\frac{3}{2}})}{dx} = \frac{4}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{x}.$$

According to Theorem B the value of the area is

$$\int_4^9 \sqrt{4x} \, dx = F(9) - F(4) = \frac{4}{3}(9^{\frac{3}{2}}) - \frac{4}{3}(4^{\frac{3}{2}}) = \frac{4}{3}(27 - 8) = 25.33.$$

Example 2. The area computed in Example 2 of Sec. 20-1 was denoted by

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx.$$

We find that for the function

$$F(x) = \sin x - x \cos x,$$

the derivative

$$\frac{dF(x)}{dx} = \cos x - (\cos x - x \sin x) = x \sin x,$$

and therefore

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin x \, dx &= F\left(\frac{\pi}{2}\right) - F(0) \\ &= \left(\sin \frac{\pi}{2} - \frac{\pi}{2} \cos \frac{\pi}{2}\right) - (\sin 0 - 0 \cdot \cos 0) \\ &= (1 - 0) - (0 - 0) = 1. \end{aligned}$$

So far, as in the preceding examples, we are able to find the function $F(x)$ only by trial and error, by testing various choices to see whether the derivative of the chosen $F(x)$ is $f(x)$. In the next section a few methods for finding $F(x)$ when $f(x)$ is given will be developed using the definitions which follow in this section. Because of the connection between the problem of finding the function $F(x)$ and the area problem whose solution was denoted by the symbol $\int_a^b f(x) \, dx$, called a definite integral, the following definitions are introduced.

Each function $F(x)$ such that $\frac{dF(x)}{dx} = f(x)$, or such that $dF(x) = f(x) \, dx$, is called a particular integral of $f(x) \, dx$.

Since every other function with the same derivative $f(x)$ can be expressed in the form $F(x) + C$, where C is a constant, we state the following definition.

If $F(x)$ is a particular integral of $f(x) dx$, the expression $F(x) + C$ is called the **general or indefinite integral** of $f(x) dx$, and is denoted by the symbol $\int f(x) dx$. C is called the **constant of integration**.

The symbol $\int f(x) dx$ which is defined by the relation

$$\int f(x) dx = F(x) + C$$

is read as *the indefinite integral of $f(x) dx$* , or briefly, *the integral of $f(x) dx$* .

The abbreviation $[F(x)]_a^b$ is often used for the expression $F(b) - F(a)$, whence the fundamental relation of Theorem B can be written as

$$(1) \quad \int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a).$$

The operation of finding an integral is called **integration**; the integral of $f(x) dx$ is found by **integrating**. As an illustration of how these notations are used, the results of Examples 1 and 2 of this section will be restated here.

Example 3. The problem of Example 1 was to compute the value of $\int_4^9 \sqrt{4x} dx$. We found that $\frac{1}{3}x^{\frac{3}{2}}$ was a function with the derivative $\sqrt{4x}$. We state this fact by saying that $\frac{1}{3}x^{\frac{3}{2}}$ is a particular integral of $\sqrt{4x} dx$. Then the general or indefinite integral of $\sqrt{4x} dx$ is

$$\int \sqrt{4x} dx = \frac{1}{3}x^{\frac{3}{2}} + C.$$

In other words, $\frac{1}{3}x^{\frac{3}{2}} + C$ is the most general function whose derivative is $\sqrt{4x}$. Then by relation (1),

$$\int_4^9 \sqrt{4x} dx = \left[\frac{1}{3}x^{\frac{3}{2}} \right]_4^9 = \frac{1}{3} \cdot 27 - \frac{1}{3} \cdot 8 = 25.33.$$

Example 4. The problem of Example 2 was to compute the value of $\int_0^{\frac{\pi}{2}} x \sin x dx$. We found that

$$F(x) = \sin x - x \cos x$$

was a particular integral of $f(x) dx$, so that

$$\int x \sin x dx = \sin x - x \cos x + C$$

and

$$\int_0^{\frac{\pi}{2}} x \sin x dx = [\sin x - x \cos x]_0^{\frac{\pi}{2}} = 1.$$

Example 5. Prove the statement that

$$\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x + C.$$

This equation states that the derivative of the right member equals $x^2 \cos x$ and therefore the equation can be checked by computing this derivative.

$$\begin{aligned} \frac{d[2x \cos x + (x^2 - 2) \sin x + C]}{dx} &= \frac{d(2x \cos x)}{dx} + \frac{d[(x^2 - 2) \sin x]}{dx} \\ &= 2(\cos x - x \sin x) + [2x \sin x + (x^2 - 2) \cos x] \\ &= 2 \cos x - 2x \sin x + 2x \sin x + x^2 \cos x - 2 \cos x \\ &= x^2 \cos x. \end{aligned}$$

✓ EXERCISES

Prove each of the following statements.

1. $\int (x^2 - 2x + 1) \, dx = \frac{x^3}{3} - x^2 + x + C.$
2. $\int (4x^5 - 3x^2) \, dx = \frac{2}{3}x^6 - x^3 + C.$
3. $\int (1 + x^n) \, dx = x + \frac{x^{n+1}}{n+1} + C, n = 1, 2, 3, \dots$
4. $\int (1 + x)^n \, dx = \frac{(1 + x)^{n+1}}{n+1} + C, n = 1, 2, 3, \dots$
5. $\int \frac{dx}{(1 + x)^2} = \frac{-1}{1 + x} + C.$
6. $\int \frac{x \, dx}{\sqrt{1 + x^2}} = \sqrt{1 + x^2} + C.$
7. $\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C.$
8. $\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C.$
9. $\int \cos^3 x \, dx = \frac{1}{3} \sin x (\cos^2 x + 2) + C.$
10. $\int \sin^3 x \, dx = -\frac{1}{3} \cos x (\sin^2 x + 2) + C.$
11. $\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + C.$
12. $\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + C.$

Using the statements of Exercises 1-12, compute the following definite integrals, giving your results correct to three significant figures. By drawing the corresponding graph, interpret each integral as an area.

13. $\int_1^3 (x^2 - 2x + 1) dx.$

14. $\int_{-1}^3 (x^2 - 2x + 1) dx.$

15. $\int_1^2 (4x^5 - 3x^2) dx.$

16. $\int_0^2 (1 + x^2) dx.$

17. $\int_{-1}^1 (1 + x^2) dx.$

18. $\int_{-1}^3 (1 + x)^2 dx.$

19. $\int_0^3 \frac{dx}{(1+x)^2}.$

20. $\int_0^4 \frac{x dx}{\sqrt{1+x^2}}.$

21. $\int_0^\pi \sin^2 x dx.$

22. $\int_0^\pi \cos^2 x dx.$

23. $\int_0^{\frac{\pi}{2}} \cos^3 x dx.$

24. $\int_0^\pi \sin^3 x dx.$

25. $\int_0^{\frac{\pi}{2}} \sin 2x dx.$

26. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x dx.$

27. $\int_0^{\frac{\pi}{3}} \sin 3x dx.$

28. $\int_0^{\frac{\pi}{4}} \sin \left(2x + \frac{\pi}{2} \right) dx.$

29. The effective value of a sinusoidal alternating current with a peak value of I_0 amperes is kI_0 amperes, where

$$k = \sqrt{\frac{1}{\pi} \int_0^\pi \sin^2 x dx}.$$

Compute the value of k correct to three significant figures. Compare this result with that obtained in Exercise 23 of Sec. 20-1.

20-3. The Integral Calculus. From the discussion in Sec. 20-2 it is obvious that the most important problem now is to find the indefinite integral of a given function. This problem is much more complicated than the inverse problem: to find the derivative of a given function. The system of all the methods which have been developed for obtaining integrals of given functions is called **integral calculus**. A few of the more elementary formulas and rules of this calculus will be discussed in this section.

Each formula of the differential calculus can be used to obtain a corresponding formula of the integral calculus. Thus the formula $\frac{dx^n}{dx} = nx^{n-1}$ can be used to prove that

$$(1) \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

This formula is proved by computing the derivative of the right member:

$$\frac{d\left(\frac{x^{n+1}}{n+1} + C\right)}{dx} = \frac{1}{n+1} \cdot \frac{dx^{n+1}}{dx} = \frac{1}{n+1} \cdot (n+1) x^n = x^n.$$

A particular case is obtained for $n = 0$,

$$(2) \quad \int dx = x + C.$$

In a similar way the formulas

$$(3) \quad \int \sin x \, dx = -\cos x + C,$$

$$(4) \quad \int \cos x \, dx = \sin x + C$$

can be proved.

The following rules correspond to the analogous rules of differential calculus.

A constant factor may be written before or after the sign of integration.

$$(5) \quad \int Cf(x) \, dx = C \int f(x) \, dx.$$

Example 1. $\int 4x^2 \, dx = 4 \int x^2 \, dx = \frac{4}{3}x^3 + C.$

The integral of a sum or of a difference is equal to the sum or the difference of the integrals of each term:

$$(6) \quad \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

Example 2. Integrate $\int (3x + 5\sqrt{x}) \, dx.$

$$\int (3x + 5\sqrt{x}) \, dx = \int 3x \, dx + \int 5\sqrt{x} \, dx \quad \text{By (6)}$$

$$= 3 \int x \, dx + 5 \int x^{\frac{1}{2}} \, dx \quad \text{By (5)}$$

$$= 3 \left(\frac{x^2}{2} \right) + 5 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \quad \text{By (1)}$$

$$= \frac{3}{2}x^2 + \frac{10}{3}x\sqrt{x} + C.$$

Example 3. Integrate $\int (2 \sin x + 5 \cos x) dx$.

$$\int (2 \sin x + 5 \cos x) dx = \int 2 \sin x dx + \int 5 \cos x dx \quad \text{By (6)}$$

$$= 2 \int \sin x dx + 5 \int \cos x dx \quad \text{By (5)}$$

$$= -2 \cos x + 5 \sin x + C. \quad \text{By (3) and (4)}$$

Many more complicated integration problems, to which the formulas given above do not apply directly, can be reduced to simpler problems where the formulas do apply by introducing a new variable. The following examples will illustrate how this can be done.

Example 4. Find $\int x\sqrt{1+2x} dx$.

The formulas and rules developed so far do not apply to this problem. A simplification can be obtained by introducing a new variable t such that

$$1 + 2x = t$$

or

$$x = \frac{t-1}{2}.$$

It is important that not only the new variable t is to be substituted for x in the function $x\sqrt{1+2x}$, but that the differential dx is to be replaced by an expression containing only t and dt according to the formula

$$dx = \frac{dx}{dt} dt.$$

In our case we obtain

$$dx = \frac{1}{2} dt,$$

and therefore

$$\begin{aligned} \int x\sqrt{1+2x} dx &= \int \frac{t-1}{2} \cdot \sqrt{t} \cdot \frac{1}{2} dt \\ &= \frac{1}{4} \int (t^{\frac{3}{2}} - t^{\frac{1}{2}}) dt \\ &= \frac{1}{4} \left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right] + C \\ &= \frac{1}{4} \left[\frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right] + C \\ &= \frac{1}{30} \sqrt{t} (3t^2 - 5t) + C. \end{aligned}$$

The value of the integral has now been found as a function of t . Substituting for t its original value in terms of x we obtain

$$\begin{aligned} \int x\sqrt{1+2x} dx &= \frac{1}{30} \sqrt{1+2x} [3(1+2x)^2 - 5(1+2x)] + C \\ &= \frac{1}{15} \sqrt{1+2x} [6x^2 + x - 1] + C. \end{aligned}$$

The result can be checked by differentiating, because the derivative of the result has to be $x\sqrt{1+2x}$.

Example 5. Find $\int \cos 4x \, dx$.

The function $\cos 4x$ suggests the introduction of a new variable t such that

$$t = 4x,$$

$$dt = 4dx,$$

$$dx = \frac{1}{4}dt.$$

This substitution gives

$$\begin{aligned} \int \cos 4x \, dx &= \int (\cos t) \frac{dt}{4} = \frac{1}{4} \int \cos t \, dt = \frac{1}{4} \sin t + C \\ &= \frac{1}{4} \sin 4x + C. \end{aligned}$$

Example 6. Find $\int \cos^2 x \sin x \, dx$.

By the substitution $t = \cos x$, $dt = -\sin x \, dx$ we obtain

$$\int \cos^2 x \sin x \, dx = - \int t^2 \, dt = -\frac{t^3}{3} + C = -\frac{\cos^3 x}{3} + C.$$

Example 7. Compute $\int \sin(\omega t + \theta) \, dt$, where ω and θ are constants.

The substitution

$$u = \omega t + \theta$$

gives

$$du = \omega \, dt$$

or

$$dt = \frac{1}{\omega} du.$$

Hence

$$\begin{aligned} \int \sin(\omega t + \theta) \, dt &= \int (\sin u) \frac{1}{\omega} du = -\frac{1}{\omega} \cos u + C \\ &= -\frac{1}{\omega} \cos(\omega t + \theta) + C. \end{aligned}$$

EXERCISES

Find the following indefinite integrals.

1. $\int x^5 \, dx.$

2. $\int \frac{dr}{r^4}.$

3. $\int \sqrt[3]{u} \, du.$

4. $\int (2x^3 - 3x^2 + 1) \, dx.$

5. $\int \left(\sqrt{10S} - 2S^2 + \frac{3}{S^2} \right) dS.$

6. $\int (a^2 - t^2) \, dt.$

7. $\int \sqrt{4x - 3} \, dx.$

8. $\int \frac{dx}{(x-5)^3}.$

9. $\int \frac{x \, dx}{\sqrt{x^2 + 3}}.$
10. $\int \frac{3x \, dx}{(x^2 + 1)^2}.$
11. $\int \frac{x^3 \, dx}{\sqrt{1 + x^4}}.$
12. $\int (2 + 3x)^{10} \, dx.$
13. $\int \sin 5x \, dx.$
14. $\int \cos \pi x \, dx.$
15. $\int (2 \sin x - 3 \sin 2x + 4 \sin 3x) \, dx.$
16. $\int \sin (380t + 5) \, dt.$
17. $\int [2 \cos (5t + 3) + 5 \cos (10t - 7)] \, dt.$
18. $\int (1 + 2 \cos \pi t + 4 \cos 2\pi t) \, dt.$

Find the following definite intervals.

19. $\int_0^2 x^3 \, dx.$
20. $\int_2^3 \left(x + \sqrt{x} - 2 \frac{1}{\sqrt{x}} \right) dx.$
21. $\int_0^a (a^3 - t^3) \, dt.$
22. $\int_0^2 \frac{x \, dx}{\sqrt{1 + x^2}}.$
23. $\int_{-2}^{+2} (4 - x^2) \, dx.$
24. $\int_1^5 \frac{dx}{\sqrt{x + 4}}.$
25. $\int_0^\pi \sin x \, dx.$
26. $\int_0^\pi \sin \omega t \, dt.$
27. $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2 \theta \sin \theta \, d\theta.$

20-4. Applications of the Integral Calculus. The integral calculus has many applications in engineering. We shall discuss three of these applications in this section.

There are two types of problems to which the methods of the integral calculus apply. The solution to problems of the first type is furnished by the definite integral; the solution to problems of the second type is expressed by an indefinite integral. In problems of the second type data are given which determine a particular value of the constant of integration.

As an example of the first type we have considered the problem of finding an area, and in this section we shall consider the problem of finding the work done by a variable force. As an example of the second type we shall consider the motion of an object when the force acting on the object is known.

Example 1. Show that the area S bounded by the x -axis, the line $x = a$, and the parabola $y = mx^2$ is given by the formula $S = \frac{1}{3}ab$, where $b = ma^2$.

The area is shown in Fig. 20-8. It can be computed as the definite integral

$$\int_0^a y \, dx = \int_0^a mx^2 \, dx = \left[m \frac{x^3}{3} \right]_0^a = \frac{1}{3} ma^3 = \frac{1}{3} a(ma^2) = \frac{1}{3} ab.$$

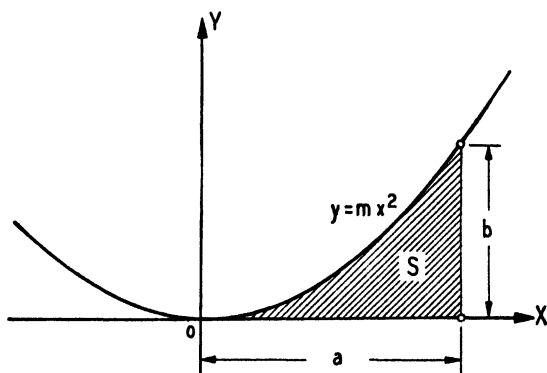


FIG. 20-8.

Assume that along a straight line (Fig. 20-9) a constant force P is acting on a small object moving along this line. The work W done by this force, if the object moves from A to B , is the product of the force and the distance AB that the object travels. Hence the work is given by the formula

$$W = P \cdot (AB).$$

However, this formula cannot be applied if the force P changes while the object is in motion.

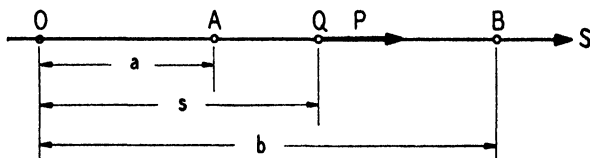


FIG. 20-9.

Let us assume that at each point Q of the segment AB , the force P has a value which is known if the distance $s = OQ$ is given. This assumption can be stated in mathematical language by saying that P is a function $F(s)$ of s ; for example, $P = \frac{k}{(s-a)^2}$, where k is a constant, is the force acting on an electric charge at Q due to another charge situated at A .

The method used to compute the work in this case will be patterned after the method used in Sec. 20-1 to compute areas. First divide the distance AB into a number of equal segments of length Δs (Fig. 20-10).

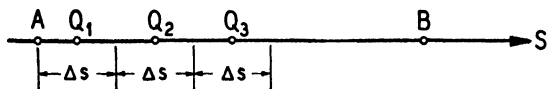


FIG. 20-10.

If the length of these segments is small, it may be assumed that the force is nearly constant along each segment. If Q_1 is an arbitrary point on the first segment whose distance from A is s_1 , then the force at all the other points of this segment has approximately the same value $P_1 = F(s_1)$ as at Q_1 , and therefore the work along this segment is roughly

$$F(s_1) \cdot \Delta s.$$

In the same way it may be inferred that the work along each of the following segments is

$$F(s_2) \cdot \Delta s, \quad F(s_3) \cdot \Delta s, \quad \dots$$

and that therefore the total work done along the distance AB is approximately

$$F(s_1) \cdot \Delta s + F(s_2) \cdot \Delta s + F(s_3) \cdot \Delta s + \dots$$

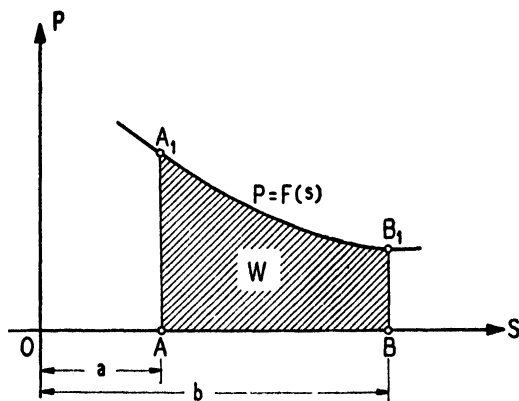


FIG. 20-11.

The work can be computed to any desired degree of precision by dividing the distance AB into an increasing number of parts.

This discussion becomes identical with that of Sec. 20-1 if the graph of the function $P = F(s)$ is plotted (Fig. 20-11). Hence we may state

that the work done if the object moves from A to B is given by the shaded area and has, therefore, the value

$$W = \int_a^b F(s) ds.$$

Thus the following theorem may be stated.

In order to find the work done by a variable force on a point moving in a straight line, choose on this line a point of reference O from which the distance s of the variable point is measured. For each point of the line find as a function of S the force P acting at the point, obtaining

$$P = F(s).$$

Then the work done during the motion from the point $s = a$ to the point $s = b$ is given by the definite integral

$$W = \int_a^b F(s) ds.$$

Example 2. An object is moving under the influence of a force which is proportional to the distance s between the object and the point O of reference and directed to the point O . What is the work required to move the object from O to a point A so that $OA = a$?

The force P is directed toward O as shown by the vector in Fig. 20-12, whence P is given by a negative number (see Sec. 6-3). Since P is proportional to s and s is positive,

$$P = F(s) = -k^2s$$

where k^2 is a positive constant. The force which must be applied from O to the right is therefore a positive one equal in absolute value to P or k^2s . Hence the work done in moving the object from O to A is

$$\int_0^a k^2s ds = \left[k^2 \frac{s^2}{2} \right]_0^a = \frac{1}{2} k^2 a^2.$$

As an example of the second type of problem mentioned in this section, we shall consider the motion of an object when the force producing

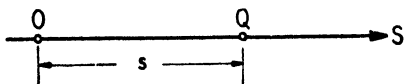


FIG. 20-13.

the motion is a known function of the time. Let us assume that a small object Q of mass m is moving along a straight line (Fig. 20-13) and that its instantaneous position

is defined by its distance s from a point of reference O . Since the object moves, s may be regarded as a function of the time t that has elapsed since the motion started. In Chapter 19 it was shown that the speed v of

the moving object Q , being the instantaneous rate of change of s with respect to the time, is given by the derivative

$$v = \frac{ds}{dt},$$

and that the acceleration a of the object, being the instantaneous rate of change of the speed with respect to the time, is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

A fundamental law of mechanics states that the force F acting on an object, provided proper units for all the quantities involved are chosen, is given by the product of mass and acceleration:

$$F = ma.$$

If F is known as a function of t , this formula can be written as

$$m \frac{dv}{dt} = F(t), \quad \text{or} \quad \frac{dv}{dt} = \frac{1}{m} F(t).$$

By this relation v is a function of t whose derivative is known, and therefore v can be found by computing an indefinite integral. Since $\frac{ds}{dt} = v$, s can be found by computing another indefinite integral after v has been found. An example will show how these ideas can be applied.

Example 3. An object is thrown vertically upward with an initial speed v_0 . Find its distance from the starting point as a function of the time.

In order to apply the formulas discussed above, the force acting on the object must be known. This force is that due to the acceleration of gravity $g = 32.2$ ft. per sec. per sec., and it has the constant value

$$F = mg$$

where m is the mass of the object. Since

$$F = ma$$

we obtain

$$ma = mg$$

or

$$a = g.$$

In order to find the speed of the object at any time t , we use the formula

$$\frac{dv}{dt} = a = g$$

from which

$$v = \int g \, dt = gt + C$$

where C is the constant of integration. This constant has in this problem a value which can be determined by using the information that when $t = 0$, $v = v_0$. Substituting these values in the formula for v , we obtain

$$v_0 = g \cdot 0 + C = C,$$

and therefore

$$v = C + gt = v_0 + gt$$

where v_0 and g are known constants.

The next step is to use the formula

$$\frac{ds}{dt} = v = v_0 + gt,$$

from which we find that

$$s = \int (v_0 + gt) dt = v_0 t + \frac{g}{2} t^2 + C'.$$

In order to find the value which has to be ascribed to C' in order to solve the present problem, we use the fact that for $t = 0$, the moving point starts with $s = 0$. Substituting these values into the formula for s , we obtain

$$0 = v_0 \cdot 0 + \frac{g}{2} \cdot 0 + C'$$

whence

$$C' = 0,$$

and

$$s = v_0 t + \frac{1}{2} g t^2.$$

To find the values of the constants of integration in this problem, we used the conditions that were known for the beginning of the motion. These conditions are called **initial conditions**. In particular, we used the fact that when $t = 0$, the velocity is v_0 , and the fact that when $t = 0$, the distance s from the starting point is zero. Initial conditions of this kind can always be used to find the constants of integration in problems of this type.

EXERCISES

1. The increase in length of a particular spring is proportional to the force acting on this spring, provided that this increase is not more than 0.5 ft. It has been found that a force of 50 lb. is required to stretch this spring by 1 ft. What is the work, in foot-pounds, required to stretch the spring 0.5 ft.?

2. A positive electric charge of Q electrostatic units which is concentrated in a small sphere exerts a force of $\frac{Q}{r^2}$ dynes on a positive electrostatic unit at the distance of r cm. Compute the work done by the electric force if the unit charge moves from a distance of 3 cm. to a distance of 10 cm. from the charge Q . (The work of a force of 1 dyne along a distance of 1 cm. is called an erg.)

3. Solve the problem of Exercise 2, assuming that the unit charge is moved from the distance r_1 cm. to the distance r_2 cm.

4. The electrical energy in joules required to increase the potential of an electrical condenser from V_1 volts to V_2 volts is given by the integral

$$C \int_{V_1}^{V_2} V dV$$

where C is the capacitance, in farads, of the condenser.

(a) Compute the energy required to increase the potential of a condenser of 0.1 microfarad from 10 to 500 volts.

(b) Compute the total energy stored in a condenser of capacitance C and potential V (total energy = energy required to increase the potential from 0 to V).

(c) Solve the problem in (b) for the special case $C = 2$ microfarads, $V = 1000$ volts (1 microfarad = 10^{-6} farad).

5. A sinusoidal force acting along a straight line may be given by the formula

$$P = F \sin \omega t.$$

If this force is acting on a small object of mass m and if the object starts from rest,

(a) compute its distance from the starting point as a function of time and (b) show that the speed of the object oscillates between 0 and $\frac{2F}{m\omega}$.

PROGRESS REPORT

Two new concepts were introduced in this chapter: the concept of the definite integral and the concept of the indefinite integral. The first was introduced in connection with the area problem, the second arose in connection with the inverse operation of differentiation. It was found that the area, or definite integral, could be computed in two ways: directly, by dividing the area into narrow trapezoidal strips, and indirectly, by subtracting two values of a corresponding indefinite integral. It was shown how a few simple indefinite integrals can be found, and finally, a few applications were considered.

APPENDIX

REVIEW OF FUNDAMENTALS OF GEOMETRY

In the following pages we shall briefly summarize a few of the terms and facts from geometry which have been used in the discussions in this book.

In plane geometry, which deals entirely with configurations in the plane, there are certain fundamental conceptions which we must assume from experience, without definition. These include the notion of a **point**, a **straight line**, and an **area**. Understanding that a straight line extends infinitely far in both directions, we define a **ray** to be a point on a line with the part of the line on one side of the point and a **segment** to be that part of a straight line between two points on the line. It is assumed that one and only one straight line can be drawn through two points.

Experience leads us to believe that two straight lines can meet at most in one point, called the **intersection** of the lines. The **angle** formed by two intersecting lines has been defined in Chapter 4. The intersection of the lines is called the **vertex** of the angle. In Chapter 4 the **degree**, a unit for the measurement of angles, was also introduced. Special names are sometimes used to refer to angles classified according to their magnitude as follows:

SIZE OF ANGLE	TERM
90°	Right angle
180°	Straight angle
Between 0° and 90°	Acute angle
Between 90° and 180°	Obtuse angle
	} Oblique angles

If two angles have a sum of 90° , they are called **complementary angles**. If two angles have a sum of 180° , they are called **supplementary angles**.

If two intersecting lines form a right angle, the lines are said to be **perpendicular**. If two lines do not intersect, they are said to be **parallel**. There are many theorems concerning parallel and perpendicular lines. The most important are:

1. Two lines which are perpendicular to the same line are parallel.
2. Two lines which are parallel to the same line are parallel.
3. If two parallel lines are cut by a third line (Fig. A-1), the **corresponding angles** are equal (for example, $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$), the **alternate interior angles** are equal (for example, $\angle 3 = \angle 6$, $\angle 4 = \angle 5$), and the **alternate exterior angles** are equal (for example, $\angle 1 = \angle 8$, $\angle 2 = \angle 7$).

From Fig. A-1, we see that $\angle 1 = \angle 4$, $\angle 2 = \angle 3$, etc. Such pairs of angles are called **vertical angles**.

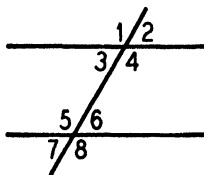


FIG. A-1.

4. Vertical angles are equal.

5. Two angles having their sides respectively parallel or perpendicular are either equal or supplementary.

If three or more intersecting straight lines form a figure which completely encloses a portion of the plane, that figure is called a **polygon**. The line segments which bound the enclosure are called the **sides** of the polygon, and the sum of their lengths is called the **perimeter** of the polygon. The **area of a polygon** is the area of the portion of the plane enclosed by the polygon. If two polygons have their corresponding angles and sides equal, they are said to be **congruent**. If two polygons have their corresponding angles equal they are said to be **similar**.

A **triangle** is a polygon having three sides. The sides of a triangle taken two at a time form the **three angles of the triangle**. An **exterior angle** of a triangle is an angle formed by one side of the triangle and the extension of another side through their point of intersection. An **altitude** of a triangle is the perpendicular line drawn from any vertex to the opposite side. A triangle one of whose angles is a right angle is a **right triangle**. A triangle none of whose angles are right angles is an **oblique triangle**. A triangle two of whose sides are equal in length is an **isosceles triangle**. A triangle all three of whose sides are equal is an **equilateral triangle**.

If the corresponding sides and angles of two triangles are equal, the triangles are **congruent**. If the corresponding angles of two triangles are equal the triangles are **similar**.

Some important theorems concerning triangles are:

1. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
2. If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side, and conversely.
3. The sum of the angles of a triangle is 180° .
4. The corresponding sides of similar triangles are proportional.
5. Two triangles are congruent if:
 - (a) Two sides and the included angle of one are equal respectively to two sides and the included angle of the other.
 - (b) Two angles and a side of one are equal respectively to two angles and the corresponding side of the other.
 - (c) Three sides of one are equal respectively to the three sides of the other.

6. Two triangles are similar if:

- (a) Two angles of one are equal respectively to two angles of the other.
- (b) An angle of one equals an angle of the other and the including sides are proportional.

7. In an isosceles triangle, the angles opposite the equal sides are equal.

8. The angles of an equilateral triangle are each equal to 60° .

9. If DE is parallel to AC in Fig. A-2, then triangle BDE is similar to triangle BAC .

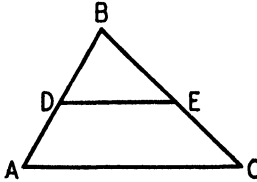


FIG. A-2.

The side opposite the right angle in a right triangle is called the **hypotenuse**. The other two sides are the **legs** of the triangle. The following theorems about right triangles are useful.

1. In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. This theorem is called the theorem of **Pythagoras**.

2. If one of the acute angles of a right triangle is 30° , then the side opposite that angle has a length equal to one-half the hypotenuse, and the legs and hypotenuse are proportional respectively to 1, $\sqrt{3}$, 2.

3. If one of the acute angles of a right triangle is 45° , the legs are equal, and the legs and hypotenuse are proportional respectively to 1, 1, $\sqrt{2}$.

4. Two right triangles are congruent if:

- (a) Two sides of one are equal respectively to two sides of the other.
- (b) An acute angle and a side of one are equal, respectively, to an acute angle and the corresponding side of the other.

5. Two right triangles are similar if:

- (a) An acute angle of one equals an acute angle of the other.
- (b) Two sides of one are proportional, respectively, to the corresponding sides of the other.

6. If ABC is a right triangle with hypotenuse AB , then CD , the altitude to the hypotenuse, forms two right triangles ACD , BCD such that $\triangle ACD$ is similar to $\triangle BCD$, $\triangle ACD$ is similar to $\triangle ABC$, $\triangle BCD$ is similar to $\triangle ABC$.

A polygon having four sides is a **quadrilateral**. A quadrilateral which has two and only two sides parallel is called a **trapezoid**. The parallel sides are called the **bases**. A **parallelogram** is a quadrilateral which has both pairs of opposite sides parallel. Either pair of parallel sides may be called **bases**. The perpendicular distance between two parallel sides of a parallelogram or trape-

zoid is called an **altitude**. A **rectangle** is a parallelogram having four right angles. A **square** is a rectangle which has its adjacent sides equal. The most important facts concerning quadrilaterals are:

1. If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram; and conversely, if a quadrilateral is a parallelogram, then its opposite sides are equal.

2. If the opposite angles of a quadrilateral are equal, then it is a parallelogram; and conversely.

3. If one pair of opposite sides of a quadrilateral are equal and parallel, then it is a parallelogram.

Many questions in plane geometry and in other more advanced work are simplified somewhat by using the notion of the locus of a point. If C is a given condition and S is a set of points such that (a) every point which satisfies C is a point of S , and (b) every point which does not satisfy C is not in S , then S is called the **locus of a point satisfying C** .

The locus of a point in the plane which satisfies the condition that the point is at a fixed distance from a fixed point in the plane is called a **circle** (Fig. A-3). The fixed point is the **center** of the circle. The length of a circle is called the **circumference**, and the **area** of a circle is the area of the portion of the plane enclosed by the circle. Either of the two parts of a circle between two points on the circle is called an **arc** of the circle. A **tangent** is a line which touches a circle in one and only one point, called the **point of tangency**. A **secant** is a line which cuts a circle in two points. A **chord** is a line segment joining two points on a circle. A **diameter** is a chord which passes through the center of a circle, and a **radius** is a line segment from the center of a circle to any point on the circle. Any arc of a circle which is cut off by the two sides of an angle is called an **arc intercepted by or subtended by the angle**. An angle formed by two radii of a circle is called a **central angle**. The figure formed by a central angle and its intercepted arc is called a **sector** of the circle. An **inscribed angle** is an angle formed by two chords which meet at a point on the circle.

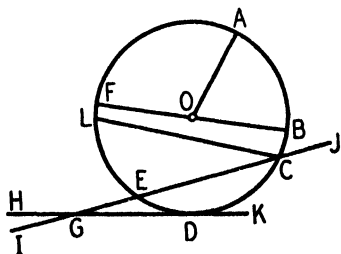


FIG. A-3.

In the circle in Fig. A-3, O is the center, and the portion of the circle between A and B , or between A and F , is an arc. The line KH is a tangent with point of tangency D ; the line IJ is a secant; EC and FB are chords; FB is also a diameter; OF , OA , and OB are radii; CD and ED are arcs intercepted by angle

KGJ ; AB is the arc intercepted by the central angle BOA ; and ABO is a sector. Angle LCE is an inscribed angle.

Some of the more useful theorems concerning the circle are:

1. A tangent is a perpendicular to the radius drawn to the point of tangency.
2. An inscribed angle which intercepts a semi-circle is a right angle.
3. On the same circle or on circles with the same radius, central angles are proportional to the lengths of their intercepted arcs.
4. The length of a chord of a circle is less than or equal to the length of the diameter.
5. The ratio of the circumference to the length of a diameter of a circle is the same for all circles. This ratio, which is denoted by π , is an irrational number having approximately the value 3.14159.

Solid geometry deals with configurations in space. A few of its terms and facts are summarized in the following section.

Any straight line passing through two points of a plane lies entirely in the plane. A plane can be determined by three points, by two intersecting lines, by a line and a point not on the line, or by two parallel lines. If two planes intersect, their intersection is a straight line.

A line L is perpendicular to a plane if every line in the plane which passes through the intersection of L and the plane is perpendicular to L . Lines which are perpendicular to the same plane are parallel. Planes which are perpendicular to the same or parallel lines are parallel to each other.

If a line and a plane do not intersect, they are said to be **parallel**. If two planes do not intersect, they are said to be **parallel**. Let two planes, P_1 and P_2 , intersect in a line L , and let L_1 be a line in P_1 which meets L at P and is perpendicular to L ; let L_2 be a line in P_2 which meets L at P and is perpendicular to L . Then if L_1 and L_2 are perpendicular, the planes P_1 and P_2 are said to be perpendicular.

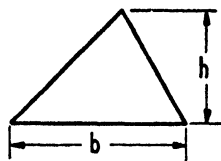
If C is a plane curve and L is a line which passes through C but does not lie in the plane of C , the surface found by moving L about C always parallel to its original position is called a **cylindrical surface**. Each position of the moving line L is called an **element** of the surface. All the elements are parallel. When two parallel planes cut all the elements of a cylindrical surface, a **cylinder** is formed and the parallel sections are called **bases** of the cylinder. A **right circular cylinder** is a cylinder having bases which are perpendicular to the elements and which cut the surface in circles. The **axis** of a right circular cylinder is the line between the centers of the bases.

If two segments in space have a common end point, the angle between their projections on a horizontal plane is called the **horizontal angle** between the segments.

FORMULAS FROM GEOMETRY

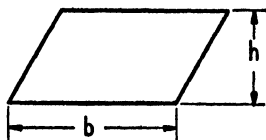
For a triangle with side b and altitude to that side h ,

$$\text{Area} = \frac{1}{2}bh.$$



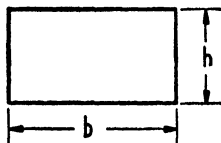
For a parallelogram with side b and altitude to that side h ,

$$\text{Area} = bh.$$



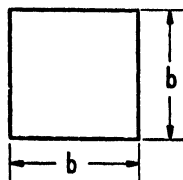
For a rectangle of length b and width h ,

$$\text{Area} = bh.$$



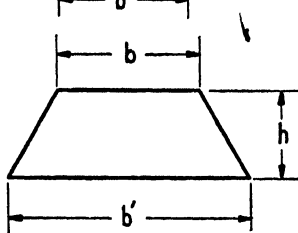
For a square of side b ,

$$\text{Area} = b^2.$$



For a trapezoid with parallel bases b and b' and altitude h ,

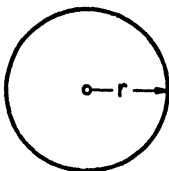
$$\text{Area} = \frac{1}{2}h(b + b').$$



For a circle with radius r ,

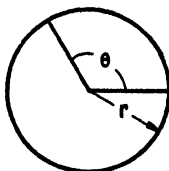
$$\text{Circumference} = 2\pi r,$$

$$\text{Area} = \pi r^2.$$



In a circle of radius r , a sector with central angle θ measured in radians has an

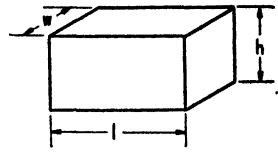
$$\text{Area} = \frac{1}{2}r^2\theta.$$



For a rectangular solid of length l , height h , and width w ,

$$\text{Total surface area} = 2(lw + lh + wh),$$

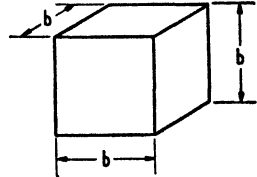
$$\text{Volume} = lwh.$$



For a cube with the length of one edge b ,

$$\text{Total surface area} = 6b^2,$$

$$\text{Volume} = b^3.$$

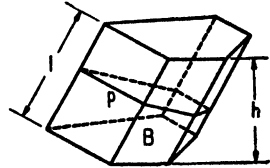


For a prism with lateral edge l , altitude h , perimeter of a right section p , and area of the base B ,

$$\text{Lateral area} = pl,$$

$$\text{Total area} = pl + 2B,$$

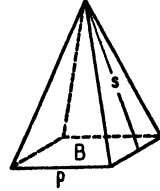
$$\text{Volume} = Bh.$$



For a regular pyramid with slant height s , perimeter of the base p , and area of the base B ,

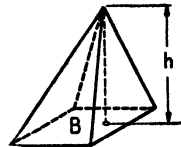
$$\text{Lateral area} = \frac{1}{2}ps,$$

$$\text{Total area} = \frac{1}{2}ps + B.$$



For a pyramid with altitude h and area of the base B .

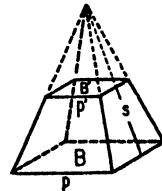
$$\text{Volume} = \frac{1}{3}Bh.$$



For a frustum of a regular pyramid with slant height s , bases of perimeter p and p' , and areas B and B' respectively,

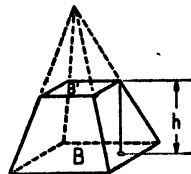
$$\text{Lateral area} = \frac{1}{2}s(p + p'),$$

$$\text{Total area} = \frac{1}{2}s(p + p') + B + B'.$$



For a frustum of a pyramid with altitude h and areas of the bases B and B' ,

$$\text{Volume} = \frac{1}{3}h(B + B' + \sqrt{BB'}).$$

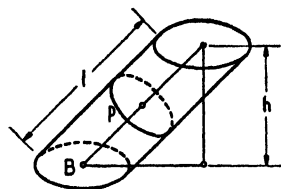


For a cylinder with lateral edge l , altitude h , perimeter of a right section p , and base area B ,

$$\text{Lateral area} = pl,$$

$$\text{Total area} = pl + 2B,$$

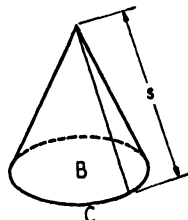
$$\text{Volume} = hB.$$



For a right circular cone with slant height s , circumference of the base C , and area of the base B ,

$$\text{Lateral area} = \frac{1}{2}sC,$$

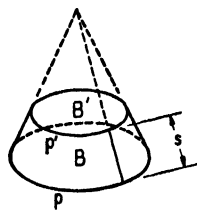
$$\text{Total area} = \frac{1}{2}sC + B.$$



For a frustum of a right circular cone with slant height s , bases of perimeter p and p' and area B and B' , respectively,

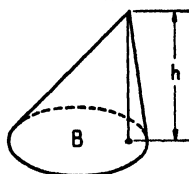
$$\text{Lateral area} = \frac{1}{2}s(p + p'),$$

$$\text{Total area} = \frac{1}{2}s(p + p') + B + B'.$$



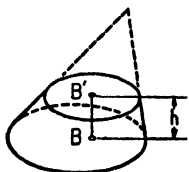
For a cone with altitude h and base area B ,

$$\text{Volume} = \frac{1}{3}Bh.$$



For a frustum of a cone with altitude h and bases of area B and B' ,

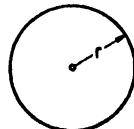
$$\text{Volume} = \frac{1}{3}h(B + B' + \sqrt{BB'}).$$



For a sphere of radius r ,

$$\text{Surface area} = 4\pi r^2,$$

$$\text{Volume} = \frac{4}{3}\pi r^3.$$



In a sphere of radius r , a zone with altitude h has

$$\text{Area} = 2\pi hr.$$

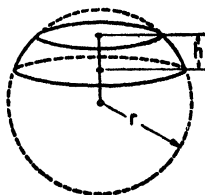


TABLE 1—SQUARES AND SQUARE ROOTS

N	N ²	\sqrt{N}	$\sqrt{10N}$
1.00	1.0000	1.00000	3.16228
1.01	1.0201	1.00499	3.17805
1.02	1.0404	1.00995	3.19374
1.03	1.0609	1.01489	3.20936
1.04	1.0816	1.01980	3.22490
1.05	1.1025	1.02470	3.24037
1.06	1.1236	1.02956	3.25576
1.07	1.1449	1.03441	3.27109
1.08	1.1664	1.03923	3.28634
1.09	1.1881	1.04403	3.30151
1.10	1.2100	1.04881	3.31662
1.11	1.2321	1.05357	3.33167
1.12	1.2544	1.05830	3.34664
1.13	1.2769	1.06301	3.36155
1.14	1.2996	1.06771	3.37639
1.15	1.3225	1.07238	3.39116
1.16	1.3456	1.07703	3.40588
1.17	1.3689	1.08167	3.42053
1.18	1.3924	1.08628	3.43511
1.19	1.4161	1.09087	3.44964
1.20	1.4400	1.09545	3.46410
1.21	1.4641	1.10000	3.47851
1.22	1.4884	1.10454	3.49285
1.23	1.5129	1.10905	3.50714
1.24	1.5376	1.11355	3.52136
1.25	1.5625	1.11803	3.53553
1.26	1.5876	1.12250	3.54965
1.27	1.6129	1.12694	3.56371
1.28	1.6384	1.13137	3.57771
1.29	1.6641	1.13578	3.59166
1.30	1.6900	1.14018	3.60555
1.31	1.7161	1.14455	3.61939
1.32	1.7424	1.14891	3.63318
1.33	1.7689	1.15326	3.64692
1.34	1.7956	1.15758	3.66060
1.35	1.8225	1.16190	3.67423
1.36	1.8496	1.16619	3.68782
1.37	1.8769	1.17047	3.70135
1.38	1.9044	1.17473	3.71484
1.39	1.9321	1.17898	3.72827
1.40	1.9600	1.18322	3.74166
1.41	1.9881	1.18743	3.75500
1.42	2.0164	1.19164	3.76829
1.43	2.0449	1.19583	3.78153
1.44	2.0736	1.20000	3.79473
1.45	2.1025	1.20416	3.80789
1.46	2.1316	1.20830	3.82099
1.47	2.1609	1.21244	3.83406
1.48	2.1904	1.21655	3.84708
1.49	2.2201	1.22066	3.86005
1.50	2.2500	1.22474	3.87298
N	N ²	\sqrt{N}	$\sqrt{10N}$
1.50	2.2500	1.22474	3.87298
1.51	2.2801	1.22882	3.88587
1.52	2.3104	1.23288	3.89872
1.53	2.3409	1.23693	3.91162
1.54	2.3716	1.24097	3.92428
1.55	2.4025	1.24499	3.93700
1.56	2.4336	1.24900	3.94968
1.57	2.4649	1.25300	3.96232
1.58	2.4964	1.25698	3.97492
1.59	2.5281	1.26095	3.98748
1.60	2.5600	1.26491	4.00000
1.61	2.5921	1.26886	4.01248
1.62	2.6244	1.27279	4.02492
1.63	2.6569	1.27671	4.03733
1.64	2.6896	1.28062	4.04969
1.65	2.7225	1.28452	4.06202
1.66	2.7556	1.28841	4.07431
1.67	2.7889	1.29228	4.08656
1.68	2.8224	1.29615	4.09878
1.69	2.8561	1.30000	4.11096
1.70	2.8900	1.30384	4.12311
1.71	2.9241	1.30767	4.13521
1.72	2.9584	1.31149	4.14729
1.73	2.9929	1.31529	4.15933
1.74	3.0276	1.31909	4.17133
1.75	3.0625	1.32288	4.18330
1.76	3.0976	1.32665	4.19524
1.77	3.1329	1.33041	4.20714
1.78	3.1684	1.33417	4.21900
1.79	3.2041	1.33791	4.23084
1.80	3.2400	1.34164	4.24264
1.81	3.2761	1.34536	4.25441
1.82	3.3124	1.34907	4.26615
1.83	3.3489	1.35277	4.27786
1.84	3.3856	1.35647	4.28952
1.85	3.4225	1.36015	4.30116
1.86	3.4596	1.36382	4.31277
1.87	3.4969	1.36748	4.32435
1.88	3.5344	1.37113	4.33590
1.89	3.5721	1.37477	4.34741
1.90	3.6100	1.37840	4.35890
1.91	3.6481	1.38203	4.37035
1.92	3.6864	1.38564	4.38178
1.93	3.7249	1.38924	4.39318
1.94	3.7636	1.39284	4.40454
1.95	3.8025	1.39642	4.41588
1.96	3.8416	1.40000	4.42719
1.97	3.8809	1.40357	4.43847
1.98	3.9204	1.40712	4.44972
1.99	3.9601	1.41067	4.46094
2.00	4.0000	1.41421	4.47214
N	N ²	\sqrt{N}	$\sqrt{10N}$

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N²	√N	√10N
2.00	4.0000	1.41421	4.47214
2.01	4.0401	1.41774	4.48330
2.02	4.0804	1.42127	4.49444
2.03	4.1209	1.42478	4.50555
2.04	4.1616	1.42829	4.51664
2.05	4.2025	1.43178	4.52769
2.06	4.2436	1.43527	4.53872
2.07	4.2849	1.43875	4.54973
2.08	4.3264	1.44222	4.56070
2.09	4.3681	1.44568	4.57165
2.10	4.4100	1.44914	4.58258
2.11	4.4521	1.45258	4.59347
2.12	4.4944	1.45602	4.60435
2.13	4.5369	1.45945	4.61519
2.14	4.5796	1.46287	4.62601
2.15	4.6225	1.46629	4.63681
2.16	4.6656	1.46969	4.64758
2.17	4.7089	1.47309	4.65833
2.18	4.7524	1.47648	4.66905
2.19	4.7961	1.47986	4.67974
2.20	4.8400	1.48324	4.69042
2.21	4.8841	1.48661	4.70106
2.22	4.9284	1.48997	4.71169
2.23	4.9729	1.49332	4.72229
2.24	5.0176	1.49666	4.73286
2.25	5.0625	1.50000	4.74342
2.26	5.1076	1.50333	4.75395
2.27	5.1529	1.50665	4.76445
2.28	5.1984	1.50997	4.77493
2.29	5.2441	1.51327	4.78539
2.30	5.2900	1.51658	4.79583
2.31	5.3361	1.51987	4.80625
2.32	5.3824	1.52315	4.81664
2.33	5.4289	1.52643	4.82701
2.34	5.4756	1.52971	4.83735
2.35	5.5225	1.53297	4.84768
2.36	5.5696	1.53623	4.85798
2.37	5.6169	1.53948	4.86826
2.38	5.6644	1.54272	4.87852
2.39	5.7121	1.54596	4.88876
2.40	5.7600	1.54919	4.89898
2.41	5.8081	1.55242	4.90918
2.42	5.8564	1.55563	4.91935
2.43	5.9049	1.55885	4.92950
2.44	5.9536	1.56205	4.93964
2.45	6.0025	1.56525	4.94975
2.46	6.0516	1.56844	4.95984
2.47	6.1009	1.57162	4.96991
2.48	6.1504	1.57480	4.97996
2.49	6.2001	1.57797	4.98999
2.50	6.2500	1.58114	5.00000
N	N²	√N	√10N
2.50	6.2500	1.58114	5.00000
2.51	6.3001	1.58430	5.00999
2.52	6.3504	1.58745	5.01996
2.53	6.4009	1.59060	5.02991
2.54	6.4516	1.59374	5.03984
2.55	6.5025	1.59687	5.04975
2.56	6.5536	1.60000	5.05964
2.57	6.6049	1.60312	5.06952
2.58	6.6564	1.60624	5.07937
2.59	6.7081	1.60935	5.08920
2.60	6.7600	1.61245	5.09902
2.61	6.8121	1.61555	5.10882
2.62	6.8644	1.61864	5.11859
2.63	6.9169	1.62173	5.12835
2.64	6.9696	1.62481	5.13809
2.65	7.0225	1.62788	5.14782
2.66	7.0756	1.63095	5.15752
2.67	7.1289	1.63401	5.16720
2.68	7.1824	1.63707	5.17687
2.69	7.2361	1.64012	5.18652
2.70	7.2900	1.64317	5.19615
2.71	7.3441	1.64621	5.20577
2.72	7.3984	1.64924	5.21536
2.73	7.4529	1.65227	5.22494
2.74	7.5076	1.65529	5.23450
2.75	7.5625	1.65831	5.24404
2.76	7.6176	1.66132	5.25357
2.77	7.6729	1.66433	5.26308
2.78	7.7284	1.66733	5.27257
2.79	7.7841	1.67033	5.28205
2.80	7.8400	1.67332	5.29150
2.81	7.8961	1.67631	5.30094
2.82	7.9524	1.67929	5.31037
2.83	8.0089	1.68226	5.31977
2.84	8.0656	1.68523	5.32917
2.85	8.1225	1.68819	5.33854
2.86	8.1796	1.69115	5.34790
2.87	8.2369	1.69411	5.35724
2.88	8.2944	1.69706	5.36656
2.89	8.3521	1.70000	5.37587
2.90	8.4100	1.70294	5.38516
2.91	8.4681	1.70587	5.39444
2.92	8.5264	1.70880	5.40370
2.93	8.5849	1.71172	5.41295
2.94	8.6436	1.71464	5.42218
2.95	8.7025	1.71756	5.43139
2.96	8.7616	1.72047	5.44059
2.97	8.8209	1.72337	5.44977
2.98	8.8804	1.72627	5.45894
2.99	8.9401	1.72916	5.46809
3.00	9.0000	1.73205	5.47723
N	N²	√N	√10N

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	\sqrt{N}	$\sqrt{10N}$
3.00	9.0000	1.73205	5.47723
3.01	9.0601	1.73494	5.48635
3.02	9.1204	1.73781	5.49545
3.03	9.1809	1.74069	5.50454
3.04	9.2416	1.74356	5.51362
3.05	9.3025	1.74642	5.52268
3.06	9.3636	1.74929	5.53173
3.07	9.4249	1.75214	5.54076
3.08	9.4864	1.75499	5.54977
3.09	9.5481	1.75784	5.55878
3.10	9.6100	1.76068	5.56776
3.11	9.6721	1.76352	5.57674
3.12	9.7344	1.76635	5.58570
3.13	9.7969	1.76918	5.59464
3.14	9.8596	1.77200	5.60357
3.15	9.9225	1.77482	5.61249
3.16	9.9856	1.77764	5.62139
3.17	10.0489	1.78045	5.63028
3.18	10.1124	1.78326	5.63915
3.19	10.1761	1.78606	5.64801
3.20	10.2400	1.78885	5.65685
3.21	10.3041	1.79165	5.66569
3.22	10.3684	1.79444	5.67450
3.23	10.4329	1.79722	5.68331
3.24	10.4976	1.80000	5.69210
3.25	10.5625	1.80278	5.70088
3.26	10.6276	1.80555	5.70964
3.27	10.6929	1.80831	5.71839
3.28	10.7584	1.81108	5.72713
3.29	10.8241	1.81384	5.73585
3.30	10.8900	1.81659	5.74456
3.31	10.9561	1.81934	5.75326
3.32	11.0224	1.82209	5.76194
3.33	11.0889	1.82483	5.77062
3.34	11.1556	1.82757	5.77927
3.35	11.2225	1.83030	5.78792
3.36	11.2896	1.83303	5.79655
3.37	11.3569	1.83576	5.80517
3.38	11.4244	1.83848	5.81378
3.39	11.4921	1.84120	5.82237
3.40	11.5600	1.84391	5.83095
3.41	11.6281	1.84662	5.83952
3.42	11.6964	1.84932	5.84808
3.43	11.7649	1.85203	5.85662
3.44	11.8336	1.85472	5.86515
3.45	11.9025	1.85742	5.87367
3.46	11.9716	1.86011	5.88218
3.47	12.0409	1.86279	5.89067
3.48	12.1104	1.86548	5.89915
3.49	12.1801	1.86815	5.90762
3.50	12.2500	1.87083	5.91608
3.51	12.3201	1.87350	5.92453
3.52	12.3904	1.87617	5.93296
3.53	12.4609	1.87883	5.94138
3.54	12.5316	1.88149	5.94979
3.55	12.6025	1.88414	5.95819
3.56	12.6736	1.88680	5.96657
3.57	12.7449	1.88944	5.97495
3.58	12.8164	1.89209	5.98331
3.59	12.8881	1.89473	5.99166
3.60	12.9600	1.89737	6.00000
3.61	13.0321	1.90000	6.00833
3.62	13.1044	1.90263	6.01664
3.63	13.1769	1.90526	6.02495
3.64	13.2496	1.90788	6.03324
3.65	13.3225	1.91050	6.04152
3.66	13.3956	1.91311	6.04979
3.67	13.4689	1.91572	6.05805
3.68	13.5424	1.91833	6.06630
3.69	13.6161	1.92094	6.07454
3.70	13.6900	1.92354	6.08276
3.71	13.7641	1.92614	6.09098
3.72	13.8384	1.92873	6.09918
3.73	13.9129	1.93132	6.10737
3.74	13.9876	1.93391	6.11555
3.75	14.0625	1.93649	6.12372
3.76	14.1376	1.93907	6.13188
3.77	14.2129	1.94165	6.14003
3.78	14.2884	1.94422	6.14817
3.79	14.3641	1.94679	6.15630
3.80	14.4400	1.94936	6.16441
3.81	14.5161	1.95192	6.17252
3.82	14.5924	1.95448	6.18061
3.83	14.6689	1.95704	6.18870
3.84	14.7456	1.95959	6.19677
3.85	14.8225	1.96214	6.20484
3.86	14.8996	1.96469	6.21289
3.87	14.9769	1.96723	6.22093
3.88	15.0544	1.96977	6.22896
3.89	15.1321	1.97231	6.23699
3.90	15.2100	1.97484	6.24500
3.91	15.2881	1.97737	6.25300
3.92	15.3664	1.97990	6.26099
3.93	15.4449	1.98242	6.26897
3.94	15.5236	1.98494	6.27694
3.95	15.6025	1.98746	6.28490
3.96	15.6816	1.98997	6.29285
3.97	15.7609	1.99249	6.30079
3.98	15.8404	1.99499	6.30872
3.99	15.9201	1.99750	6.31664
4.00	16.0000	2.00000	6.32456
N	N ²	\sqrt{N}	$\sqrt{10N}$

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	\sqrt{N}	$\sqrt{10N}$
4.00	16.0000	2.00000	6.32456
4.01	16.0801	2.00250	6.33246
4.02	16.1604	2.00499	6.34035
4.03	16.2409	2.00749	6.34823
4.04	16.3216	2.00998	6.35610
4.05	16.4025	2.01246	6.36396
4.06	16.4836	2.01494	6.37181
4.07	16.5649	2.01742	6.37966
4.08	16.6464	2.01990	6.38749
4.09	16.7281	2.02237	6.39531
4.10	16.8100	2.02485	6.40312
4.11	16.8921	2.02731	6.41093
4.12	16.9744	2.02978	6.41872
4.13	17.0569	2.03224	6.42651
4.14	17.1396	2.03470	6.43428
4.15	17.2225	2.03715	6.44205
4.16	17.3056	2.03961	6.44981
4.17	17.3889	2.04206	6.45755
4.18	17.4724	2.04450	6.46529
4.19	17.5561	2.04695	6.47302
4.20	17.6400	2.04939	6.48074
4.21	17.7241	2.05183	6.48845
4.22	17.8084	2.05426	6.49615
4.23	17.8929	2.05670	6.50384
4.24	17.9776	2.05913	6.51153
4.25	18.0625	2.06155	6.51920
4.26	18.1476	2.06398	6.52687
4.27	18.2329	2.06640	6.53452
4.28	18.3184	2.06882	6.54217
4.29	18.4041	2.07123	6.54981
4.30	18.4900	2.07364	6.55744
4.31	18.5761	2.07605	6.56506
4.32	18.6624	2.07846	6.57267
4.33	18.7489	2.08087	6.58027
4.34	18.8356	2.08327	6.58787
4.35	18.9225	2.08567	6.59545
4.36	19.0096	2.08806	6.60303
4.37	19.0969	2.09045	6.61060
4.38	19.1844	2.09284	6.61816
4.39	19.2721	2.09523	6.62571
4.40	19.3600	2.09762	6.63325
4.41	19.4481	2.10000	6.64078
4.42	19.5364	2.10238	6.64831
4.43	19.6249	2.10476	6.65582
4.44	19.7136	2.10713	6.66333
4.45	19.8025	2.10950	6.67083
4.46	19.8916	2.11187	6.67832
4.47	19.9809	2.11424	6.68581
4.48	20.0704	2.11660	6.69328
4.49	20.1601	2.11896	6.70075
4.50	20.2500	2.12132	6.70820
N	N ²	\sqrt{N}	$\sqrt{10N}$
4.50	20.2500	2.12132	6.70820
4.51	20.3401	2.12368	6.71565
4.52	20.4304	2.12603	6.72309
4.53	20.5209	2.12838	6.73053
4.54	20.6116	2.13073	6.73795
4.55	20.7025	2.13307	6.74537
4.56	20.7936	2.13542	6.75278
4.57	20.8849	2.13776	6.76018
4.58	20.9764	2.14009	6.76757
4.59	21.0681	2.14243	6.77495
4.60	21.1600	2.14476	6.78233
4.61	21.2521	2.14709	6.78970
4.62	21.3444	2.14942	6.79706
4.63	21.4369	2.15174	6.80441
4.64	21.5296	2.15407	6.81175
4.65	21.6225	2.15639	6.81909
4.66	21.7156	2.15870	6.82642
4.67	21.8089	2.16102	6.83374
4.68	21.9024	2.16333	6.84105
4.69	21.9961	2.16564	6.84836
4.70	22.0900	2.16795	6.85565
4.71	22.1841	2.17025	6.86294
4.72	22.2784	2.17256	6.87023
4.73	22.3729	2.17486	6.87750
4.74	22.4676	2.17715	6.88477
4.75	22.5625	2.17945	6.89202
4.76	22.6576	2.18174	6.89928
4.77	22.7529	2.18403	6.90652
4.78	22.8484	2.18632	6.91375
4.79	22.9441	2.18861	6.92098
4.80	23.0400	2.19089	6.92820
4.81	23.1361	2.19317	6.93542
4.82	23.2324	2.19545	6.94262
4.83	23.3289	2.19773	6.94982
4.84	23.4256	2.20000	6.95701
4.85	23.5225	2.20227	6.96419
4.86	23.6196	2.20454	6.97137
4.87	23.7169	2.20681	6.97854
4.88	23.8144	2.20907	6.98570
4.89	23.9121	2.21133	6.99285
4.90	24.0100	2.21359	7.00000
4.91	24.1081	2.21585	7.00714
4.92	24.2064	2.21811	7.01427
4.93	24.3049	2.22036	7.02140
4.94	24.4036	2.22261	7.02851
4.95	24.5025	2.22486	7.03562
4.96	24.6016	2.22711	7.04273
4.97	24.7009	2.22935	7.04982
4.98	24.8004	2.23159	7.05691
4.99	24.9001	2.23383	7.06399
5.00	25.0000	2.23607	7.07107
N	N ²	\sqrt{N}	$\sqrt{10N}$

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	√N	√10N
5.00	25.0000	2.23607	7.07107
5.01	25.1001	2.23830	7.07814
5.02	25.2004	2.24054	7.08520
5.03	25.3009	2.24277	7.09225
5.04	25.4016	2.24499	7.09930
5.05	25.5025	2.24722	7.10634
5.06	25.6036	2.24944	7.11337
5.07	25.7049	2.25167	7.12039
5.08	25.8064	2.25389	7.12741
5.09	25.9081	2.25610	7.13442
5.10	26.0100	2.25832	7.14143
5.11	26.1121	2.26053	7.14843
5.12	26.2144	2.26274	7.15542
5.13	26.3169	2.26495	7.16240
5.14	26.4196	2.26716	7.16938
5.15	26.5225	2.26936	7.17635
5.16	26.6256	2.27156	7.18331
5.17	26.7289	2.27376	7.19027
5.18	26.8324	2.27596	7.19722
5.19	26.9361	2.27816	7.20417
5.20	27.0400	2.28035	7.21110
5.21	27.1441	2.28254	7.21803
5.22	27.2484	2.28473	7.22496
5.23	27.3529	2.28692	7.23187
5.24	27.4576	2.28910	7.23878
5.25	27.5625	2.29129	7.24569
5.26	27.6676	2.29347	7.25259
5.27	27.7729	2.29565	7.25948
5.28	27.8784	2.29783	7.26636
5.29	27.9841	2.30000	7.27324
5.30	28.0900	2.30217	7.28011
5.31	28.1961	2.30434	7.28697
5.32	28.3024	2.30651	7.29383
5.33	28.4089	2.30868	7.30068
5.34	28.5156	2.31084	7.30753
5.35	28.6225	2.31301	7.31437
5.36	28.7296	2.31517	7.32120
5.37	28.8369	2.31733	7.32803
5.38	28.9444	2.31948	7.33485
5.39	29.0521	2.32164	7.34166
5.40	29.1600	2.32379	7.34847
5.41	29.2681	2.32594	7.35527
5.42	29.3764	2.32809	7.36206
5.43	29.4849	2.33024	7.36885
5.44	29.5936	2.33238	7.37564
5.45	29.7025	2.33452	7.38241
5.46	29.8116	2.33666	7.38918
5.47	29.9209	2.33880	7.39594
5.48	30.0304	2.34094	7.40270
5.49	30.1401	2.34307	7.40945
5.50	30.2500	2.34521	7.41620
N	N ²	√N	√10N
5.50	30.2500	2.34521	7.41620
5.51	30.3601	2.34734	7.42294
5.52	30.4704	2.34947	7.42967
5.53	30.5809	2.35160	7.43640
5.54	30.6916	2.35372	7.44312
5.55	30.8025	2.35584	7.44983
5.56	30.9136	2.35797	7.45654
5.57	31.0249	2.36008	7.46324
5.58	31.1364	2.36220	7.46994
5.59	31.2481	2.36432	7.47663
5.60	31.3600	2.36643	7.48331
5.61	31.4721	2.36854	7.48999
5.62	31.5844	2.37065	7.49667
5.63	31.6969	2.37276	7.50333
5.64	31.8096	2.37487	7.50999
5.65	31.9225	2.37697	7.51665
5.66	32.0356	2.37908	7.52330
5.67	32.1489	2.38118	7.52994
5.68	32.2624	2.38328	7.53658
5.69	32.3761	2.38537	7.54321
5.70	32.4900	2.38747	7.54983
5.71	32.6041	2.38956	7.55645
5.72	32.7184	2.39165	7.56307
5.73	32.8329	2.39374	7.56968
5.74	32.9476	2.39583	7.57628
5.75	33.0625	2.39792	7.58288
5.76	33.1776	2.40000	7.58947
5.77	33.2929	2.40208	7.59605
5.78	33.4084	2.40416	7.60263
5.79	33.5241	2.40624	7.60920
5.80	33.6400	2.40832	7.61577
5.81	33.7561	2.41039	7.62234
5.82	33.8724	2.41247	7.62889
5.83	33.9889	2.41454	7.63544
5.84	34.1056	2.41661	7.64199
5.85	34.2225	2.41868	7.64853
5.86	34.3396	2.42074	7.65506
5.87	34.4569	2.42281	7.66159
5.88	34.5744	2.42487	7.66812
5.89	34.6921	2.42693	7.67463
5.90	34.8100	2.42899	7.68115
5.91	34.9281	2.43105	7.68765
5.92	35.0464	2.43311	7.69415
5.93	35.1649	2.43516	7.70065
5.94	35.2836	2.43721	7.70714
5.95	35.4025	2.43926	7.71362
5.96	35.5216	2.44131	7.72010
5.97	35.6409	2.44336	7.72658
5.98	35.7604	2.44540	7.73305
5.99	35.8801	2.44745	7.73951
6.00	36.0000	2.44949	7.74597
N	N ²	√N	√10N

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	√N	√10N
6.00	36.0000	2.44949	7.74597
6.01	36.1201	2.45153	7.75242
6.02	36.2404	2.45357	7.75887
6.03	36.3609	2.45561	7.76531
6.04	36.4816	2.45764	7.77174
6.05	36.6025	2.45967	7.77817
6.06	36.7236	2.46171	7.78460
6.07	36.8449	2.46374	7.79102
6.08	36.9664	2.46577	7.79744
6.09	37.0881	2.46779	7.80385
6.10	37.2100	2.46982	7.81025
6.11	37.3321	2.47184	7.81665
6.12	37.4544	2.47386	7.82304
6.13	37.5769	2.47588	7.82943
6.14	37.6996	2.47790	7.83582
6.15	37.8225	2.47992	7.84219
6.16	37.9456	2.48193	7.84857
6.17	38.0689	2.48395	7.85493
6.18	38.1924	2.48596	7.86130
6.19	38.3161	2.48797	7.86766
6.20	38.4400	2.48998	7.87401
6.21	38.5641	2.49199	7.88036
6.22	38.6884	2.49399	7.88670
6.23	38.8129	2.49600	7.89303
6.24	38.9376	2.49800	7.89937
6.25	39.0625	2.50000	7.90569
6.26	39.1876	2.50200	7.91202
6.27	39.3129	2.50400	7.91833
6.28	39.4384	2.50599	7.92465
6.29	39.5641	2.50799	7.93095
6.30	39.6900	2.50998	7.93725
6.31	39.8161	2.51197	7.94355
6.32	39.9424	2.51396	7.94984
6.33	40.0689	2.51595	7.95613
6.34	40.1956	2.51794	7.96241
6.35	40.3225	2.51992	7.96869
6.36	40.4496	2.52190	7.97496
6.37	40.5769	2.52389	7.98123
6.38	40.7044	2.52587	7.98749
6.39	40.8321	2.52784	7.99375
6.40	40.9600	2.52982	8.00000
6.41	41.0881	2.53180	8.00625
6.42	41.2164	2.53377	8.01249
6.43	41.3449	2.53574	8.01873
6.44	41.4736	2.53772	8.02496
6.45	41.6025	2.53969	8.03119
6.46	41.7316	2.54165	8.03741
6.47	41.8609	2.54362	8.04363
6.48	41.9904	2.54558	8.04984
6.49	42.1201	2.54755	8.05605
6.50	42.2500	2.54951	8.06226
N	N ²	√N	√10N
6.50	42.2500	2.54951	8.06226
6.51	42.3801	2.55147	8.06846
6.52	42.5104	2.55343	8.07465
6.53	42.6409	2.55539	8.08084
6.54	42.7716	2.55734	8.08703
6.55	42.9025	2.55930	8.09321
6.56	43.0336	2.56125	8.09938
6.57	43.1649	2.56320	8.10555
6.58	43.2964	2.56515	8.11172
6.59	43.4281	2.56710	8.11788
6.60	43.5600	2.56905	8.12404
6.61	43.6921	2.57099	8.13019
6.62	43.8244	2.57294	8.13634
6.63	43.9569	2.57488	8.14248
6.64	44.0896	2.57682	8.14862
6.65	44.2225	2.57876	8.15475
6.66	44.3556	2.58070	8.16088
6.67	44.4889	2.58263	8.16701
6.68	44.6224	2.58457	8.17313
6.69	44.7561	2.58650	8.17924
6.70	44.8900	2.58844	8.18535
6.71	45.0241	2.59037	8.19146
6.72	45.1584	2.59230	8.19756
6.73	45.2929	2.59422	8.20366
6.74	45.4276	2.59615	8.20975
6.75	45.5625	2.59808	8.21584
6.76	45.6976	2.60000	8.22192
6.77	45.8329	2.60192	8.22800
6.78	45.9684	2.60384	8.23408
6.79	46.1041	2.60576	8.24015
6.80	46.2400	2.60768	8.24621
6.81	46.3761	2.60960	8.25227
6.82	46.5124	2.61151	8.25833
6.83	46.6489	2.61343	8.26438
6.84	46.7856	2.61534	8.27043
6.85	46.9225	2.61725	8.27647
6.86	47.0596	2.61916	8.28251
6.87	47.1969	2.62107	8.28855
6.88	47.3344	2.62298	8.29458
6.89	47.4721	2.62488	8.30060
6.90	47.6100	2.62679	8.30662
6.91	47.7481	2.62869	8.31264
6.92	47.8864	2.63059	8.31865
6.93	48.0249	2.63249	8.32466
6.94	48.1636	2.63439	8.33067
6.95	48.3025	2.63629	8.33667
6.96	48.4416	2.63818	8.34266
6.97	48.5809	2.64008	8.34865
6.98	48.7204	2.64197	8.35464
6.99	48.8601	2.64386	8.36062
7.00	49.0000	2.64575	8.36660
N	N ²	√N	√10N

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	√N	√10N
7.00	49.0000	2.64575	8.36660
7.01	49.1401	2.64764	8.37257
7.02	49.2804	2.64953	8.37854
7.03	49.4209	2.65141	8.38451
7.04	49.5616	2.65330	8.39047
7.05	49.7025	2.65518	8.39643
7.06	49.8436	2.65707	8.40238
7.07	49.9849	2.65895	8.40833
7.08	50.1264	2.66083	8.41427
7.09	50.2681	2.66271	8.42021
7.10	50.4100	2.66458	8.42615
7.11	50.5521	2.66646	8.43208
7.12	50.6944	2.66833	8.43801
7.13	50.8369	2.67021	8.44393
7.14	50.9796	2.67208	8.44985
7.15	51.1225	2.67395	8.45577
7.16	51.2656	2.67582	8.46168
7.17	51.4089	2.67769	8.46759
7.18	51.5524	2.67955	8.47349
7.19	51.6961	2.68142	8.47939
7.20	51.8400	2.68328	8.48528
7.21	51.9841	2.68514	8.49117
7.22	52.1284	2.68701	8.49706
7.23	52.2729	2.68887	8.50294
7.24	52.4176	2.69072	8.50882
7.25	52.5625	2.69258	8.51469
7.26	52.7076	2.69444	8.52056
7.27	52.8529	2.69629	8.52643
7.28	52.9984	2.69815	8.53229
7.29	53.1441	2.70000	8.53815
7.30	53.2900	2.70185	8.54400
7.31	53.4361	2.70370	8.54985
7.32	53.5824	2.70555	8.55570
7.33	53.7289	2.70740	8.56154
7.34	53.8756	2.70924	8.56738
7.35	54.0225	2.71109	8.57321
7.36	54.1696	2.71293	8.57904
7.37	54.3169	2.71477	8.58487
7.38	54.4644	2.71662	8.59069
7.39	54.6121	2.71846	8.59651
7.40	54.7600	2.72029	8.60233
7.41	54.9081	2.72213	8.60814
7.42	55.0564	2.72397	8.61394
7.43	55.2049	2.72580	8.61974
7.44	55.3536	2.72764	8.62554
7.45	55.5025	2.72947	8.63134
7.46	55.6516	2.73130	8.63713
7.47	55.8009	2.73313	8.64292
7.48	55.9504	2.73496	8.64870
7.49	56.1001	2.73679	8.65448
7.50	56.2500	2.73861	8.66025
N	N ²	√N	√10N
7.50	56.2500	2.73861	8.66025
7.51	56.4001	2.74044	8.66603
7.52	56.5504	2.74226	8.67179
7.53	56.7009	2.74408	8.67756
7.54	56.8516	2.74591	8.68332
7.55	57.0025	2.74773	8.68907
7.56	57.1536	2.74955	8.69483
7.57	57.3049	2.75136	8.70057
7.58	57.4564	2.75318	8.70632
7.59	57.6081	2.75500	8.71206
7.60	57.7600	2.75681	8.71780
7.61	57.9121	2.75862	8.72353
7.62	58.0644	2.76043	8.72926
7.63	58.2169	2.76225	8.73499
7.64	58.3696	2.76405	8.74071
7.65	58.5225	2.76586	8.74643
7.66	58.6756	2.76767	8.75214
7.67	58.8289	2.76948	8.75785
7.68	58.9824	2.77128	8.76356
7.69	59.1361	2.77308	8.76926
7.70	59.2900	2.77489	8.77496
7.71	59.4441	2.77669	8.78066
7.72	59.5984	2.77849	8.78635
7.73	59.7529	2.78029	8.79204
7.74	59.9076	2.78209	8.79773
7.75	60.0625	2.78388	8.80341
7.76	60.2176	2.78568	8.80909
7.77	60.3729	2.78747	8.81476
7.78	60.5284	2.78927	8.82043
7.79	60.6841	2.79106	8.82610
7.80	60.8400	2.79285	8.83176
7.81	60.9961	2.79464	8.83742
7.82	61.1524	2.79643	8.84308
7.83	61.3089	2.79821	8.84873
7.84	61.4656	2.80000	8.85438
7.85	61.6225	2.80179	8.86002
7.86	61.7796	2.80357	8.86566
7.87	61.9369	2.80535	8.87130
7.88	62.0944	2.80713	8.87694
7.89	62.2521	2.80891	8.88257
7.90	62.4100	2.81069	8.88819
7.91	62.5681	2.81247	8.89382
7.92	62.7264	2.81425	8.89944
7.93	62.8849	2.81603	8.90505
7.94	63.0436	2.81780	8.91067
7.95	63.2025	2.81957	8.91628
7.96	63.3616	2.82135	8.92188
7.97	63.5209	2.82312	8.92749
7.98	63.6804	2.82489	8.93308
7.99	63.8401	2.82666	8.93868
8.00	64.0000	2.82843	8.94427
N	N ²	√N	√10N

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	\sqrt{N}	$\sqrt{10N}$
8.00	64.0000	2.82843	8.94427
8.01	64.1601	2.83019	8.94986
8.02	64.3204	2.83196	8.95545
8.03	64.4809	2.83373	8.96103
8.04	64.6416	2.83549	8.96660
8.05	64.8025	2.83725	8.97218
8.06	64.9636	2.83901	8.97775
8.07	65.1249	2.84077	8.98332
8.08	65.2864	2.84253	8.98888
8.09	65.4481	2.84429	8.99444
8.10	65.6100	2.84605	9.00000
8.11	65.7721	2.84781	9.00555
8.12	65.9344	2.84956	9.01110
8.13	66.0969	2.85132	9.01665
8.14	66.2596	2.85307	9.02219
8.15	66.4225	2.85482	9.02774
8.16	66.5856	2.85657	9.03327
8.17	66.7489	2.85832	9.03881
8.18	66.9124	2.86007	9.04434
8.19	67.0761	2.86182	9.04986
8.20	67.2400	2.86356	9.05539
8.21	67.4041	2.86531	9.06091
8.22	67.5684	2.86705	9.06642
8.23	67.7329	2.86880	9.07193
8.24	67.8976	2.87054	9.07744
8.25	68.0625	2.87228	9.08295
8.26	68.2276	2.87402	9.08845
8.27	68.3929	2.87576	9.09395
8.28	68.5584	2.87750	9.09945
8.29	68.7241	2.87924	9.10494
8.30	68.8900	2.88097	9.11043
8.31	69.0561	2.88271	9.11592
8.32	69.2224	2.88444	9.12140
8.33	69.3889	2.88617	9.12688
8.34	69.5556	2.88791	9.13236
8.35	69.7225	2.88964	9.13783
8.36	69.8896	2.89137	9.14330
8.37	70.0569	2.89310	9.14877
8.38	70.2244	2.89482	9.15423
8.39	70.3921	2.89655	9.15969
8.40	70.5600	2.89828	9.16515
8.41	70.7281	2.90000	9.17061
8.42	70.8964	2.90172	9.17606
8.43	71.0649	2.90345	9.18150
8.44	71.2336	2.90517	9.18695
8.45	71.4025	2.90689	9.19239
8.46	71.5716	2.90861	9.19783
8.47	71.7409	2.91033	9.20326
8.48	71.9104	2.91204	9.20869
8.49	72.0801	2.91376	9.21412
8.50	72.2500	2.91548	9.21954
N	N ²	\sqrt{N}	$\sqrt{10N}$
8.50	72.2500	2.91548	9.21954
8.51	72.4201	2.91719	9.22497
8.52	72.5904	2.91890	9.23038
8.53	72.7609	2.92062	9.23580
8.54	72.9316	2.92233	9.24121
8.55	73.1025	2.92404	9.24662
8.56	73.2736	2.92575	9.25203
8.57	73.4449	2.92746	9.25743
8.58	73.6164	2.92916	9.26283
8.59	73.7881	2.93087	9.26823
8.60	73.9600	2.93258	9.27362
8.61	74.1321	2.93428	9.27901
8.62	74.3044	2.93598	9.28440
8.63	74.4769	2.93769	9.28978
8.64	74.6496	2.93939	9.29516
8.65	74.8225	2.94109	9.30054
8.66	74.9956	2.94279	9.30591
8.67	75.1689	2.94449	9.31128
8.68	75.3424	2.94618	9.31665
8.69	75.5161	2.94788	9.32202
8.70	75.6900	2.94958	9.32738
8.71	75.8641	2.95127	9.33274
8.72	76.0384	2.95296	9.33809
8.73	76.2129	2.95466	9.34345
8.74	76.3876	2.95635	9.34880
8.75	76.5625	2.95804	9.35414
8.76	76.7376	2.95973	9.35949
8.77	76.9129	2.96142	9.36483
8.78	77.0884	2.96311	9.37017
8.79	77.2641	2.96479	9.37550
8.80	77.4400	2.96648	9.38083
8.81	77.6161	2.96816	9.38616
8.82	77.7924	2.96985	9.39149
8.83	77.9689	2.97153	9.39681
8.84	78.1456	2.97321	9.40213
8.85	78.3225	2.97489	9.40744
8.86	78.4996	2.97658	9.41276
8.87	78.6769	2.97825	9.41807
8.88	78.8544	2.97993	9.42338
8.89	79.0321	2.98161	9.42868
8.90	79.2100	2.98329	9.43398
8.91	79.3881	2.98496	9.43928
8.92	79.5664	2.98664	9.44458
8.93	79.7449	2.98831	9.44987
8.94	79.9236	2.98998	9.45516
8.95	80.1025	2.99166	9.46044
8.96	80.2816	2.99333	9.46573
8.97	80.4609	2.99500	9.47101
8.98	80.6404	2.99666	9.47629
8.99	80.8201	2.99833	9.48156
9.00	81.0000	3.00000	9.48683
N	N ²	\sqrt{N}	$\sqrt{10N}$

TABLE 1—SQUARES AND SQUARE ROOTS—Continued

N	N ²	\sqrt{N}	$\sqrt{10N}$
9.00	81.0000	3.00000	9.48683
9.01	81.1801	3.00167	9.49210
9.02	81.3604	3.00333	9.49737
9.03	81.5409	3.00500	9.50263
9.04	81.7216	3.00666	9.50789
9.05	81.9025	3.00832	9.51315
9.06	82.0836	3.00998	9.51840
9.07	82.2649	3.01164	9.52365
9.08	82.4464	3.01330	9.52890
9.09	82.6281	3.01496	9.53415
9.10	82.8100	3.01662	9.53939
9.11	82.9921	3.01828	9.54463
9.12	83.1744	3.01993	9.54987
9.13	83.3569	3.02159	9.55510
9.14	83.5396	3.02324	9.56033
9.15	83.7225	3.02490	9.56556
9.16	83.9056	3.02655	9.57079
9.17	84.0889	3.02820	9.57601
9.18	84.2724	3.02985	9.58123
9.19	84.4561	3.03150	9.58645
9.20	84.6400	3.03315	9.59165
9.21	84.8241	3.03480	9.59687
9.22	85.0084	3.03645	9.60208
9.23	85.1929	3.03809	9.60729
9.24	85.3776	3.03974	9.61249
9.25	85.5625	3.04138	9.61769
9.26	85.7476	3.04302	9.62289
9.27	85.9329	3.04467	9.62808
9.28	86.1184	3.04631	9.63328
9.29	86.3041	3.04795	9.63846
9.30	86.4900	3.04959	9.64365
9.31	86.6761	3.05123	9.64883
9.32	86.8624	3.05287	9.65401
9.33	87.0489	3.05450	9.65919
9.34	87.2356	3.05614	9.66437
9.35	87.4225	3.05778	9.66954
9.36	87.6096	3.05941	9.67471
9.37	87.7969	3.06105	9.67988
9.38	87.9844	3.06268	9.68504
9.39	88.1721	3.06431	9.69020
9.40	88.3600	3.06594	9.69536
9.41	88.5481	3.06757	9.70052
9.42	88.7364	3.06920	9.70567
9.43	88.9249	3.07083	9.71082
9.44	89.1136	3.07246	9.71597
9.45	89.3025	3.07409	9.72111
9.46	89.4916	3.07571	9.72625
9.47	89.6809	3.07734	9.73139
9.48	89.8704	3.07896	9.73653
9.49	90.0601	3.08058	9.74166
9.50	90.2500	3.08221	9.74679
N	N ²	\sqrt{N}	$\sqrt{10N}$
9.50	90.2500	3.08221	9.74679
9.51	90.4401	3.08383	9.75192
9.52	90.6304	3.08545	9.75705
9.53	90.8209	3.08707	9.76217
9.54	91.0116	3.08869	9.76729
9.55	91.2025	3.09031	9.77241
9.56	91.3936	3.09192	9.77753
9.57	91.5849	3.09354	9.78264
9.58	91.7764	3.09516	9.78775
9.59	91.9681	3.09677	9.79285
9.60	92.1600	3.09839	9.79796
9.61	92.3521	3.10000	9.80306
9.62	92.5444	3.10161	9.80816
9.63	92.7369	3.10322	9.81326
9.64	92.9296	3.10483	9.81835
9.65	93.1225	3.10644	9.82344
9.66	93.3156	3.10805	9.82853
9.67	93.5089	3.10966	9.83362
9.68	93.7024	3.11127	9.83870
9.69	93.8961	3.11288	9.84378
9.70	94.0900	3.11448	9.84886
9.71	94.2841	3.11609	9.85393
9.72	94.4784	3.11769	9.85901
9.73	94.6729	3.11929	9.86408
9.74	94.8676	3.12090	9.86914
9.75	95.0625	3.12250	9.87421
9.76	95.2576	3.12410	9.87927
9.77	95.4529	3.12570	9.88433
9.78	95.6484	3.12730	9.88939
9.79	95.8441	3.12890	9.89444
9.80	96.0400	3.13050	9.89949
9.81	96.2361	3.13209	9.90454
9.82	96.4324	3.13369	9.90959
9.83	96.6289	3.13528	9.91464
9.84	96.8256	3.13688	9.91968
9.85	97.0225	3.13847	9.92472
9.86	97.2196	3.14006	9.92975
9.87	97.4169	3.14166	9.93479
9.88	97.6144	3.14325	9.93982
9.89	97.8121	3.14484	9.94485
9.90	98.0100	3.14643	9.94987
9.91	98.2081	3.14802	9.95490
9.92	98.4064	3.14960	9.95992
9.93	98.6049	3.15119	9.96494
9.94	98.8036	3.15278	9.96995
9.95	99.0025	3.15436	9.97497
9.96	99.2016	3.15595	9.97998
9.97	99.4009	3.15753	9.98499
9.98	99.6004	3.15911	9.98999
9.99	99.8001	3.16070	9.99500
10.00	100.000	3.16228	10.0000
N	N ²	\sqrt{N}	$\sqrt{10N}$

TABLE 2—COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
0	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
N	0	1	2	3	4	5	6	7	8	9

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TABLE 2—COMMON LOGARITHMS—Continued

N	0	1	2	3	4	5	6	7	8	9
60	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
N	0	1	2	3	4	5	6	7	8	9

TABLE 3

NATURAL TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	Sin	Tan	Cot	Cos	Deg.	Deg.	Sin	Tan	Cot	Cos	Deg.
0.0	0.00000	0.00000	∞	1.0000	90.0	.5	.07846	.07870	12.706	.9969	.5
.1	.00175	.00175	573.0	1.0000	.9	.6	.08020	.08046	12.429	.9968	.4
.2	.00349	.00349	286.5	1.0000	8	.7	.08194	.08221	12.163	.9966	.3
.3	.00524	.00524	191.0	1.0000	.7	.8	.08368	.08397	11.909	.9965	.2
.4	.00698	.00698	143.24	1.0000	6	.9	.08542	.08573	11.664	.9963	.1
.5	.00873	.00873	114.59	1.0000	.5	5.0	0.08716	0.08749	11.430	0.9962	85.0
.6	.01047	.01047	95.49	0.9999	4	1	.08889	.08925	11.205	.9960	.9
.7	.01222	.01222	81.85	.9999	.3	.2	.09063	.09101	10.988	.9959	.8
.8	.01396	.01396	71.62	.9999	2	3	.09237	.09277	10.780	.9957	.7
.9	.01571	.01571	63.66	.9999	.1	.4	.09411	.09453	10.579	.9956	.6
1.0	0.01745	0.01746	57.29	0.9998	89.0	5	.09585	.09629	10.385	.9954	.5
.1	.01920	.01920	52.08	.9998	9	6	.09758	.09805	10.190	.9952	.4
.2	.02094	.02095	47.74	.9998	8	7	.09932	.09981	10.019	.9951	.3
.3	.02269	.02269	44.07	.9997	.7	8	.10106	.10158	9.845	.9949	.2
.4	.02443	.02444	40.92	.9997	6	9	.10279	.10334	9.677	.9947	.1
.5	.02618	.02619	38.19	.9997	5	6.0	0.10453	0.10510	9.514	0.9945	84.0
.6	.02792	.02793	35.80	.9996	4	1	.10626	.10687	9.357	.9943	.9
.7	.02967	.02968	33.69	.9996	3	.2	.10800	.10863	9.205	.9942	.8
.8	.03141	.03143	31.82	.9995	2	3	.10973	.11040	9.058	.9940	.7
.9	.03316	.03317	30.14	.9995	1	4	.11147	.11217	8.915	.9938	.6
2.0	0.03490	0.03492	28.64	0.9994	88.0	.5	.11320	.11394	8.777	.9936	.5
.1	.03664	.03667	27.27	.9993	9	6	.11494	.11570	8.643	.9934	.4
.2	.03839	.03842	26.03	.9993	.8	7	.11667	.11747	8.513	.9932	.3
.3	.04013	.04016	24.90	.9992	7	.8	.11840	.11924	8.386	.9930	.2
.4	.04188	.04191	23.86	.9991	6	9	.12014	.12101	8.264	.9928	.1
.5	.04362	.04366	22.90	.9990	.5	7.0	0.12187	0.12278	8.144	0.9925	83.0
.6	.04536	.04541	22.02	.9990	4	1	.12360	.12456	8.028	.9923	.9
.7	.04711	.04716	21.20	.9989	.3	2	.12533	.12633	7.916	.9921	.8
.8	.04885	.04891	20.45	.9988	.2	3	.12706	.12810	7.806	.9919	.7
.9	.05059	.05066	19.74	.9987	.1	4	.12880	.12988	7.700	.9917	.6
3.0	0.05234	0.05241	19.081	0.9986	87.0	5	.13053	.13165	7.596	.9914	.5
.1	.05408	.05416	18.464	.9985	.9	6	.13226	.13343	7.495	.9912	.4
.2	.05582	.05591	17.886	.9984	8	.7	.13399	.13521	7.396	.9910	.3
.3	.05756	.05766	17.343	.9983	7	8	.13572	.13698	7.300	.9907	.2
.4	.05931	.05941	16.832	.9982	.6	.9	.13744	.13876	7.207	.9905	.1
.5	.06105	.06116	16.350	.9981	5	8.0	0.13917	0.14054	7.115	0.9903	82.0
.6	.06279	.06291	15.895	.9980	.4	1	.14090	.14232	7.026	.9900	.9
.7	.06453	.06467	15.464	.9979	.3	2	.14263	.14410	6.940	.9898	.8
.8	.06627	.06642	15.056	.9978	.2	.3	.14436	.14588	6.855	.9895	.7
.9	.06802	.06817	14.669	.9977	.1	.4	.14608	.14767	6.772	.9893	.6
4.0	0.06976	0.06993	14.301	0.9976	86.0	5	.14781	.14945	6.691	.9890	.5
.1	.07150	.07168	13.951	.9974	.9	.6	.14964	.15124	6.612	.9888	.4
.2	.07324	.07344	13.617	.9973	.8	.7	.15126	.15302	6.535	.9885	.3
.3	.07498	.07519	13.300	.9972	.7	.8	.15299	.15481	6.460	.9882	.2
.4	.07672	.07695	12.996	.9971	.6	.9	.15471	.15660	6.386	.9880	.1
Deg.	Cos	Cot	Tan	Sin	Deg.	Deg.	Cos	Cot	Tan	Sin	Deg.

TABLE 3—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	Sin	Tan	Cot	Cos	Deg.	Deg.	Sin	Tan	Cot	Cos	Deg.
9.0	0.15643	0.15838	6.314	0.9877	81.0	5	.2334	.2401	4.165	.9724	.5
.1	.15816	.16017	6.243	.9874	.9	6	.2351	.2419	4.134	.9720	.4
.2	.15988	.16196	6.174	.9871	.8	7	.2368	.2438	4.102	.9715	.3
.3	.16160	.16376	6.107	.9869	.7	8	.2385	.2456	4.071	.9711	.2
.4	.16333	.16555	6.041	.9866	.6	9	.2402	.2475	4.041	.9707	.1
5	.16505	.16734	5.976	.9863	.5	14.0	0.2419	0.2493	4.011	0.9703	76.0
.6	.16677	.16914	5.912	.9860	.4	1	.2436	.2512	3.981	.9699	.9
.7	.16849	.17093	5.850	.9857	.3	2	.2453	.2530	3.952	.9694	.8
.8	.17021	.17273	5.789	.9854	.2	3	.2470	.2549	3.923	.9690	.7
.9	.17193	.17453	5.730	.9851	.1	4	.2487	.2568	3.895	.9686	.6
10.0	0.1736	0.1763	5.671	0.9848	80.0	5	.2504	.2586	3.867	.9681	.5
.1	.1754	.1781	5.614	.9845	.9	6	.2521	.2605	3.839	.9677	.4
.2	.1771	.1799	5.558	.9842	.8	7	.2538	.2623	3.812	.9673	.3
.3	.1788	.1817	5.503	.9839	.7	8	.2554	.2642	3.785	.9668	.2
.4	.1805	.1835	5.449	.9836	.6	9	.2571	.2661	3.758	.9664	.1
.5	.1822	.1853	5.396	.9833	.5	15.0	0.2588	0.2.79	3.732	0.9659	75.0
.6	.1840	.1871	5.343	.9829	.4	1	.2605	.2698	3.706	.9655	.9
.7	.1857	.1890	5.292	.9826	.3	2	.2622	.2717	3.681	.9650	.8
.8	.1874	.1908	5.242	.9823	.2	3	.2639	.2736	3.655	.9646	.7
.9	.1891	.1926	5.193	.9820	.1	4	.2656	.2754	3.630	.9641	.6
11.0	0.1908	0.1944	5.145	0.9816	79.0	.5	.2672	.2773	3.606	.9636	.5
.1	.1925	.1962	5.097	.9813	.9	6	.2689	.2792	3.582	.9632	.4
.2	.1942	.1980	5.050	.9810	.8	7	.2706	.2811	3.558	.9627	.3
.3	.1959	.1998	5.005	.9806	.7	8	.2723	.2830	3.534	.9622	.2
.4	.1977	.2016	4.959	.9803	.6	9	.2740	.2849	3.511	.9617	.1
.5	.1994	.2035	4.915	.9799	.5	16.0	0.2756	0.2867	3.487	0.9613	74.0
.6	.2011	.2053	4.872	.9796	.4	1	.2773	.2886	3.465	.9608	.9
.7	.2028	.2071	4.829	.9792	.3	2	.2790	.2905	3.442	.9603	.8
.8	.2045	.2089	4.787	.9789	.2	3	.2807	.2924	3.420	.9598	.7
.9	.2062	.2107	4.745	.9785	.1	4	.2823	.2943	3.398	.9593	.6
12.0	0.2079	0.2126	4.705	0.9781	78.0	.5	.2840	.2962	3.376	.9588	.5
.1	.2096	.2144	4.665	.9778	.9	6	.2857	.2981	3.354	.9583	.4
.2	.2113	.2162	4.625	.9774	.8	7	.2874	.3000	3.333	.9578	.3
.3	.2130	.2180	4.586	.9770	.7	8	.2890	.3019	3.312	.9573	.2
.4	.2147	.2199	4.548	.9767	.6	9	.2907	.3038	3.291	.9568	.1
.5	.2164	.2217	4.511	.9763	.5	17.0	0.2924	0.3057	3.271	0.9563	73.0
.6	.2181	.2235	4.474	.9759	.4	1	.2940	.3076	3.251	.9558	.9
.7	.2198	.2254	4.437	.9755	.3	2	.2957	.3096	3.230	.9553	.8
.8	.2215	.2272	4.402	.9751	.2	3	.2974	.3115	3.211	.9548	.7
.9	.2233	.2290	4.366	.9748	.1	4	.2990	.3134	3.191	.9542	.6
13.0	0.2250	0.2309	4.331	0.9744	77.0	.5	.3007	.3153	3.172	.9537	.5
.1	.2267	.2327	4.297	.9740	.9	6	.3024	.3172	3.152	.9532	.4
.2	.2284	.2345	4.264	.9736	.8	.7	.3040	.3191	3.133	.9527	.3
.3	.2300	.2364	4.230	.9732	.7	8	.3057	.3211	3.115	.9521	.2
.4	.2317	.2382	4.198	.9728	.6	9	.3074	.3230	3.096	.9516	.1
Deg.	Cos	Cot	Tan	Sin	Deg	Deg	Cos	Cot	Tan	Sin	Deg.

TABLE 3—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	Sin	Tan	Cot	Cos	Deg.	Deg.	Sin	Tan	Cot	Cos	Deg.
18.0	0.3090	0.3249	3.078	0.9511	72 0	.5	.3827	.4142	2.414	.9239	.5
.1	.3107	.3269	3.060	.9505	.9	6	.3843	.4163	2.402	.9232	.4
.2	.3123	.3288	3.042	.9500	.8	.7	.3859	.4183	2.391	.9225	.3
.3	.3140	.3307	3.024	.9494	.7	.8	.3875	.4204	2.379	.9219	.2
.4	.3156	.3327	3.006	.9489	6	.9	.3891	.4224	2.367	.9212	.1
.5	.3173	.3346	2.989	.9483	5	23 0	0.3907	0.4245	2.356	0.9205	67.0
.6	.3190	.3365	2.971	.9478	.4	1	.3923	.4265	2.344	.9198	.9
.7	.3206	.3385	2.954	.9472	3	.2	.3939	.4286	2.333	.9191	.8
.8	.3223	.3404	2.937	.9466	2	3	.3955	.4307	2.322	.9184	.7
.9	.3239	.3424	2.921	.9461	.1	4	.3971	.4327	2.311	.9178	.6
19 0	0.3256	0.3443	2.904	0.9455	71 0	.5	.3987	.4348	2.300	.9171	.5
.1	.3272	.3463	2.888	.9449	.9	.6	.4003	.4369	2.289	.9164	.4
.2	.3289	.3482	2.872	.9444	.8	.7	.4019	.4390	2.278	.9157	.3
.3	.3305	.3502	2.856	.9438	.7	8	.4035	.4411	2.267	.9150	.2
.4	.3322	.3522	2.840	.9432	6	.9	.4051	.4431	2.257	.9143	.1
.5	.3338	.3541	2.824	.9426	.5	24.0	0.4067	0.4452	2.246	0.9135	66 0
.6	.3355	.3561	2.808	.9421	.4	1	.4083	.4473	2.236	.9128	.9
.7	.3371	.3581	2.793	.9415	3	.2	.4099	.4494	2.225	.9121	.8
.8	.3387	.3600	2.778	.9409	2	.3	.4115	.4515	2.215	.9114	.7
.9	.3404	.3620	2.762	.9403	.1	4	.4131	.4536	2.204	.9107	.6
20 0	0.3420	0.3640	2.747	0.9397	70 0	5	.4147	.4557	2.194	.9100	.5
.1	.3437	.3659	2.733	.9391	.9	.6	.4163	.4578	2.184	.9092	.4
.2	.3453	.3679	2.716	.9385	.8	.7	.4179	.4599	2.174	.9085	.3
.3	.3469	.3699	2.703	.9379	.7	8	.4195	.4621	2.164	.9078	.2
.4	.3486	.3719	2.689	.9373	.6	.9	.4210	.4642	2.154	.9070	.1
.5	.3502	.3739	2.675	.9367	.5	25 0	0.4226	0.4663	2.145	0.9063	65 0
.6	.3518	.3759	2.660	.9361	.4	1	.4242	.4684	2.135	.9056	.9
.7	.3535	.3779	2.646	.9354	.3	2	.4258	.4706	2.125	.9048	.8
.8	.3551	.3799	2.633	.9348	2	.3	.4274	.4727	2.116	.9041	.7
.9	.3567	.3819	2.619	.9342	.1	4	.4289	.4748	2.106	.9033	.6
21 0	0.3584	0.3839	2.605	0.9336	69.0	.5	.4305	.4770	2.097	.9026	.5
.1	.3600	.3859	2.592	.9330	.9	6	.4321	.4791	2.087	.9018	.4
.2	.3616	.3879	2.578	.9323	.8	7	.4337	.4813	2.078	.9011	.3
.3	.3633	.3899	2.565	.9317	.7	8	.4352	.4834	2.069	.9003	.2
.4	.3649	.3919	2.552	.9311	.6	.9	.4368	.4856	2.059	.8996	.1
.5	.3665	.3939	2.539	.9304	.5	26.0	0.4384	0.4877	2.050	0.8988	64.0
.6	.3681	.3959	2.526	.9298	.4	1	.4399	.4899	2.041	.8980	.9
.7	.3697	.3979	2.513	.9291	.3	.2	.4415	.4921	2.032	.8973	.8
.8	.3714	.4000	2.500	.9285	.2	.3	.4431	.4942	2.023	.8965	.7
.9	.3730	.4020	2.488	.9278	.1	4	.4446	.4964	2.014	.8957	.6
22.0	0.3746	0.4040	2.475	0.9272	68.0	.5	.4462	.4986	2.006	.8949	.5
.1	.3762	.4061	2.463	.9265	.9	.6	.4478	.5008	1.997	.8942	.4
.2	.3778	.4081	2.450	.9259	.8	.7	.4493	.5029	1.988	.8934	.3
.3	.3795	.4101	2.438	.9252	.7	8	.4509	.5051	1.980	.8926	.2
.4	.3811	.4122	2.426	.9245	.6	.9	.4524	.5073	1.971	.8918	.1
Deg.	Cos	Cot	Tan	Sin	Deg.	Deg.	Cos	Cot	Tan	Sin	Deg.

TABLE 3—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	Sin	Tan	Cot	Cos	Deg.	Deg.	Sin	Tan	Cot	Cos	Deg.
27 0	0 4540	0 5095	1 963	0 8910	63 0	5	.5225	.6128	1 6319	.8526	.5
.1	.4555	.5117	1.954	.8902	9	.6	.5240	.6152	1.6255	.8517	.4
.2	.4571	.5139	1.946	.8894	8	.7	.5255	.6176	1.6191	.8508	.3
.3	.4586	.5161	1 937	.8886	7	.8	.5270	.6200	1.6128	.8499	.2
.4	.4602	.5184	1 929	.8878	6	.9	.5284	.6224	1.6066	.8490	.1
.5	.4617	.5206	1.921	.8870	5	32.0	0.5299	0 6249	1.6003	0 8480	58.0
.6	.4633	.5228	1 913	.8862	4	1	.5314	.6273	1.5941	.8471	.9
.7	.4648	.5250	1 905	.8854	3	2	.5329	.6297	1.5880	.8462	.8
.8	.4664	.5272	1.897	.8846	2	.3	.5344	.6322	1.5818	.8453	.7
.9	.4679	.5295	1.889	.8838	1	4	.5358	.6346	1 5757	.8443	.6
28 0	0 4695	0 5317	1 881	0.8829	62 0	.5	.5373	.6371	1.5697	.8434	.5
.1	.4710	.5340	1 873	.8821	9	.6	.5388	.6395	1 5637	.8425	.4
.2	.4726	.5362	1 865	.8813	8	.7	.5402	.6420	1 5577	.8415	.3
.3	.4741	.5384	1 857	.8805	.7	8	.5417	.6445	1 5517	.8406	.2
.4	.4756	.5407	1 849	.8796	6	9	.5432	.6469	1.5458	.8396	.1
.5	.4772	.5430	1 842	.8788	5	33 0	0 5446	0 6494	1.5399	0 8387	57.0
.6	.4787	.5452	1 834	.8780	4	.1	.5461	.6519	1 5340	.8377	.9
.7	.4802	.5475	1 827	.8771	3	.2	.5476	.6544	1 5282	.8368	.8
.8	.4818	.5498	1 819	.8763	2	.3	.5490	.6569	1 5224	.8358	.7
.9	.4833	.5520	1 811	.8755	.1	.4	.5505	.6594	1.5166	.8348	.6
29 0	0 4848	0 5543	1 804	0.8746	61 0	5	.5519	.6619	1 5108	.8339	.5
.1	.4863	.5566	1 797	.8738	9	.6	.5534	.6644	1 5051	.8329	.4
.2	.4879	.5589	1 789	.8729	8	.7	.5548	.6669	1.4994	.8320	.3
.3	.4894	.5612	1 782	.8721	7	8	.5563	.6694	1 4938	.8310	.2
.4	.4909	.5635	1 775	.8712	6	9	.5577	.6720	1 4882	.8300	.1
.5	.4924	.5658	1 767	.8704	5	34 0	0 5592	0 6745	1 4826	0 8290	56 0
.6	.4939	.5681	1 760	.8695	4	1	.5606	.6771	1.4770	.8281	.9
.7	.4955	.5704	1 753	.8686	3	2	.5621	.6796	1.4715	.8271	.8
.8	.4970	.5727	1 746	.8678	2	.3	.5635	.6822	1 4659	.8261	.7
.9	.4985	.5750	1.739	.8669	.1	1	.5650	.6847	1 4605	.8251	.6
30 0	0 5000	0 5774	1 7321	0 8660	60 0	.5	.5664	.6873	1 4550	.8241	.5
.1	.5015	.5797	1 7251	.8652	9	.6	.5678	.6899	1 4496	.8231	.4
.2	.5030	.5820	1 7182	.8643	.8	7	.5693	.6924	1 4442	.8221	.3
.3	.5045	.5844	1.7113	.8634	7	.8	.5707	.6950	1 4388	.8211	.2
.4	.5060	.5867	1 7045	.8625	.6	9	.5721	.6976	1 4335	.8202	.1
.5	.5075	.5890	1 6977	.8616	.5	35 0	0 5736	0 7002	1 4281	0.8192	55 0
.6	.5090	.5914	1 6909	.8607	4	.1	.5750	.7028	1.4229	.8181	.9
.7	.5105	.5938	1 6842	.8599	3	2	.5764	.7054	1 4176	.8171	.8
.8	.5120	.5961	1 6775	.8590	.2	.3	.5779	.7080	1.4124	.8161	.7
.9	.5135	.5985	1 6709	.8581	.1	.4	.5793	.7107	1.4071	.8151	.6
31 0	0.5150	0 6009	1 6643	0 8572	59.0	.5	.5807	.7133	1.4019	.8141	.5
.1	.5165	.6032	1.6577	.8563	.9	.6	.5821	.7159	1 3968	.8131	.4
.2	.5180	.6056	1 6512	.8554	.8	.7	.5835	.7186	1.3916	.8121	.3
.3	.5195	.6080	1 6447	.8545	.7	.8	.5850	.7212	1 3865	.8111	.2
.4	.5210	.6104	1 6383	.8536	6	.9	.5864	.7239	1.3814	.8100	.1
Deg.	Cos	Cot	Tan	Sin	Deg.	Deg.	Cos	Cot	Tan	Sin	Deg.

TABLE 3—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	Sin	Tan	Cot	Cos	Deg.	Deg.	Sin	Tan	Cot	Cos	Deg.
36.0	0.5878	0.7265	1.3764	0.8090	54.0	.5	.6494	.8541	1.1708	.7604	.5
.1	.5892	.7292	1.3713	.8080	.9	.6	.6508	.8571	1.1667	.7593	.4
.2	.5906	.7319	1.3663	.8070	.8	.7	.6521	.8601	1.1626	.7581	.3
.3	.5920	.7346	1.3613	.8059	.7	.8	.6534	.8632	1.1585	.7570	.2
.4	.5934	.7373	1.3564	.8049	.6	.9	.6547	.8662	1.1544	.7559	.1
.5	.5948	.7400	1.3514	.8039	.5	41.0	0.6561	0.8693	1.1504	0.7547	49.0
.6	.5962	.7427	1.3465	.8028	.4	.1	.6574	.8724	1.1463	.7536	.9
.7	.5976	.7454	1.3416	.8018	.3	.2	.6587	.8754	1.1423	.7524	.8
.8	.5990	.7481	1.3367	.8007	.2	.3	.6600	.8785	1.1383	.7513	.7
.9	.6004	.7508	1.3319	.7997	.1	.4	.6613	.8816	1.1343	.7501	.6
37.0	0.6018	0.7536	1.3270	0.7986	53.0	.5	.6626	.8847	1.1303	.7490	.5
.1	.6032	.7563	1.3222	.7976	.9	.6	.6639	.8878	1.1263	.7478	.4
.2	.6046	.7590	1.3175	.7965	.8	.7	.6652	.8910	1.1224	.7466	.3
.3	.6060	.7618	1.3127	.7955	.7	.8	.6665	.8941	1.1184	.7455	.2
.4	.6074	.7646	1.3079	.7944	.6	.9	.6678	.8972	1.1145	.7443	.1
.5	.6088	.7673	1.3032	.7934	.5	42.0	0.6691	0.9004	1.1106	0.7431	48.0
.6	.6101	.7701	1.2985	.7923	.4	.1	.6704	.9036	1.1067	.7420	.9
.7	.6115	.7729	1.2938	.7912	.3	.2	.6717	.9067	1.1028	.7408	.8
.8	.6129	.7757	1.2892	.7902	.2	.3	.6730	.9099	1.0990	.7396	.7
.9	.6143	.7785	1.2846	.7891	.1	.4	.6743	.9131	1.0951	.7385	.6
38.0	0.6157	0.7813	1.2799	0.7880	52.0	.5	.6756	.9163	1.0913	.7373	.5
.1	.6170	.7841	1.2753	.7869	.9	.6	.6769	.9195	1.0875	.7361	.4
.2	.6184	.7869	1.2708	.7859	.8	.7	.6782	.9228	1.0837	.7349	.3
.3	.6198	.7898	1.2662	.7848	.7	.8	.6794	.9260	1.0799	.7337	.2
.4	.6211	.7926	1.2617	.7837	.6	.9	.6807	.9293	1.0761	.7325	.1
.5	.6225	.7954	1.2572	.7826	.5	43.0	0.6820	0.9325	1.0724	0.7314	47.0
.6	.6239	.7983	1.2527	.7815	.4	.1	.6833	.9358	1.0686	.7302	.9
.7	.6252	.8012	1.2482	.7804	.3	.2	.6845	.9391	1.0649	.7290	.8
.8	.6266	.8040	1.2437	.7793	.2	.3	.6858	.9424	1.0612	.7278	.7
.9	.6280	.8069	1.2393	.7782	.1	.4	.6871	.9457	1.0575	.7266	.6
39.0	0.6293	0.8098	1.2349	0.7771	51.0	.5	.6884	.9490	1.0538	.7254	.5
.1	.6307	.8127	1.2305	.7760	.9	.6	.6896	.9523	1.0501	.7242	.4
.2	.6320	.8156	1.2261	.7749	.8	.7	.6909	.9556	1.0464	.7230	.3
.3	.6334	.8185	1.2218	.7738	.7	.8	.6921	.9590	1.0428	.7218	.2
.4	.6347	.8214	1.2174	.7727	.6	.9	.6934	.9623	1.0392	.7206	.1
.5	.6361	.8243	1.2131	.7716	.5	44.0	0.6947	0.9657	1.0355	0.7193	46.0
.6	.6374	.8273	1.2088	.7705	.4	.1	.6959	.9691	1.0319	.7181	.9
.7	.6388	.8302	1.2045	.7694	.3	.2	.6972	.9725	1.0283	.7169	.8
.8	.6401	.8332	1.2002	.7683	.2	.3	.6984	.9759	1.0247	.7157	.7
.9	.6414	.8361	1.1960	.7672	.1	.4	.6997	.9793	1.0212	.7145	.6
40.0	0.6428	0.8391	1.1918	0.7660	50.0	.5	.7009	.9827	1.0176	.7133	.5
.1	.6441	.8421	1.1875	.7649	.9	.6	.7022	.9861	1.0141	.7120	.4
.2	.6455	.8451	1.1833	.7638	.8	.7	.7034	.9896	1.0105	.7108	.3
.3	.6468	.8481	1.1792	.7627	.7	.8	.7046	.9930	1.0070	.7096	.2
.4	.6481	.8511	1.1750	.7615	.6	.9	.7059	.9965	1.0035	.7083	.1
						45.0	0.7071	1.0000	1.0000	0.7071	45.0
Deg.	Cos	Cot	Tan	Sin	Deg.	Deg.	Cos	Cot	Tan	Sin	Deg.

TABLE 4

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
0.0	— ∞	— ∞	∞	0.0000	90.0
.1	7.2419	7.2419	2.7581	0.0000	.9
.2	7.5429	7.5429	2.4571	0.0000	.8
.3	7.7190	7.7190	2.2810	0.0000	.7
.4	7.8439	7.8439	2.1561	0.0000	.6
.5	7.9408	7.9409	2.0591	0.0000	.5
.6	8.0200	8.0200	1.9800	0.0000	.4
.7	8.0870	8.0870	1.9130	0.0000	.3
.8	8.1450	8.1450	1.8550	0.0000	.2
.9	8.1961	8.1962	1.8038	9.9999	.1
1.0	8.2419	8.2419	1.7581	9.9999	89.0
.1	8.2832	8.2833	1.7167	9.9999	.9
.2	8.3210	8.3211	1.6789	9.9999	.8
.3	8.3558	8.3559	1.6441	9.9999	.7
.4	8.3880	8.3881	1.6119	9.9999	.6
.5	8.4179	8.4181	1.5819	9.9999	.5
.6	8.4459	8.4461	1.5539	9.9998	.4
.7	8.4723	8.4725	1.5275	9.9998	.3
.8	8.4971	8.4973	1.5027	9.9998	.2
.9	8.5206	8.5208	1.4792	9.9998	.1
2.0	8.5428	8.5431	1.4569	9.9997	88.0
.1	8.5640	8.5643	1.4357	9.9997	.9
.2	8.5842	8.5845	1.4155	9.9997	.8
.3	8.6035	8.6038	1.3962	9.9996	.7
.4	8.6220	8.6223	1.3777	9.9996	.6
.5	8.6397	8.6401	1.3599	9.9996	.5
.6	8.6567	8.6571	1.3429	9.9996	.4
.7	8.6731	8.6736	1.3264	9.9995	.3
.8	8.6889	8.6894	1.3106	9.9995	.2
.9	8.7041	8.7046	1.2954	9.9994	.1
3.0	8.7188	8.7194	1.2806	9.9994	87.0
.1	8.7330	8.7337	1.2663	9.9994	.9
.2	8.7468	8.7475	1.2525	9.9993	.8
.3	8.7602	8.7609	1.2391	9.9993	.7
.4	8.7731	8.7739	1.2261	9.9992	.6
.5	8.7857	8.7865	1.2135	9.9992	.5
.6	8.7979	8.7988	1.2012	9.9991	.4
.7	8.8098	8.8107	1.1893	9.9991	.3
.8	8.8213	8.8223	1.1777	9.9990	.2
.9	8.8326	8.8336	1.1664	9.9990	.1
4.0	8.8436	8.8446	1.1554	9.9989	86.0
.1	8.8543	8.8554	1.1446	9.9989	.9
.2	8.8647	8.8659	1.1341	9.9988	.8
.3	8.8749	8.8762	1.1238	9.9988	.7
.4	8.8849	8.8862	1.1138	9.9987	.6
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L Sin	L. Tan	L. Cot	L. Cos	Deg.
4.5	8.8946	8.8960	1.1040	9.9987	86.5
.6	8.9042	8.9056	1.0944	9.9986	.4
.7	8.9135	8.9150	1.0850	9.9985	.3
.8	8.9226	8.9241	1.0759	9.9985	.2
.9	8.9315	8.9331	1.0669	9.9984	.1
5.0	8.9403	8.9420	1.0580	9.9983	85.0
.1	8.9489	8.9506	1.0494	9.9983	.9
.2	8.9573	8.9591	1.0409	9.9982	.8
.3	8.9655	8.9674	1.0326	9.9981	.7
.4	8.9736	8.9756	1.0244	9.9981	.6
.5	8.9816	8.9836	1.0164	9.9980	.5
.6	8.9894	8.9915	1.0085	9.9979	.4
.7	8.9970	8.9992	1.0008	9.9978	.3
.8	9.0046	9.0068	0.9932	9.9978	.2
.9	9.0120	9.0143	0.9857	9.9977	.1
6.0	9.0192	9.0216	0.9784	9.9976	84.0
.1	9.0264	9.0289	0.9711	9.9975	.9
.2	9.0334	9.0360	0.9640	9.9975	.8
.3	9.0403	9.0430	0.9570	9.9974	.7
.4	9.0472	9.0499	0.9501	9.9973	.6
.5	9.0539	9.0567	0.9433	9.9972	.5
.6	9.0605	9.0633	0.9367	9.9971	.4
.7	9.0670	9.0699	0.9301	9.9970	.3
.8	9.0734	9.0764	0.9236	9.9969	.2
.9	9.0797	9.0828	0.9172	9.9968	.1
7.0	9.0859	9.0891	0.9109	9.9968	83.0
.1	9.0920	9.0954	0.9046	9.9967	.9
.2	9.0981	9.1015	0.8985	9.9966	.8
.3	9.1040	9.1076	0.8924	9.9965	.7
.4	9.1099	9.1135	0.8865	9.9964	.6
.5	9.1157	9.1194	0.8806	9.9963	.5
.6	9.1214	9.1252	0.8748	9.9962	.4
.7	9.1271	9.1310	0.8690	9.9961	.3
.8	9.1326	9.1367	0.8633	9.9960	.2
.9	9.1381	9.1423	0.8577	9.9959	.1
8.0	9.1436	9.1478	0.8522	9.9958	82.0
.1	9.1489	9.1533	0.8467	9.9956	.9
.2	9.1542	9.1587	0.8413	9.9955	.8
.3	9.1594	9.1640	0.8360	9.9954	.7
.4	9.1646	9.1693	0.8307	9.9953	.6
.5	9.1697	9.1745	0.8255	9.9952	.5
.6	9.1747	9.1797	0.8203	9.9951	.4
.7	9.1797	9.1848	0.8152	9.9950	.3
.8	9.1847	9.1898	0.8102	9.9949	.2
.9	9.1895	9.1948	0.8052	9.9947	.1
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
9.0	9.1943	9.1997	0.8003	9.9946	81.0
.1	9.1991	9.2046	0.7954	9.9945	.9
.2	9.2038	9.2094	0.7906	9.9944	.8
.3	9.2085	9.2142	0.7858	9.9943	.7
.4	9.2131	9.2189	0.7811	9.9941	.6
.5	9.2176	9.2236	0.7764	9.9940	.5
.6	9.2221	9.2282	0.7718	9.9939	.4
.7	9.2266	9.2328	0.7672	9.9937	.3
.8	9.2310	9.2374	0.7626	9.9936	.2
.9	9.2353	9.2419	0.7581	9.9935	.1
10.0	9.2397	9.2463	0.7537	9.9934	80.0
.1	9.2439	9.2507	0.7493	9.9932	.9
.2	9.2482	9.2551	0.7449	9.9931	.8
.3	9.2524	9.2594	0.7406	9.9929	.7
.4	9.2565	9.2637	0.7363	9.9928	.6
.5	9.2606	9.2680	0.7320	9.9927	.5
.6	9.2647	9.2722	0.7278	9.9925	.4
.7	9.2687	9.2764	0.7236	9.9924	.3
.8	9.2727	9.2805	0.7195	9.9922	.2
.9	9.2767	9.2846	0.7154	9.9921	.1
11.0	9.2806	9.2887	0.7113	9.9919	79.0
.1	9.2845	9.2927	0.7073	9.9918	.9
.2	9.2883	9.2967	0.7033	9.9916	.8
.3	9.2921	9.3006	0.6994	9.9915	.7
.4	9.2959	9.3046	0.6954	9.9913	.6
.5	9.2997	9.3085	0.6915	9.9912	.5
.6	9.3034	9.3123	0.6877	9.9910	.4
.7	9.3070	9.3162	0.6838	9.9909	.3
.8	9.3107	9.3200	0.6800	9.9907	.2
.9	9.3143	9.3237	0.6763	9.9906	.1
12.0	9.3179	9.3275	0.6725	9.9904	78.0
.1	9.3214	9.3312	0.6688	9.9902	.9
.2	9.3250	9.3349	0.6651	9.9901	.8
.3	9.3284	9.3385	0.6615	9.9899	.7
.4	9.3319	9.3422	0.6578	9.9897	.6
.5	9.3353	9.3458	0.6542	9.9896	.5
.6	9.3387	9.3493	0.6507	9.9894	.4
.7	9.3421	9.3529	0.6471	9.9892	.3
.8	9.3455	9.3564	0.6436	9.9891	.2
.9	9.3488	9.3599	0.6401	9.9889	.1
13.0	9.3521	9.3634	0.6366	9.9887	77.0
.1	9.3554	9.3668	0.6332	9.9885	.9
.2	9.3586	9.3702	0.6298	9.9884	.8
.3	9.3618	9.3736	0.6264	9.9882	.7
.4	9.3650	9.3770	0.6230	9.9880	.6
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
13.5	9.3682	9.3804	0.6196	9.9878	77.5
.6	9.3713	9.3837	0.6163	9.9876	.4
.7	9.3745	9.3870	0.6130	9.9875	.3
.8	9.3775	9.3903	0.6097	9.9873	.2
.9	9.3806	9.3935	0.6065	9.9871	.1
14.0	9.3837	9.3968	0.6032	9.9869	76.0
.1	9.3867	9.4000	0.6000	9.9867	.9
.2	9.3897	9.4032	0.5968	9.9865	.8
.3	9.3927	9.4064	0.5936	9.9863	.7
.4	9.3957	9.4095	0.5905	9.9861	.6
.5	9.3986	9.4127	0.5873	9.9859	.5
.6	9.4015	9.4158	0.5842	9.9857	.4
.7	9.4044	9.4189	0.5811	9.9855	.3
.8	9.4073	9.4220	0.5780	9.9853	.2
.9	9.4102	9.4250	0.5750	9.9851	.1
15.0	9.4130	9.4281	0.5719	9.9849	75.0
.1	9.4158	9.4311	0.5689	9.9847	.9
.2	9.4186	9.4341	0.5659	9.9845	.8
.3	9.4214	9.4371	0.5629	9.9843	.7
.4	9.4242	9.4400	0.5600	9.9841	.6
.5	9.4269	9.4430	0.5570	9.9839	.5
.6	9.4296	9.4459	0.5541	9.9837	.4
.7	9.4323	9.4488	0.5512	9.9835	.3
.8	9.4350	9.4517	0.5483	9.9833	.2
.9	9.4377	9.4546	0.5454	9.9831	.1
16.0	9.4403	9.4575	0.5425	9.9828	74.0
.1	9.4430	9.4603	0.5397	9.9826	.9
.2	9.4456	9.4632	0.5368	9.9824	.8
.3	9.4482	9.4660	0.5340	9.9822	.7
.4	9.4508	9.4688	0.5312	9.9820	.6
.5	9.4533	9.4716	0.5284	9.9817	.5
.6	9.4559	9.4744	0.5256	9.9815	.4
.7	9.4584	9.4771	0.5229	9.9813	.3
.8	9.4609	9.4799	0.5201	9.9811	.2
.9	9.4634	9.4826	0.5174	9.9808	.1
17.0	9.4659	9.4853	0.5147	9.9806	73.0
.1	9.4684	9.4880	0.5120	9.9804	.9
.2	9.4709	9.4907	0.5093	9.9801	.8
.3	9.4733	9.4934	0.5066	9.9799	.7
.4	9.4757	9.4961	0.5039	9.9797	.6
.5	9.4781	9.4987	0.5013	9.9794	.5
.6	9.4805	9.5014	0.4986	9.9792	.4
.7	9.4829	9.5040	0.4960	9.9789	.3
.8	9.4853	9.5066	0.4934	9.9787	.2
.9	9.4876	9.5092	0.4908	9.9785	.1
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
18.0	9.4900	9.5118	0.4882	9.9782	72.0
.1	9.4923	9.5143	0.4857	9.9780	.9
.2	9.4946	9.5169	0.4831	9.9777	.8
.3	9.4969	9.5195	0.4805	9.9775	.7
.4	9.4992	9.5220	0.4780	9.9772	.6
.5	9.5015	9.5245	0.4755	9.9770	.5
.6	9.5037	9.5270	0.4730	9.9767	.4
.7	9.5060	9.5295	0.4705	9.9764	.3
.8	9.5082	9.5320	0.4680	9.9762	.2
.9	9.5104	9.5345	0.4655	9.9759	.1
19.0	9.5126	9.5370	0.4630	9.9757	71.0
.1	9.5148	9.5394	0.4606	9.9754	.9
.2	9.5170	9.5419	0.4581	9.9751	.8
.3	9.5192	9.5443	0.4557	9.9749	.7
.4	9.5213	9.5467	0.4533	9.9746	.6
.5	9.5235	9.5491	0.4509	9.9743	.5
.6	9.5256	9.5516	0.4484	9.9741	.4
.7	9.5278	9.5539	0.4461	9.9738	.3
.8	9.5299	9.5563	0.4437	9.9735	.2
.9	9.5320	9.5587	0.4413	9.9733	.1
20.0	9.5341	9.5611	0.4389	9.9730	70.0
.1	9.5361	9.5634	0.4366	9.9727	.9
.2	9.5332	9.5658	0.4342	9.9724	.8
.3	9.5402	9.5681	0.4319	9.9722	.7
.4	9.5423	9.5704	0.4296	9.9719	.6
.5	9.5443	9.5727	0.4273	9.9716	.5
.6	9.5463	9.5750	0.4250	9.9713	.4
.7	9.5484	9.5773	0.4227	9.9710	.3
.8	9.5504	9.5796	0.4204	9.9707	.2
.9	9.5523	9.5819	0.4181	9.9704	.1
21.0	9.5543	9.5842	0.4158	9.9702	69.0
.1	9.5563	9.5864	0.4136	9.9699	.9
.2	9.5583	9.5887	0.4113	9.9696	.8
.3	9.5602	9.5909	0.4091	9.9693	.7
.4	9.5621	9.5932	0.4068	9.9690	.6
.5	9.5641	9.5954	0.4046	9.9687	.5
.6	9.5660	9.5976	0.4024	9.9684	.4
.7	9.5679	9.5998	0.4002	9.9681	.3
.8	9.5698	9.6020	0.3980	9.9678	.2
.9	9.5717	9.6042	0.3958	9.9675	.1
22.0	9.5736	9.6064	0.3936	9.9672	68.0
.1	9.5754	9.6086	0.3914	9.9669	.9
.2	9.5773	9.6108	0.3892	9.9666	.8
.3	9.5792	9.6129	0.3871	9.9662	.7
.4	9.5810	9.6151	0.3849	9.9659	.6
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
22.5	9.5828	9.6172	0.3828	9.9656	68.5
.6	9.5847	9.6194	0.3806	9.9653	.4
.7	9.5865	9.6215	0.3785	9.9650	.3
.8	9.5883	9.6236	0.3764	9.9647	.2
.9	9.5901	9.6257	0.3743	9.9643	.1
23.0	9.5919	9.6279	0.3721	9.9640	67.0
.1	9.5937	9.6300	0.3700	9.9637	.9
.2	9.5954	9.6321	0.3679	9.9634	.8
.3	9.5972	9.6341	0.3659	9.9631	.7
.4	9.5990	9.6362	0.3638	9.9627	.6
.5	9.6007	9.6383	0.3617	9.9624	.5
.6	9.6024	9.6404	0.3596	9.9621	.4
.7	9.6042	9.6424	0.3576	9.9617	.3
.8	9.6059	9.6445	0.3555	9.9614	.2
.9	9.6076	9.6465	0.3535	9.9611	.1
24.0	9.6093	9.6486	0.3514	9.9607	66.0
.1	9.6110	9.6506	0.3494	9.9604	.9
.2	9.6127	9.6527	0.3473	9.9601	.8
.3	9.6144	9.6547	0.3453	9.9597	.7
.4	9.6161	9.6567	0.3433	9.9594	.6
.5	9.6177	9.6587	0.3413	9.9590	.5
.6	9.6194	9.6607	0.3393	9.9587	.4
.7	9.6210	9.6627	0.3373	9.9583	.3
.8	9.6227	9.6647	0.3353	9.9580	.2
.9	9.6243	9.6667	0.3333	9.9576	.1
25.0	9.6259	9.6687	0.3313	9.9573	65.0
.1	9.6276	9.6706	0.3294	9.9569	.9
.2	9.6292	9.6726	0.3274	9.9566	.8
.3	9.6308	9.6746	0.3254	9.9562	.7
.4	9.6324	9.6765	0.3235	9.9558	.6
.5	9.6340	9.6785	0.3215	9.9555	.5
.6	9.6356	9.6804	0.3196	9.9551	.4
.7	9.6371	9.6824	0.3176	9.9548	.3
.8	9.6387	9.6843	0.3157	9.9544	.2
.9	9.6403	9.6863	0.3137	9.9540	.1
26.0	9.6418	9.6882	0.3118	9.9537	64.0
.1	9.6434	9.6901	0.3099	9.9533	.9
.2	9.6449	9.6920	0.3080	9.9529	.8
.3	9.6465	9.6939	0.3061	9.9525	.7
.4	9.6480	9.6958	0.3042	9.9522	.6
.5	9.6495	9.6977	0.3023	9.9518	.5
.6	9.6510	9.6996	0.3004	9.9514	.4
.7	9.6526	9.7015	0.2985	9.9510	.3
.8	9.6541	9.7034	0.2966	9.9506	.2
.9	9.6556	9.7053	0.2947	9.9503	.1
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
27.0	9.6570	9.7072	0.2928	9.9499	63.0
.1	9.6585	9.7090	0.2910	9.9495	.9
.2	9.6600	9.7109	0.2891	9.9491	.8
.3	9.6615	9.7128	0.2872	9.9487	.7
.4	9.6629	9.7146	0.2854	9.9483	.6
.5	9.6644	9.7165	0.2835	9.9479	.5
.6	9.6659	9.7183	0.2817	9.9475	.4
.7	9.6673	9.7202	0.2798	9.9471	.3
.8	9.6687	9.7220	0.2780	9.9467	.2
.9	9.6702	9.7238	0.2762	9.9463	.1
28.0	9.6716	9.7257	0.2743	9.9459	62.0
.1	9.6730	9.7275	0.2725	9.9455	.9
.2	9.6744	9.7293	0.2707	9.9451	.8
.3	9.6759	9.7311	0.2689	9.9447	.7
.4	9.6773	9.7330	0.2670	9.9443	.6
.5	9.6787	9.7348	0.2652	9.9439	.5
.6	9.6801	9.7366	0.2634	9.9435	.4
.7	9.6814	9.7384	0.2616	9.9431	.3
.8	9.6828	9.7402	0.2598	9.9427	.2
.9	9.6842	9.7420	0.2580	9.9422	.1
29.0	9.6856	9.7438	0.2562	9.9418	61.0
.1	9.6869	9.7455	0.2545	9.9414	.9
.2	9.6883	9.7473	0.2527	9.9410	.8
.3	9.6896	9.7491	0.2509	9.9406	.7
.4	9.6910	9.7509	0.2491	9.9401	.6
.5	9.6923	9.7526	0.2474	9.9397	.5
.6	9.6937	9.7544	0.2456	9.9393	.4
.7	9.6950	9.7562	0.2438	9.9388	.3
.8	9.6963	9.7579	0.2421	9.9384	.2
.9	9.6977	9.7597	0.2403	9.9380	.1
30.0	9.6990	9.7614	0.2386	9.9375	60.0
.1	9.7003	9.7632	0.2368	9.9371	.9
.2	9.7016	9.7649	0.2351	9.9367	.8
.3	9.7029	9.7667	0.2333	9.9362	.7
.4	9.7042	9.7684	0.2316	9.9358	.6
.5	9.7055	9.7701	0.2299	9.9353	.5
.6	9.7068	9.7719	0.2281	9.9349	.4
.7	9.7080	9.7736	0.2264	9.9344	.3
.8	9.7093	9.7753	0.2247	9.9340	.2
.9	9.7106	9.7771	0.2229	9.9335	.1
31.0	9.7118	9.7788	0.2212	9.9331	59.0
.1	9.7131	9.7805	0.2195	9.9326	.9
.2	9.7144	9.7822	0.2178	9.9322	.8
.3	9.7156	9.7839	0.2161	9.9317	.7
.4	9.7168	9.7856	0.2144	9.9312	.6
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
31.5	9.7181	9.7873	0.2127	9.9308	59.5
.6	9.7193	9.7890	0.2110	9.9303	.4
.7	9.7205	9.7907	0.2093	9.9298	.3
.8	9.7218	9.7924	0.2076	9.9294	.2
.9	9.7230	9.7941	0.2059	9.9289	.1
32.0	9.7242	9.7958	0.2042	9.9284	58.0
.1	9.7254	9.7975	0.2025	9.9279	.9
.2	9.7266	9.7992	0.2008	9.9275	.8
.3	9.7278	9.8008	0.1992	9.9270	.7
.4	9.7290	9.8025	0.1975	9.9265	.6
.5	9.7302	9.8042	0.1958	9.9260	.5
.6	9.7314	9.8059	0.1941	9.9255	.4
.7	9.7326	9.8075	0.1925	9.9251	.3
.8	9.7338	9.8092	0.1908	9.9246	.2
.9	9.7349	9.8109	0.1891	9.9241	.1
33.0	9.7361	9.8125	0.1875	9.9236	57.0
.1	9.7373	9.8142	0.1858	9.9231	.9
.2	9.7384	9.8158	0.1842	9.9226	.8
.3	9.7396	9.8175	0.1825	9.9221	.7
.4	9.7407	9.8191	0.1809	9.9216	.6
.5	9.7419	9.8208	0.1792	9.9211	.5
.6	9.7430	9.8224	0.1776	9.9206	.4
.7	9.7442	9.8241	0.1759	9.9201	.3
.8	9.7453	9.8257	0.1743	9.9196	.2
.9	9.7464	9.8274	0.1726	9.9191	.1
34.0	9.7476	9.8290	0.1710	9.9186	56.0
.1	9.7487	9.8306	0.1694	9.9181	.9
.2	9.7498	9.8323	0.1677	9.9175	.8
.3	9.7509	9.8339	0.1661	9.9170	.7
.4	9.7520	9.8355	0.1645	9.9165	.6
.5	9.7531	9.8371	0.1629	9.9160	.5
.6	9.7542	9.8388	0.1612	9.9155	.4
.7	9.7553	9.8404	0.1596	9.9149	.3
.8	9.7564	9.8420	0.1580	9.9144	.2
.9	9.7575	9.8436	0.1564	9.9139	.1
35.0	9.7586	9.8452	0.1548	9.9134	55.0
.1	9.7597	9.8468	0.1532	9.9128	.9
.2	9.7607	9.8484	0.1516	9.9123	.8
.3	9.7618	9.8501	0.1499	9.9118	.7
.4	9.7629	9.8517	0.1483	9.9112	.6
.5	9.7640	9.8533	0.1467	9.9107	.5
.6	9.7650	9.8549	0.1451	9.9101	.4
.7	9.7661	9.8565	0.1435	9.9096	.3
.8	9.7671	9.8581	0.1419	9.9091	.2
.9	9.7682	9.8597	0.1403	9.9085	.1
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
36.0	9.7692	9.8613	0.1387	9.9080	54.0
.1	9.7703	9.8629	0.1371	9.9074	.9
.2	9.7713	9.8644	0.1356	9.9069	.8
.3	9.7723	9.8660	0.1340	9.9063	.7
.4	9.7734	9.8676	0.1324	9.9057	.6
.5	9.7744	9.8692	0.1308	9.9052	.5
.6	9.7754	9.8708	0.1292	9.9046	.4
.7	9.7764	9.8724	0.1276	9.9041	.3
.8	9.7774	9.8740	0.1260	9.9035	.2
.9	9.7785	9.8755	0.1245	9.9029	.1
37.0	9.7795	9.8771	0.1229	9.9023	53.0
.1	9.7805	9.8787	0.1213	9.9018	.9
.2	9.7815	9.8803	0.1197	9.9012	.8
.3	9.7825	9.8818	0.1182	9.9006	.7
.4	9.7835	9.8834	0.1166	9.9000	.6
.5	9.7844	9.8850	0.1150	9.8995	.5
.6	9.7854	9.8865	0.1135	9.8989	.4
.7	9.7864	9.8881	0.1119	9.8983	.3
.8	9.7874	9.8897	0.1103	9.8977	.2
.9	9.7884	9.8912	0.1088	9.8971	.1
38.0	9.7893	9.8928	0.1072	9.8965	52.0
.1	9.7903	9.8944	0.1056	9.8959	.9
.2	9.7913	9.8959	0.1041	9.8953	.8
.3	9.7922	9.8975	0.1025	9.8947	.7
.4	9.7932	9.8990	0.1010	9.8941	.6
.5	9.7941	9.9006	0.0994	9.8935	.5
.6	9.7951	9.9022	0.0978	9.8929	.4
.7	9.7960	9.9037	0.0963	9.8923	.3
.8	9.7970	9.9053	0.0947	9.8917	.2
.9	9.7979	9.9068	0.0932	9.8911	.1
39.0	9.7989	9.9084	0.0916	9.8905	51.0
.1	9.7998	9.9099	0.0901	9.8899	.9
.2	9.8007	9.9115	0.0885	9.8893	.8
.3	9.8017	9.9130	0.0870	9.8887	.7
.4	9.8026	9.9146	0.0854	9.8880	.6
.5	9.8035	9.9161	0.0839	9.8874	.5
.6	9.8044	9.9176	0.0824	9.8868	.4
.7	9.8053	9.9192	0.0808	9.8862	.3
.8	9.8063	9.9207	0.0793	9.8855	.2
.9	9.8072	9.9223	0.0777	9.8849	.1
40.0	9.8081	9.9238	0.0762	9.8843	50.0
.1	9.8090	9.9254	0.0746	9.8836	.9
.2	9.8099	9.9269	0.0731	9.8830	.8
.3	9.8108	9.9284	0.0716	9.8823	.7
.4	9.8117	9.9300	0.0700	9.8817	.6
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 4—Continued

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS FOR DECIMAL FRACTIONS OF A DEGREE

Deg.	L. Sin	L. Tan	L. Cot	L. Cos	Deg.
40.5	9.8125	9.9315	0.0685	9.8810	50.5
.6	9.8134	9.9330	0.0670	9.8804	.4
.7	9.8143	9.9346	0.0654	9.8797	.3
.8	9.8152	9.9361	0.0639	9.8791	.2
.9	9.8161	9.9376	0.0624	9.8784	.1
41.0	9.8169	9.9392	0.0608	9.8778	49.0
.1	9.8178	9.9407	0.0593	9.8771	.9
.2	9.8187	9.9422	0.0578	9.8765	.8
.3	9.8195	9.9438	0.0562	9.8758	.7
.4	9.8204	9.9453	0.0547	9.8751	.6
.5	9.8213	9.9468	0.0532	9.8745	.5
.6	9.8221	9.9483	0.0517	9.8738	.4
.7	9.8230	9.9499	0.0501	9.8731	.3
.8	9.8238	9.9514	0.0486	9.8724	.2
.9	9.8247	9.9529	0.0471	9.8718	.1
42.0	9.8255	9.9544	0.0456	9.8711	48.0
.1	9.8264	9.9560	0.0440	9.8704	.9
.2	9.8272	9.9575	0.0425	9.8697	.8
.3	9.8280	9.9590	0.0410	9.8690	.7
.4	9.8289	9.9605	0.0395	9.8683	.6
.5	9.8297	9.9621	0.0379	9.8676	.5
.6	9.8305	9.9636	0.0364	9.8669	.4
.7	9.8313	9.9651	0.0349	9.8662	.3
.8	9.8322	9.9666	0.0334	9.8655	.2
.9	9.8330	9.9681	0.0319	9.8648	.1
43.0	9.8338	9.9697	0.0303	9.8641	47.0
.1	9.8346	9.9712	0.0288	9.8634	.9
.2	9.8354	9.9727	0.0273	9.8627	.8
.3	9.8362	9.9742	0.0258	9.8620	.7
.4	9.8370	9.9757	0.0243	9.8613	.6
.5	9.8378	9.9772	0.0228	9.8606	.5
.6	9.8386	9.9788	0.0212	9.8598	.4
.7	9.8394	9.9803	0.0197	9.8591	.3
.8	9.8402	9.9818	0.0182	9.8584	.2
.9	9.8410	9.9833	0.0167	9.8577	.1
44.0	9.8418	9.9848	0.0152	9.8569	46.0
.1	9.8426	9.9864	0.0136	9.8562	.9
.2	9.8433	9.9879	0.0121	9.8555	.8
.3	9.8441	9.9894	0.0106	9.8547	.7
.4	9.8449	9.9909	0.0091	9.8540	.6
.5	9.8457	9.9924	0.0076	9.8532	.5
.6	9.8464	9.9939	0.0061	9.8525	.4
.7	9.8472	9.9955	0.0045	9.8517	.3
.8	9.8480	9.9970	0.0030	9.8510	.2
.9	9.8487	9.9985	0.0015	9.8502	.1
45.0	9.8495	0.0000	0.0000	9.8495	45.0
Deg.	L. Cos	L. Cot	L. Tan	L. Sin	Deg.

TABLE 5
NATURAL TRIGONOMETRIC FUNCTIONS FOR ANGLES IN RADIANS

Rad.	Sin	Tan	Cot	Cos
.00	.00000	.00000	∞	1.00000
.01	.01000	.01000	99.997	0.99995
.02	.02000	.02000	49.993	.99980
.03	.03000	.03001	33.323	.99955
.04	.03999	.04002	24.987	.99920
.05	.04998	.05004	19.983	.99875
.06	.05996	.06007	16.647	.99820
.07	.06994	.07011	14.262	.99755
.08	.07991	.08017	12.473	.99680
.09	.08988	.09024	11.081	.99595
.10	.09983	.10033	9.9666	.99500
.11	.10978	.11045	9.0542	.99396
.12	.11971	.12058	8.2933	.99281
.13	.12963	.13074	7.6489	.99156
.14	.13954	.14092	7.0961	.99022
.15	.14944	.15114	6.6166	.98877
.16	.15932	.16138	6.1966	.98723
.17	.16918	.17166	5.8256	.98558
.18	.17903	.18197	5.4954	.98384
.19	.18886	.19232	5.1997	.98200
.20	.19867	.20271	4.9332	.98007
.21	.20846	.21314	4.6917	.97803
.22	.21823	.22362	4.4719	.97590
.23	.22798	.23414	4.2709	.97367
.24	.23770	.24472	4.0864	.97134
.25	.24740	.25534	3.9163	.96891
.26	.25708	.26602	3.7591	.96639
.27	.26673	.27676	3.6133	.96377
.28	.27636	.28755	3.4776	.96106
.29	.28595	.29841	3.3511	.95824
.30	.29552	.30934	3.2327	.95534
.31	.30506	.32033	3.1218	.95233
.32	.31457	.33139	3.0176	.94924
.33	.32404	.34252	2.9195	.94604
.34	.33349	.35374	2.8270	.94275
.35	.34290	.36503	2.7395	.93937
.36	.35227	.37640	2.6567	.93590
.37	.36162	.38786	2.5782	.93233
.38	.37092	.39941	2.5037	.92866
.39	.38019	.41105	2.4328	.92491
Rad.	Sin	Tan	Cot	Cos

TABLE 5—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR ANGLES IN RADIANS

Rad.	Sin	Tan	Cot	Cos
.40	.38942	.42279	2.3652	.92106
.41	.39861	.43463	2.3008	.91712
.42	.40776	.44657	2.2393	.91309
.43	.41687	.45862	2.1804	.90897
.44	.42594	.47078	2.1241	.90475
.45	.43497	.48306	2.0702	.90045
.46	.44395	.49545	2.0184	.89605
.47	.45289	.50797	1.9686	.89157
.48	.46178	.52061	1.9208	.88699
.49	.47063	.53339	1.8748	.88233
.50	.47943	.54630	1.8305	.87758
.51	.48818	.55936	1.7878	.87274
.52	.49688	.57256	1.7465	.86782
.53	.50553	.58592	1.7067	.86281
.54	.51414	.59943	1.6683	.85771
.55	.52269	.61311	1.6310	.85252
.56	.53119	.62695	1.5950	.84726
.57	.53963	.64097	1.5601	.84190
.58	.54802	.65517	1.5263	.83646
.59	.55636	.66956	1.4935	.83094
.60	.56464	.68414	1.4617	.82534
.61	.57287	.69892	1.4308	.81965
.62	.58104	.71391	1.4007	.81388
.63	.58914	.72911	1.3715	.80803
.64	.59720	.74454	1.3431	.80210
.65	.60519	.76020	1.3154	.79608
.66	.61312	.77610	1.2885	.78999
.67	.62099	.79225	1.2622	.78382
.68	.62879	.80866	1.2366	.77757
.69	.63654	.82534	1.2116	.77125
.70	.64422	.84229	1.1872	.76484
.71	.65183	.85953	1.1634	.75836
.72	.65938	.87707	1.1402	.75181
.73	.66687	.89492	1.1174	.74517
.74	.67429	.91309	1.0952	.73847
.75	.68164	.93160	1.0734	.73169
.76	.68892	.95045	1.0521	.72484
.77	.69614	.96967	1.0313	.71791
.78	.70328	.98926	1.0109	.71091
.79	.71035	1.0092	.99084	.70385
Rad.	Sin	Tan	Cot	Cos

TABLE 5—*Continued*
NATURAL TRIGONOMETRIC FUNCTIONS FOR ANGLES IN RADIANS

Rad.	Sin	Tan	Cot	Cos
.80	.71736	1.0296	.97121	.69671
.81	.72429	1.0505	.95197	.68950
.82	.73115	1.0717	.93309	.68222
.83	.73793	1.0934	.91455	.67488
.84	.74464	1.1156	.89635	.66746
.85	.75128	1.1383	.87848	.65998
.86	.75784	1.1616	.86091	.65244
.87	.76433	1.1853	.84365	.64483
.88	.77074	1.2097	.82668	.63715
.89	.77707	1.2346	.80998	.62941
.90	.78333	1.2602	.79355	.62161
.91	.78950	1.2864	.77738	.61375
.92	.79560	1.3133	.76146	.60582
.93	.80162	1.3409	.74578	.59783
.94	.80756	1.3692	.73034	.58979
.95	.81342	1.3984	.71511	.58168
.96	.81919	1.4284	.70010	.57352
.97	.82489	1.4592	.68531	.56530
.98	.83050	1.4910	.67071	.55702
.99	.83603	1.5237	.65631	.54869
1.00	.84147	1.5574	.64209	.54030
1.01	.84683	1.5922	.62806	.53186
1.02	.85211	1.6281	.61420	.52337
1.03	.85730	1.6652	.60051	.51482
1.04	.86240	1.7036	.58699	.50622
1.05	.86742	1.7433	.57362	.49757
1.06	.87236	1.7844	.56040	.48887
1.07	.87720	1.8270	.54734	.48012
1.08	.88196	1.8712	.53441	.47133
1.09	.88663	1.9171	.52162	.46249
1.10	.89121	1.9648	.50897	.45360
1.11	.89570	2.0143	.49644	.44466
1.12	.90010	2.0660	.48404	.43568
1.13	.90441	2.1198	.47175	.42666
1.14	.90863	2.1759	.45959	.41759
1.15	.91276	2.2345	.44753	.40849
1.16	.91680	2.2958	.43558	.39934
1.17	.92075	2.3600	.42373	.39015
1.18	.92461	2.4273	.41199	.38092
1.19	.92837	2.4979	.40034	.37166
Rad.	Sin	Tan	Cot	Cos

TABLE 5—Continued

NATURAL TRIGONOMETRIC FUNCTIONS FOR ANGLES IN RADIANS

Rad.	Sin	Tan	Cot	Cos
1.20	.93204	2.5722	.38878	.36236
1.21	.93562	2.6503	.37731	.35302
1.22	.93910	2.7328	.36593	.34365
1.23	.94249	2.8198	.35463	.33424
1.24	.94578	2.9119	.34341	.32480
1.25	.94898	3.0096	.33227	.31532
1.26	.95209	3.1133	.32121	.30582
1.27	.95510	3.2236	.31021	.29628
1.28	.95802	3.3413	.29928	.28672
1.29	.96084	3.4672	.28842	.27712
1.30	.96356	3.6021	.27762	.26750
1.31	.96618	3.7471	.26687	.25785
1.32	.96872	3.9033	.25619	.24818
1.33	.97115	4.0723	.24556	.23848
1.34	.97348	4.2556	.23498	.22875
1.35	.97572	4.4552	.22446	.21901
1.36	.97786	4.6734	.21398	.20924
1.37	.97991	4.9131	.20354	.19945
1.38	.98185	5.1774	.19315	.18964
1.39	.98370	5.4707	.18279	.17981
1.40	.98545	5.7979	.17248	.16997
1.41	.98710	6.1654	.16220	.16010
1.42	.98865	6.5811	.15195	.15023
1.43	.99010	7.0555	.14173	.14033
1.44	.99146	7.6018	.13155	.13042
1.45	.99271	8.2381	.12139	.12050
1.46	.99387	8.9886	.11125	.11057
1.47	.99492	9.8874	.10114	.10063
1.48	.99588	10.983	.09105	.09067
1.49	.99674	12.350	.08097	.08071
1.50	.99749	14.101	.07091	.07074
1.51	.99815	16.428	.06087	.06076
1.52	.99871	19.670	.05084	.05077
1.53	.99917	24.498	.04082	.04079
1.54	.99953	32.461	.03081	.03079
1.55	.99978	48.078	.02080	.02079
1.56	.99994	92.621	.01080	.01080
1.57	1.00000	1255.8	.00080	.00080
1.58	.99996	—108.65	— .00920	— .00920
1.59	.99982	—52.067	— .01921	— .01920
Rad.	Sin	Tan	Cot	Cos

TABLE 5—*Continued*
NATURAL TRIGONOMETRIC FUNCTIONS FOR ANGLES IN RADIANS

Rad.	Sin	Tan	Cot	Cos
1.60	.99957	—34.233	— .02921	— .02920
1.61	.99923	—25.495	— .03922	— .03919
1.62	.99879	—20.307	— .04924	— .04918
1.63	.99825	—16.871	— .05927	— .05917
1.64	.99761	—14.427	— .06931	— .06915
1.65	.99687	—12.599	— .07937	— .07912
1.66	.99602	—11.181	— .08944	— .08909
1.67	.99508	—10.047	— .09953	— .09904
1.68	.99404	—9.1208	— .10964	— .10899
1.69	.99290	—8.3492	— .11977	— .11892
1.70	.99166	—7.6966	— .12993	— .12884
1.71	.99033	—7.1373	— .14011	— .13875
1.72	.98889	—6.6524	— .15032	— .14865
1.73	.98735	—6.2281	— .16056	— .15853
1.74	.98572	—5.8535	— .17084	— .16840
1.75	.98399	—5.5204	— .18115	— .17825
1.76	.98215	—5.2221	— .19149	— .18808
1.77	.98022	—4.9534	— .20188	— .19789
1.78	.97820	—4.7101	— .21231	— .20768
1.79	.97607	—4.4887	— .22278	— .21745
1.80	.97385	—4.2863	— .23330	— .22720
1.81	.97153	—4.1005	— .24387	— .23693
1.82	.96911	—3.9294	— .25449	— .24663
1.83	.96659	—3.7712	— .26517	— .25631
1.84	.96398	—3.6245	— .27590	— .26596
1.85	.96128	—3.4881	— .28669	— .27559
1.86	.95847	—3.3608	— .29755	— .28519
1.87	.95557	—3.2419	— .30846	— .29476
1.88	.95258	—3.1304	— .31945	— .30430
1.89	.94949	—3.0257	— .33051	— .31381
1.90	.94630	—2.9271	— .34164	— .32329
1.91	.94302	—2.8341	— .35284	— .33274
1.92	.93965	—2.7463	— .36413	— .34215
1.93	.93618	—2.6632	— .37549	— .35153
1.94	.93262	—2.5843	— .38695	— .36087
1.95	.92896	—2.5095	— .39849	— .37018
1.96	.92521	—2.4383	— .41012	— .37945
1.97	.92137	—2.3705	— .42185	— .38868
1.98	.91744	—2.3058	— .43368	— .39788
1.99	.91341	—2.2441	— .44562	— .40703
2.00	.90930	—2.1850	— .45766	— .41615
Rad.	Sin	Tan	Cot	Cos

TABLE 6—DEGREES TO RADIANS *

Degr.	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332
2	0.0349	0.0367	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0489	0.0506
3	0.0524	0.0541	0.0559	0.0576	0.0593	0.0611	0.0628	0.0646	0.0663	0.0681
4	0.0698	0.0716	0.0733	0.0750	0.0768	0.0785	0.0803	0.0820	0.0838	0.0855
5	0.0873	0.0890	0.0908	0.0925	0.0942	0.0960	0.0977	0.0995	0.1012	0.1030
6	0.1047	0.1065	0.1082	0.1100	0.1117	0.1134	0.1152	0.1169	0.1187	0.1204
7	0.1222	0.1239	0.1257	0.1274	0.1292	0.1309	0.1326	0.1344	0.1361	0.1379
8	0.1396	0.1414	0.1431	0.1449	0.1466	0.1484	0.1501	0.1518	0.1536	0.1553
9	0.1571	0.1588	0.1606	0.1623	0.1641	0.1658	0.1676	0.1693	0.1710	0.1728
10	0.1745	0.1763	0.1780	0.1798	0.1815	0.1833	0.1850	0.1868	0.1885	0.1902
11	0.1920	0.1937	0.1955	0.1972	0.1990	0.2007	0.2025	0.2042	0.2059	0.2077
12	0.2094	0.2112	0.2129	0.2147	0.2164	0.2182	0.2199	0.2217	0.2234	0.2251
13	0.2269	0.2286	0.2304	0.2321	0.2339	0.2356	0.2374	0.2391	0.2409	0.2426
14	0.2443	0.2461	0.2478	0.2496	0.2513	0.2531	0.2548	0.2566	0.2583	0.2601
15	0.2618	0.2635	0.2653	0.2670	0.2688	0.2705	0.2723	0.2740	0.2758	0.2775
16	0.2793	0.2810	0.2827	0.2845	0.2862	0.2880	0.2897	0.2915	0.2932	0.2950
17	0.2967	0.2985	0.3002	0.3019	0.3037	0.3054	0.3072	0.3089	0.3107	0.3124
18	0.3142	0.3159	0.3176	0.3194	0.3211	0.3229	0.3246	0.3264	0.3281	0.3299
19	0.3316	0.3334	0.3351	0.3368	0.3386	0.3403	0.3421	0.3438	0.3456	0.3473
20	0.3491	0.3508	0.3526	0.3543	0.3560	0.3578	0.3595	0.3613	0.3630	0.3648
21	0.3665	0.3683	0.3700	0.3718	0.3735	0.3752	0.3770	0.3787	0.3805	0.3822
22	0.3840	0.3857	0.3875	0.3892	0.3910	0.3927	0.3944	0.3962	0.3979	0.3997
23	0.4014	0.4032	0.4049	0.4067	0.4084	0.4102	0.4119	0.4136	0.4154	0.4171
24	0.4189	0.4206	0.4224	0.4241	0.4259	0.4276	0.4294	0.4311	0.4328	0.4346
25	0.4363	0.4381	0.4398	0.4416	0.4433	0.4451	0.4468	0.4485	0.4503	0.4520
26	0.4538	0.4555	0.4573	0.4590	0.4608	0.4625	0.4643	0.4660	0.4677	0.4695
27	0.4712	0.4730	0.4747	0.4765	0.4782	0.4800	0.4817	0.4835	0.4852	0.4869
28	0.4887	0.4904	0.4922	0.4939	0.4957	0.4974	0.4992	0.5009	0.5027	0.5044
29	0.5061	0.5079	0.5096	0.5114	0.5131	0.5149	0.5166	0.5184	0.5201	0.5219
30	0.5236	0.5253	0.5271	0.5288	0.5306	0.5323	0.5341	0.5358	0.5376	0.5393
31	0.5411	0.5428	0.5445	0.5463	0.5480	0.5498	0.5515	0.5533	0.5550	0.5568
32	0.5585	0.5603	0.5620	0.5637	0.5655	0.5672	0.5690	0.5707	0.5725	0.5742
33	0.5760	0.5777	0.5794	0.5812	0.5829	0.5847	0.5864	0.5882	0.5899	0.5917
34	0.5934	0.5952	0.5969	0.5986	0.6004	0.6021	0.6039	0.6056	0.6074	0.6091
35	0.6109	0.6126	0.6144	0.6161	0.6178	0.6196	0.6213	0.6231	0.6248	0.6266
36	0.6283	0.6301	0.6318	0.6336	0.6353	0.6370	0.6388	0.6405	0.6423	0.6440
37	0.6458	0.6475	0.6493	0.6510	0.6528	0.6545	0.6562	0.6580	0.6597	0.6615
38	0.6632	0.6650	0.6667	0.6685	0.6702	0.6720	0.6737	0.6754	0.6772	0.6789
39	0.6807	0.6824	0.6842	0.6859	0.6877	0.6894	0.6912	0.6929	0.6946	0.6964
40	0.6981	0.6999	0.7016	0.7034	0.7051	0.7069	0.7086	0.7103	0.7121	0.7138
41	0.7156	0.7173	0.7191	0.7208	0.7226	0.7243	0.7261	0.7278	0.7295	0.7313
42	0.7330	0.7348	0.7365	0.7383	0.7400	0.7418	0.7435	0.7453	0.7470	0.7487
43	0.7505	0.7522	0.7540	0.7557	0.7575	0.7592	0.7610	0.7627	0.7645	0.7662
44	0.7679	0.7697	0.7714	0.7732	0.7749	0.7767	0.7784	0.7802	0.7819	0.7837
45	0.7854	0.7871	0.7889	0.7906	0.7924	0.7941	0.7959	0.7976	0.7994	0.8011
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'
90° = 1.5708 radians	30° = $\frac{\pi}{6}$, 45° = $\frac{\pi}{4}$, 60° = $\frac{\pi}{3}$, 90° = $\frac{\pi}{2}$ radians									
180° = 3.1416 radians	120° = $\frac{2\pi}{3}$, 135° = $\frac{3\pi}{4}$, 150° = $\frac{5\pi}{6}$, 180° = π radians									
270° = 4.7124 radians	210° = $\frac{7\pi}{6}$, 225° = $\frac{5\pi}{4}$, 240° = $\frac{4\pi}{3}$, 270° = $\frac{3\pi}{2}$ radians									
360° = 6.2832 radians	300° = $\frac{5\pi}{3}$, 315° = $\frac{7\pi}{4}$, 330° = $\frac{11\pi}{6}$, 360° = 2π radians									

* Ralph G. Hudson, *The Engineers' Manual*, Second Edition, New York, John Wiley & Sons, Inc., 1939, pp. 276-277. Reprinted by permission.

TABLE 6—DEGREES TO RADIANs—Continued

Degs.	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
45	0.7854	0.7871	0.7889	0.7906	0.7924	0.7941	0.7959	0.7976	0.7994	0.8011
46	0.8029	0.8046	0.8063	0.8081	0.8098	0.8116	0.8133	0.8151	0.8168	0.8186
47	0.8203	0.8221	0.8238	0.8255	0.8273	0.8290	0.8308	0.8325	0.8343	0.8360
48	0.8378	0.8395	0.8412	0.8430	0.8447	0.8465	0.8482	0.8500	0.8517	0.8535
49	0.8552	0.8570	0.8587	0.8604	0.8622	0.8639	0.8657	0.8674	0.8692	0.8709
50	0.8727	0.8744	0.8762	0.8779	0.8796	0.8814	0.8831	0.8849	0.8866	0.8884
51	0.8901	0.8919	0.8936	0.8954	0.8971	0.8988	0.9006	0.9023	0.9041	0.9058
52	0.9076	0.9093	0.9111	0.9128	0.9146	0.9163	0.9180	0.9198	0.9215	0.9233
53	0.9250	0.9268	0.9285	0.9303	0.9320	0.9338	0.9355	0.9372	0.9390	0.9407
54	0.9425	0.9442	0.9460	0.9477	0.9495	0.9512	0.9529	0.9547	0.9564	0.9582
55	0.9599	0.9617	0.9634	0.9652	0.9669	0.9687	0.9704	0.9721	0.9739	0.9756
56	0.9774	0.9791	0.9809	0.9826	0.9844	0.9861	0.9879	0.9896	0.9913	0.9931
57	0.9948	0.9966	0.9983	1.0001	1.0018	1.0036	1.0053	1.0071	1.0088	1.0105
58	1.0123	1.0140	1.0158	1.0175	1.0193	1.0210	1.0228	1.0245	1.0263	1.0280
59	1.0297	1.0315	1.0332	1.0350	1.0367	1.0385	1.0402	1.0420	1.0437	1.0455
60	1.0472	1.0489	1.0507	1.0524	1.0542	1.0559	1.0577	1.0594	1.0612	1.0629
61	1.0647	1.0664	1.0681	1.0699	1.0716	1.0734	1.0751	1.0769	1.0786	1.0804
62	1.0821	1.0838	1.0856	1.0873	1.0891	1.0908	1.0926	1.0943	1.0961	1.0978
63	1.0996	1.1013	1.1030	1.1048	1.1065	1.1083	1.1100	1.1118	1.1135	1.1153
64	1.1170	1.1188	1.1205	1.1222	1.1240	1.1257	1.1275	1.1292	1.1310	1.1327
65	1.1345	1.1362	1.1380	1.1397	1.1414	1.1432	1.1449	1.1467	1.1484	1.1502
66	1.1519	1.1537	1.1554	1.1572	1.1589	1.1606	1.1624	1.1641	1.1659	1.1676
67	1.1694	1.1711	1.1729	1.1746	1.1764	1.1781	1.1798	1.1816	1.1833	1.1851
68	1.1868	1.1886	1.1903	1.1921	1.1938	1.1956	1.1973	1.1990	1.2008	1.2025
69	1.2043	1.2060	1.2078	1.2095	1.2113	1.2130	1.2147	1.2165	1.2182	1.2200
70	1.2217	1.2235	1.2252	1.2270	1.2287	1.2305	1.2322	1.2339	1.2357	1.2374
71	1.2392	1.2409	1.2427	1.2444	1.2462	1.2479	1.2497	1.2514	1.2531	1.2549
72	1.2566	1.2584	1.2601	1.2619	1.2636	1.2654	1.2671	1.2689	1.2706	1.2723
73	1.2741	1.2758	1.2776	1.2793	1.2811	1.2828	1.2846	1.2863	1.2881	1.2898
74	1.2915	1.2933	1.2950	1.2968	1.2985	1.3003	1.3020	1.3038	1.3055	1.3073
75	1.3090	1.3107	1.3125	1.3142	1.3160	1.3177	1.3195	1.3212	1.3230	1.3247
76	1.3265	1.3282	1.3299	1.3317	1.3334	1.3352	1.3369	1.3387	1.3404	1.3422
77	1.3439	1.3456	1.3474	1.3491	1.3509	1.3526	1.3544	1.3561	1.3579	1.3596
78	1.3614	1.3631	1.3648	1.3666	1.3683	1.3701	1.3718	1.3736	1.3753	1.3771
79	1.3788	1.3806	1.3823	1.3840	1.3858	1.3875	1.3893	1.3910	1.3928	1.3945
80	1.3963	1.3980	1.3998	1.4015	1.4032	1.4050	1.4067	1.4085	1.4102	1.4120
81	1.4137	1.4155	1.4172	1.4190	1.4207	1.4224	1.4242	1.4259	1.4277	1.4294
82	1.4312	1.4329	1.4347	1.4364	1.4382	1.4399	1.4416	1.4434	1.4451	1.4469
83	1.4486	1.4504	1.4521	1.4539	1.4556	1.4573	1.4591	1.4608	1.4626	1.4643
84	1.4661	1.4678	1.4696	1.4713	1.4731	1.4748	1.4765	1.4783	1.4800	1.4818
85	1.4835	1.4853	1.4870	1.4888	1.4905	1.4923	1.4940	1.4957	1.4975	1.4992
86	1.5010	1.5027	1.5045	1.5062	1.5080	1.5097	1.5115	1.5132	1.5149	1.5167
87	1.5184	1.5202	1.5219	1.5237	1.5254	1.5272	1.5289	1.5307	1.5324	1.5341
88	1.5359	1.5376	1.5394	1.5411	1.5429	1.5446	1.5464	1.5481	1.5499	1.5516
89	1.5533	1.5551	1.5568	1.5586	1.5603	1.5621	1.5638	1.5656	1.5673	1.5691
90	1.5708	1.5725	1.5743	1.5760	1.5778	1.5795	1.5813	1.5830	1.5848	1.5865
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'
90° = 1.5708 radians	30° = $\frac{\pi}{6}$, 45° = $\frac{\pi}{4}$, 60° = $\frac{\pi}{3}$, 90° = $\frac{\pi}{2}$ radians									
180° = 3.1416 radians	120° = $\frac{2\pi}{3}$, 135° = $\frac{3\pi}{4}$, 150° = $\frac{5\pi}{6}$, 180° = π radians									
270° = 4.7124 radians	210° = $\frac{7\pi}{6}$, 225° = $\frac{5\pi}{4}$, 240° = $\frac{4\pi}{3}$, 270° = $\frac{3\pi}{2}$ radians									
360° = 6.2832 radians	300° = $\frac{5\pi}{3}$, 315° = $\frac{7\pi}{4}$, 330° = $\frac{11\pi}{6}$, 360° = 2π radians									

TABLE 7—CONSTANTS WITH THEIR COMMON LOGARITHMS

	Number	Logarithm
Base of Naperian logarithms	$e = 2.71828183$	0.4342945
Modulus of common logs., $\log_{10} e =$	$u = 0.43429448$	9.6377843-10
Reciprocal of modulus	$\frac{1}{u} = 2.30258509$.3622157
Circumference of a circle in degrees . . .	$= 360$	2.5563025
Circumference of a circle in minutes . . .	$= 21600$	4.3344538
Circumference of a circle in seconds . . .	$= 1296000$	6.1126050
Radian expressed in degrees	$= 57.29578$	1.7581226
Radian expressed in minutes	$= 3437.7468$	3.5362739
Radian expressed in seconds	$= 206264.806$	5.3144251
Ratio of a circumference to diameter . . .	$\pi = 3.14159265$	0.4971499
$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 8328$	$g = 981$	2.9916690

Number	Logarithm	Number	Logarithm
$2\pi = 6.28318531$	0.7981799	$\pi^2 = 9.86960440$	0.9942997
$4\pi = 12.56637061$	1.0992099	$\frac{1}{\pi^2} = 0.10132118$	9.0057003-10
$\frac{\pi}{2} = 1.57079633$	0.1961199	$\sqrt{\pi} = 1.77245385$	0.2485749
$\frac{\pi}{3} = 1.04719755$	0.0200286	$\frac{1}{\sqrt{\pi}} = 0.56418958$	9.7514251-10
$\frac{4\pi}{3} = 4.18879020$	0.6220886	$\sqrt{\frac{3}{\pi}} = 0.97720502$	9.9899857-10
$\frac{\pi}{4} = 0.78539816$	9.8950899-10	$\sqrt{\frac{4}{\pi}} = 1.12837917$	0.0524551
$\frac{\pi}{6} = 0.52359878$	9.7189986-10	$\sqrt[3]{\pi} = 1.46459189$	0.1657166
$\frac{1}{\pi} = 0.31830989$	9.5028501-10	$\frac{1}{\sqrt[3]{\pi}} = 0.68278406$	9.8342834-10
$\frac{1}{2\pi} = 0.15915494$	9.2018201-10	$\sqrt[3]{\pi^2} = 2.14502940$	0.3314332
$\frac{3}{\pi} = 0.95492966$	9.9799714-10	$\sqrt{\frac{3}{4\pi}} = 0.62035049$	9.7926371-10
$\frac{4}{\pi} = 1.27323954$	0.1049101	$\sqrt[3]{\frac{\pi}{6}} = 0.80599598$	9.9063329-10

Number	Logarithm
If the radius $r = 1$, the length of the arc is	
for 1 degree $= \frac{\pi}{180} = 0.01745329$	8.2418774-10
for 1 minute $= \frac{\pi}{10800} = 0.00029089$	6.4637261-10
for 1 second $= \frac{\pi}{648000} = 0.00000485$	4.6855749-10
$\sin 1'' = 0.00000485$	4.6855749-10

TABLE 8

SYMBOLS AND UNITS

Greek Alphabet

A, α	Alpha	I, ι	Iota	P, ρ	Rho
B, β	Beta	K, κ	Kappa	Σ , σ , s	Sigma
Γ , γ	Gamma	Λ , λ	Lambda	T, τ	Tau
Δ , δ	Delta	M, μ	Mu	Υ , ν	Upsilon
E, ϵ	Epsilon	N, ν	Nu	Φ , ϕ , φ	Phi
Z, ζ	Zeta	Ξ , ξ	Xi	X, χ	Chi
H, η	Eta	O, \omicron	Omicron	Ψ , ψ	Psi
Θ , θ , ϑ	Theta	Π , π	Pi	Ω , ω	Omega

Prefixed Used with Units

Prefixed	Abbreviation	Meaning	Scientific or Engineering Notation
micromicro	$\mu\mu$	0.000 000 000 001	10^{-12}
millimicro	m μ	0.000 000 001	10^{-9}
micro	μ	0 000 001	10^{-6}
milli	m	0.001	10^{-3}
centi	c	0.01	10^{-2}
deci	d	0 1	10^{-1}
deka	dk	10	10
hekto	h	100	10^2
kilo	k	1000	10^3
mega	M	1000 000	10^6

Table 8—Continued
Quantities and Units Used in the Exercises and Examples

Quantity	Symbol	Basic Unit	Common Multiples
Length	l	{ meter foot	millimeter, centimeter, deci- meter, kilometer inch, yard, mile
Mass	m	{ gram slug	milligram, centigram, kilogram
Time	t	second	minute, hour
Force	F	{ dyne pound	ton
Velocity	v	{ meters per second feet per second	centimeters per second miles per hour
Acceleration	a	{ meters per second per second feet per second per second	centimeters per second per second miles per hour per hour
Temperature	T	degree	
Charge	Q	coulomb	
Potential	E or V	volt	microvolt, millivolt, kilovolt
Current	I	ampere	microampere, milliamper
Frequency	f	cycle per second	kilocycle per second, megacycle per second
Resistance	R	ohm	microhm, megohm
Conductance	G	mho	micromho, millimho
Inductance	L	henry	microhenry, millihenry
Capacitance	C	farad	micromicrofarad, microfarad
Energy	W	joule	
Power	P	watt	microwatt, milliwatt, kilowatt
Magnetic flux density	B	gauss	kilogauss
Magnetic field intensity	H	oersted	

ANSWERS TO THE ODD EXERCISES

Chapter 1

Page 4. Sec. 1-2

- | | | |
|------------------|----------------|----------------|
| 1. $5 > 3$. | 3. $-1 < 0$. | 5. $-6 > -8$. |
| 7. $6 > -6$. | 9. $5 > -3$. | 11. $1 > -3$. |
| 13. $-8 > -10$. | 15. $-3 < 4$. | |

Pages 9-10. Sec. 1-8

1. (a) -9 , (b) -19 , (c) 9 , (d) 85 , (e) 18 .
3. (a) 14 , (b) 108 , (c) 23 , (d) 33 , (e) -26 .
5. (a) -13 , (b) 13 , (c) -5 , (d) -19 , (e) -19 .
7. (a) 1 , (b) -35 , (c) -35 , (d) -20 , (e) -42 .
9. (a) -22 , (b) -84 , (c) -96 , (d) 12 , (e) -78 .
11. (a) -132 , (b) -132 , (c) 0 , (d) 300 , (e) -231 .
13. (a) 0 , (b) -6 , (c) 1 , (d) -2 , (e) -2 .
15. (a) $\frac{1}{2}$, (b) not defined, (c) $-\frac{9}{2}$, (d) -9 , (e) $-\frac{1}{3}$.
17. (a) 14 , (b) -9 , (c) -10 , (d) -6 , (e) 10 .

Page 11. Sec. 1-9

- | | | |
|-------------|-----------|------------|
| 1. -4 . | 3. 8 . | 5. 8 . |
| 7. 3 . | 9. 3 . | 11. 16 . |
| 13. -9 . | 15. 8 . | 17. 63 . |
| 19. -44 . | | |

Pages 14-16. Sec. 1-10

- | | | |
|---|--|-----------------------|
| 1. (a) $\frac{4}{9}$, (b) $\frac{4}{3}$, (c) $\frac{51}{2}$. | 3. (a) $-\frac{32}{257}$, (b) $\frac{200}{235}$, (c) $\frac{37}{12}$. | |
| 5. 198 . | 7. 1680 . | 9. 1848 . |
| 11. 588 . | 13. 13 . | 15. 250 . |
| 17. 13 . | 19. 11 . | 21. $\frac{3}{8}$. |
| 23. $\frac{11}{20}$. | 25. $\frac{49}{80}$. | 27. $\frac{95}{36}$. |
| 29. $\frac{2}{5}$. | 31. -2 . | 33. 1 . |
| 35. $\frac{4608}{343}$. | 37. 6 . | 39. $\frac{12}{35}$. |
| 41. $\frac{245}{283}$. | 43. $\frac{162}{73}$. | 45. $\frac{13}{16}$. |
| 47. $\frac{50}{9}$. | 49. 5.4 cents, 18 cents. | 51. 37.5 in. |
| 53. $960,000$ lb. | 55. 63.7 amp. | 57. 87.5 milliamp. |
| 59. $146\frac{2}{3}$ cu. ft., $916\frac{2}{3}$ gal. | 61. 150 watts, 0.9 cent. | |

Page 20. Sec. 1-11

- | | | |
|---|----------------------------------|----------------------------------|
| 1. $10^7, 10^2, 10^0, 10^{-1}, 10^{-4}$. | 3. 3^7 . | |
| 5. $\frac{1}{2}$ | 7. $2^6 \cdot 3^3$. | 9. $3 \cdot 5 \cdot 7$. |
| 11. $-\frac{3^3}{2}$. | 13. $-\frac{1}{2^2 \cdot 5^2}$. | 15. $2 \cdot 5$. |
| 17. $-\frac{1}{2^3}$. | 19. $\frac{2}{3^3}$. | 21. $-\frac{3 \cdot 5^2}{2^6}$. |
| 23. $\frac{7 \cdot 2^2}{3}$. | 25. $\frac{1}{7}$. | 27. $\frac{7 \cdot 2^2}{3^3}$. |

Page 22. Sec. 1-12

- | | |
|---|---|
| 1. 6.94×10^{11} days. | 3. 6.06×10^{23} atoms. |
| 5. 3×10^{-23} gram. | 7. 5×10^{-3} cm. |
| 9. 2×10^9 light-years. | 11. 6,600,000,000,000,000,000,000 tons. |
| 13. 0.000,000,000,000,4 cm. | 15. 0.000,000,000,000,000,000,227 dyne. |
| 17. 600,000,000,000,000,000,000,000 protons per gram of matter. | |
| 19. 6,000,000,000 times per sec. | 21. 0.00008 cm. |
| 23. 27,600 km. per sec. | 25. 112,000,000 mi. |

Page 25. Sec. 1-13

- | | | |
|--------|--------|--------|
| 1. 3. | 3. 3. | 5. 1. |
| 7. 1. | 9. 1. | 11. 2. |
| 13. 1. | 15. 3. | 17. 1. |
| 19. 1. | 21. 1. | 23. 3. |
| 25. 3. | | |

Pages 28-29. Sec. 1-16

- | | | |
|------------------------------|------------------------------|-----------------------------|
| 1. 9.90×10^2 . | 3. -7.09×10^2 . | 5. 9.100×10^{-1} . |
| 7. 6.65×10 . | 9. 8.6×10^3 . | 11. 7.9×10 . |
| 13. 2.59. | 15. 2.88×10^{-3} . | 17. 9.2×10^2 . |
| 19. 5.89×10^2 . | 21. 7.3. | 23. 3.0. |
| 25. 7.4×10^5 . | 27. 1.3×10 . | 29. 3.7×10^{-1} . |
| 31. 2.08×10^3 . | 33. 2.6×10^2 . | 35. -9.59×10^5 . |
| 37. 9.676×10^2 . | 39. 4.829×10^{-3} . | 41. 5.8×10^{-8} . |
| 43. 5.5×10^{-10} . | 45. 2.6×10^{-1} . | 47. 3.59×10^{11} . |
| 49. 3.817×10^{-1} . | | |

Page 34. Sec. 1-20

- | | | |
|-----------------------------|-----------------------------|--------------------------|
| 1. 8.00×10 . | 3. 7.34×10 . | 5. 6.50×10 . |
| 7. 1.495×10 . | 9. 6.93×10 . | 11. 8.66×10^3 . |
| 13. 1.956×10^3 . | 15. 8.40×10^{-5} . | 17. 4.50×10^4 . |
| 19. 6.71. | 21. 2.11×10^4 . | 23. 2.72. |
| 25. 4.09×10^{-2} . | | |

Page 35. Sec. 1-21

- | | | |
|-----------------------------|-----------------------------|-----------------------------|
| 1. 3.00. | 3. 8.89×10^{-1} . | 5. 2.10×10 . |
| 7. 1.728×10^2 . | 9. 1.321×10^{-4} . | 11. 2.46×10^{-1} . |
| 13. 3.84. | 15. 5.05×10^{-1} . | 17. 1.127×10^5 . |
| 19. 1.33. | 21. 8.89×10^{-1} . | 23. 3.01×10^{-4} . |
| 25. 4.99×10^{-1} . | | |

Page 36. Sec. 1-22

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|-----------------------------|-----------------------------|-----------------------------|
| 1. 2.56×10^2 . | 3. 7.92×10^3 . | 5. 9.72×10^3 . |
| 7. 8.06×10^3 . | 9. 1.318×10^4 . | 11. 2.36×10^3 . |
| 13. 4.52×10^{-5} . | 15. 7.68×10^{-1} . | 17. 8.03×10^3 . |
| 19. 2.20×10^5 . | 21. 8.03×10^5 . | 23. 9.37×10^{-5} . |

Pages 37-38. Sec. 1-23

- | | | |
|-----------------------------|-----------------------------|-------------------------|
| 1. 5.00. | 3. 5.00×10 . | 5. 2.853×10 . |
| 7. 3.40. | 9. 9.45. | 11. 3.11×10 . |
| 13. 3.10×10^{-1} . | 15. 2.51×10^{-2} . | 17. 2.84. |
| 19. 2.54×10^2 . | 21. 8.93×10^2 . | 23. 2.225×10 . |
| 25. 3.82. | | |

Pages 39-40. Sec. 1-25

- | | | |
|--------------------------|----------------------------|--------------------------|
| 1. 1.033. | 3. 2.85×10^2 . | 5. 4.99×10^2 . |
| 7. 1.851. | 9. 6.49×10^{-2} . | 11. 4.50×10^6 . |
| 13. 9.46. | 15. 3.59×10^4 . | 17. 5.90×10 . |
| 19. 1.49×10^2 . | | |

Chapter 2

Pages 43-44. Sec. 2-2

- | | | |
|------------------------------------|----------------------------|-------------------------------|
| 1. $6a$. | 3. $6x + 6y$. | 5. $8E_1 - 2E_2 - 8E_3$. |
| 7. $-3V_1 - 3V_2 + 7V_3$. | 9. $2x + 6y$. | 11. $3ab + 7cd$. |
| 13. $4R_1 + 2R_2 + 5R_3$. | 15. $-2E + 2RI + 21ZI$. | 17. $3R_1 + 5R_2$. |
| 19. $x + 7y + z + 6w$. | 21. $7V$. | 23. $a + 2b$. |
| 25. $-2E + 2IX$. | 27. $20ax + 20by - 10cz$. | 29. $2x^2 + 8xy - 6y^2$. |
| 31. $4I_1 - 2I_2 - 3I_3$. | 33. $-3Z_1 - 8Z_2 - 15$. | |
| 35. $3R_1 + 8R_2 - 13R_3 - 4R_4$. | 37. $2x + 6y$. | |
| 39. $2ac - 2by$. | 41. 0. | 43. $-E - 3I_1R_1 - I_2R_2$. |
| 45. $-4x + 5y + z - 43$. | 47. $-5x + 7y$. | 49. $E + RI - 3IZ$. |

Pages 44-45. Sec. 2-3

- | | | |
|-------------------------|------------------------|---------------------------|
| 1. $-30x^3y^2$. | 3. $48a^{10}$. | 5. $15I^2R$. |
| 7. $54a^3b^3c^3d^3$. | 9. $21EI^3R$. | 11. $-6W^2X^3Y^3$. |
| 13. $-30R_1R_2R_3I^3$. | 15. $64i^2r^2s^2t^4$. | 17. $-126x^2y^2z^3w^2$. |
| 19. $252I^3r^2t^2$. | 21. $-162x^4y^4z^4$. | 23. $24a^4b^5c^5d^4e^2$. |
| 25. $56x^6y^6$. | 27. $30a^7b^7$. | 29. $R_a^3R_b^3R_c^3$. |

Pages 45-46. Sec. 2-4

1. $x^2 + 2x$.
3. $6R_1R_2 + 3R_1 + 10R_2 + 5$.
5. $3I^2RX + 3I^2RZ$.
7. $a^2 - b^2$.
9. $15I^2 + 14I - 8$.
11. $48r^4 - 6r^2Z - 15Z^2$.
13. $14x^2y^3z + 21xy^3z^2 + 35x^2y^2z^2$.
15. $12a^2bc + 18a^2bd + 42ab^2c + 54ab^2d$.
17. $48R^2I_1^2 - 16R^2I_1I_2 + 8E_0RI_1$.
19. $15P^3 + 31P^2 + 16P + 10$.
21. $6I^3R + 12I^2R + 4IR^2 + 6I + 8R^2 + 12$.
23. $a^2 - 2ab + b^2 - c^2$.
25. $3x^4 - 10x^3 - x^2 + 2$.
27. $6m^2n^2 - 17mn^2p + m^2np - 14n^2p^2 - 41mnp^2 - 15m^2p^3$.
29. $6x^3 + 4x^2y + 12x^2y^2 + 2xy^2 + 9x^2y + 9xy^3 + 3y^3$.

Page 46. Sec. 2-5

1. $5a^4$.
3. $\frac{a}{4Z}$.
5. a^2 .
7. $\frac{3A^4}{C^2}$.
9. $\frac{L(x-y)}{4}$.
11. $(a+b)yt$.
13. $7a^2b^2cd$.
15. $\frac{(R^2 + 2X - 7)^3T^{n-3}}{R}$.
17. $N^2Q^{2n-1}R^3$.
19. $M^{n-2}N^nQ$.

Page 47. Sec. 2-6

1. $R^2 + 4V$.
3. $1 + I - \frac{E}{R_1}$.
5. $\frac{2L}{R} - R$.
7. $\frac{4y^{2-n}}{x^nZ} + \frac{8x^{1-n}}{y^nZ} - \frac{7Z^2}{x^{2n}y^{n+2}}$.
9. $\frac{(a+c)^3}{4} + \frac{(a+c)^2}{k} + \frac{1}{4k}$.
11. $\frac{P^n}{M} + \frac{2P^{n-2}}{K} + \frac{1}{2P^2}$.
13. $\frac{7x^8}{24pq} - \frac{p^2x^3}{4q^2} + \frac{q}{6p}$.
15. $\frac{6y^2Z(p+1)^m}{x^m} - \frac{3x^{2-m}Z^2}{y} + \frac{p+1}{3y^2Z}$.
17. $y^3x^2Z^2(p+1)^{2m} - \frac{x^4Z^3(p+1)^m}{2} + \frac{x^{m+2}(p+1)^{m+1}}{18y}$.
19. $\frac{6R}{L} + \frac{8}{R^4Z^3L^5} - \frac{4}{R^2Z^3L^2}$.

Pages 48-49. Sec. 2-7

1. $2x + 6$.
7. $y^2b^2 - 4ybm + 4m^2$.
13. $r^2 + 8r + 15$.
19. $4c^2$.
25. $4x^2 - 8xy + 4y^2$.
31. $x^2 - 2x + 1$.
37. $\theta^2 - \frac{1}{2}\theta$.
43. $100c^2 - 100cd + 25d^2$.
49. $\frac{R^4}{36} - \frac{9Z^4}{100}$.
3. $e^3 + e^2 - e^4 + 2e$.
9. $R^2 - \frac{4}{7}R + \frac{4}{9}$.
15. $a^2 - 16a + 48$.
21. $\mu^2 + 20\mu + 100$.
27. $R^4 - x^2$.
33. $\frac{M^2}{X_p^2} - \frac{2Mi_p}{X_p} + i_p^2$.
39. $Z^2 + 12Z + 11$.
45. $9v^2 - t^2$.
5. $\lambda^2f^2 + 2\lambda fv + v^2$.
11. $I^2R^2 - 1$.
17. $2cw + 2bw + ac + ab$.
23. $3ax^2 + 6axy + 3ay^2$.
29. $\phi^4 + \frac{3}{10}\phi^2 + \frac{1}{50}$.
35. $\frac{1}{R^2} - \frac{1}{r^2}$.
41. $4ac$.
47. $\frac{E^2}{I^2} - \frac{81}{i^2}$.

Pages 50-51. Sec. 2-8

1. $a(y^2 + x^2 + z^2)$.
7. $2ab(b + 2a - 4ab)$.
13. $xyzZ^2(x^3 - y^3Z - xy)$.
17. $13RL(R^2L^2 - RL + 3)$.
21. $(x^3 + 2)(x + 1)$.
27. $(a + b)(a + y - 2)$.
33. $(Z^2 + w)(x^2 - 2)$.
39. $(N - P)^2$.
43. $\frac{1}{4}(V_t + V_0)(V_t - V_0)$.
47. $(HLI + R)(HLI - R)$.
51. $(2Z - 3)^2$.
57. $(x - 4)(x + 1)$.
63. $(W - 7)(W + 5)$.
69. $5P^3R^5(3PR - 1)(2PR - 1)$.
73. $(Z^2 - 20)(Z^2 + 5)$.
77. $(x^2y + 1)(x^2y + 3)$.
3. $17x^2(1 - 17x)$.
9. $a^2l^2(a + 3l - 5)$.
15. $3m(m^4 - 4m^2n^2 + 2n^4)$.
19. $(a - b)^2(y^2 - 5)$.
23. $p(1 + rt)(4 + q)$.
29. $2(x - 3)(y + z)$.
35. $P^2(R + E)(L + I)$.
41. $(a^2b + 1)^2$.
45. $(X_L + 4X_c)(X_L - 4X_c)$.
49. $C^2(d^2 + 1)(d + 1)(d - 1)$.
53. $(5X_L + 2X_c)^2$.
59. $(t - 17)(t + 4)$.
65. $a(d + 7)(d - 2)$.
55. $(x - 3)(x - 1)$.
61. $(a + 3d)(a + 4d)$.
67. $(3L + 1)^2$.
71. $(E + 13R)(E - 2R)$.
75. $5(L - 2)(L - 3)$.
79. $\pi(aL + 11)(aL + 6)$.
11. $W\left(\frac{L}{R} + C - W\right)$.
25. $(y + b)(4 - c)$.
31. $a^2b(a + b)(ab + 1)$.
37. $(k + 4)^2$.

Page 52. Sec. 2-9

1. $35a^2b^4$.
5. $(x + 3)(x - 2)(x + 2)^2$.
9. $x(x + y)(x - y)$.
13. $x^{13}(x + 1)$.
17. $xyZ(m + n)^2(m - n)^2$.
3. $180L^2R^3$.
7. $(Z + 3)(Z - 3)(Z + 7)$.
11. $12(E + 3)(E - 3)(E - 2)$.
15. $abc(x + y)(x - y)$.
19. $(R - 5)(R - 4)(R + 2)$.

Pages 54-56. Sec. 2-10

1. $\frac{x^2 + xy + y^2}{x^2y^2}$.
3. $\frac{2t - 5}{t}$.
5. 1.
7. $\frac{-2r}{r + 1}$.
9. 0.
11. $\frac{ad - bc}{b(b - d)}$.
13. $\frac{1}{y^n}$.
15. $\frac{x}{5x + y}$.
17. $\frac{19x - 4}{(2x + 1)(2x - 1)}$.
19. $\frac{135y^{n+1} - 50b^{n+3}c^2 - 28bc^n}{210b^6c^4y}$.
21. $\frac{-2n}{(a^n + 1)(a^n - 1)}$.
23. $\frac{r^2 + 1}{(r + 1)(r - 1)}$.
25. $\frac{4x^my^m}{(x^m - y^m)(x^m + y^m)}$.
27. 0.
29. $-\frac{6x^2 + 29x + 3}{x(x + 5)}$.
31. $\frac{b^2c - bc^2 - a^2c + ac^2 + a^2b - ab^2}{abc(a - b)(a - c)(b - c)}$.
33. $\frac{Z(6Z - 1)}{(2Z + 1)^2}$.
35. $\frac{3x^2 + 7x + 2}{x(x + 1)^2}$.
37. 0.
39. $\frac{x^2 + x + 2}{(x + 1)(x - 1)^2}$.
41. $\frac{2z^2 - 3z + 2}{(2z - 1)^2}$.
43. $\frac{x}{(x - 2)(x + 8)(x + 9)}$.
45. $\frac{-x^2 + 38x - 16}{(x - 9)(x + 2)}$.
47. $\frac{11x^2 + x - 4}{(3x + 1)(3x - 1)}$.
49. $\frac{9cd}{(c - d)^2(c + 2d)}$.

Pages 57-58. Sec. 2-11

1. $\frac{5b^3}{2a^2y}$.
3. $\frac{8x^{m+1}}{15b^3a^{n-1}}$.
5. $\frac{8Z}{3}$.
7. $\frac{bc^3d}{aej^3}$.
9. $\frac{1}{abxy}$.
11. $\frac{4r}{r - s}$.
13. $\frac{P^2 + r^2}{q}$.
15. 1.
17. 1.
19. $\frac{x + e}{x + d}$.
21. 1.
23. $\frac{x^2y^2}{r^2s^2}$.
25. $3(E + e)$.
27. $\frac{r^2 + s^2}{r}$.
29. $\frac{2(a - b)^2}{3b^2(a + b)}$.
31. $\frac{y - 3}{y - 2}$.
33. $\frac{r^2 + 1}{r}$.
35. $1 + p$.
37. $4x$.
39. $\frac{8g}{b}$.
41. $\frac{z + 1}{(z - 1)^2}$.
43. $\frac{t + 2}{t + 6}$.
45. $\frac{(I - 3)(I + 7)}{(I - 8)(I + 5)}$.
47. $\frac{2a + b - c}{2a + b + c}$.
49. $\frac{2a + b}{a + b}$.

Pages 61-63. Sec. 2-12

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|--------------------------|------------------------|-----------------------------------|
| 1. 2. | 3. $\frac{2^8}{9}$. | 5. $\frac{1^2}{119}$. |
| 7. $a^2 - 4$. | 9. $\frac{40}{109}$. | 11. $\frac{5}{2}$. |
| 13. 4. | 15. $-\frac{8^5}{3}$. | 17. 8. |
| 19. $-\frac{8^7}{18}$. | 21. $\frac{301}{74}$. | 23. 125. |
| 25. 12. | 27. 2. | 29. 8. |
| 31. 8. | 33. 4. | 35. $\frac{6(a^2 - 9)}{6a + 5}$. |
| 37. $-\frac{8^1}{2^2}$. | 39. 1. | 41. $\frac{3^9}{5^3}$. |
| 43. $a - 2b$. | 45. -3. | 47. 7. |
| 49. 1. | | |

Page 64. Sec. 2-13

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|-----------------------------------|-------------------------------------|------------------------------------|
| 1. -2, -2. | 3. -2, -3. | 5. $-\frac{3}{2}, -\frac{3}{2}$. |
| 7. $\frac{1}{2}, \frac{1}{2}$. | 9. 5, 6. | 11. $-\frac{1}{5}, -\frac{1}{5}$. |
| 13. $\frac{1}{3}, \frac{1}{3}$. | 15. $\frac{3}{10}, \frac{3}{10}$. | 17. 6, -1. |
| 19. $\frac{4}{3}, -\frac{4}{3}$. | 21. $\frac{3}{14}, -\frac{3}{14}$. | 23. 7, 7. |
| 25. 3, -2. | 27. $\frac{3}{2}, -\frac{3}{2}$. | 29. $\frac{3}{2}, -\frac{3}{2}$. |

Pages 65-67. Sec. 2-14

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|---|--|
| 1. $R_3 = R_T - R_1 - R_2$. | 3. $L_1 = L_T - L_2 + 2M$. |
| 5. $I_{x0} = I_x - Ad^2, A = \frac{I_x - I_{x0}}{d^2}$. | 7. $E = IR$. |
| 9. $R = \frac{P}{I^2}$. | 11. $d = \frac{Wh}{F}, W = \frac{Fd}{h}$. |
| 13. $w = \frac{8d^2}{l^2}$. | 15. $g = \frac{v^2 - v_0^2}{2h}, v_0^2 = v^2 - 2gh$. |
| 17. $a = \frac{Fg}{w}, w = \frac{Fg}{a}$. | 19. $t = \frac{d}{v}$. |
| 23. $l = \frac{RA}{k}$. | 21. $m = \frac{2fd}{v^2}, d = \frac{mv^2}{2f}$. |
| 29. $\phi = BA$. | 25. $b = \frac{WL}{Kd^2}$. |
| 35. $l = \frac{10^8 F}{22.5BI}, B = \frac{10^8 F}{22.5Il}$. | 27. $D = \frac{Tl}{12} + d$. |
| 39. $N_2 = -\frac{10^8 LM}{1.26N_1 A \mu}$. | 31. $N = \frac{2.534H}{D^2}$. |
| 43. $A = \frac{10^8 Cd}{8.84K}$. | 33. $r_1 = \frac{r_2 r_3}{r_4}$. |
| 47. $d^2 = \frac{L}{0.0251n^2 l}, l = \frac{L}{0.0251d^2 n^2}$. | 37. $F_1 = \frac{2\pi l F}{d}$. |
| 49. $e_2 = e_1 + \frac{i_{av.}(t_2 - t_1)}{C}, C = i_{av.} \frac{t_2 - t_1}{e_2 - e_1}$. | 41. $h = \frac{iR}{w}$. |
| | 45. $f = \frac{X_L}{2\pi L}, L = \frac{X_L}{2\pi f}$. |

Pages 69-71. Sec. 2-15

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|---------------------------------|-------------------------------|----------------|
| 1. 6090 watts. | 3. 176 watts. | 5. 11,000 ft. |
| 7. 7.0×10^{-18} watts. | 9. $25\frac{1}{4}$ cu. in. | 11. 3.1 dynes. |
| 13. 180 ohms. | 15. 4.0 ft. per sec. per sec. | |
| 17. 992 kc., 3.1416. | | |

Pages 73-74. Sec. 2-16

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|--------------------------------------|--------------------------------------|--|
| 1. -5, 8. | 3. 3, 1. | 5. 5, 2. |
| 7. 3, 4. | 9. 2, 3. | 11. 3, 5. |
| 13. 3, 2. | 15. 10, 1. | 17. $\frac{4}{18}$, $-\frac{3}{18}$. |
| 19. 8, 1. | 21. 0, -2. | 23. 3, 2. |
| 25. 4, 3. | 27. $\frac{1}{5}$, $-\frac{2}{5}$. | 29. $\frac{4}{18}$, $\frac{5}{18}$. |
| 31. $\frac{3}{7}$, $-\frac{5}{7}$. | 33. $\frac{2}{3}$, 0. | 35. 7, -1. |
| 37. 5, 4. | 39. 20, 20 | |

Pages 77-79. Sec. 2-17

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|--|--------------------------|-----------------------|
| 1. 14, 16. | | |
| 3. 105 hr. is the life of a carbon steel drill, 315 hr. is the life of a tungsten steel drill, 1575 hr. is the life of a high-speed steel drill. | | |
| 5. 30 grams of nickel, 150 grams of manganese, 820 grams of copper. | | |
| 7. A fires $\frac{4}{34}$ kg. at $\frac{1}{34}$ kg. per min., B fires $\frac{4}{17}$ kg. at $\frac{1}{17}$ kg. per min., C fires $\frac{8}{7}$ kg. at $\frac{2}{7}$ kg. per min. | | |
| 9. 9.5 grams of antimony, 79.7 grams of copper, 10.0 grams of tin, 0.794 grams of phosphorus. | | |
| 11. 36 lb., 180 lb. | 13. \$25. | 15. $\frac{1}{7}$ hr. |
| 17. 48 min. | 19. $\frac{1}{13}$ days. | 21. 6 hr. |
| 23. 73. | 25. 72. | 27. 256. |
| 29. 50%. | 31. 576 sq. yd. | |

Pages 82-83. Sec. 2-18

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|---------------------------|--|---------------|
| 1. 10 hr., 50 mi., 40 mi. | 3. After 5 hr. | 5. 22 m.p.h. |
| 7. 45 m.p.h. | 9. 6 volts. | 11. 44 amp. |
| 13. 15 ohms. | 15. $220\frac{5}{8}$ ohms. | 17. 2000 amp. |
| 19. 1.3 ohms. | 21. 6 ft. from the fulcrum on the opposite side. | |
| 23. 184 lb., 115 lb. | 25. 325 lb. | |

Chapter 3

Pages 89-92. Sec. 3-4

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|---|------------------------------------|--|
| 1. $C = f(r) = 2\pi r$. | 3. $A = f(d) = d^2$. | 5. $V = f(h, r) = \frac{1}{3}\pi hr^2$. |
| 7. $V = f(w, h, t) = wht$. | 9. $A = f(b, h) = \frac{1}{2}bh$. | 11. $V = f(A, h) = \frac{1}{2}Ah$. |
| 13. $A = f(s, r) = \pi r(s + r)$. | 15. $I = f(t) = \$1.50 \times t$. | |
| 17. $C = f(q) = 20 + 30(q - \frac{1}{2})$. | 19. $H = f(T, I, R) = 0.24I^2RT$. | |

21. -6, -8, -2. 23. $-3, \frac{1}{3}, 5$. 25. 10, 7, 2.
27. 0, 0, $(c-a)(c-b)$. 29. $(1-a)(1-b), 0, a(a-1)(a^2-b)$.
31. $\left(y + \frac{1}{y}\right)\left(\frac{1}{y^2} - y\right), \frac{35}{4}$. 33. $\frac{(1-t^2)(1+t^8)}{(1+t^2)(1+t^4)^2}$.
35. $\frac{a^2 + a - b^2 - b}{a^2 - 2ab + b^2 + a - b + 1}$. 37. $2xh + h^2$.
39. $3x^2 + 3xh + h^2$. 41. 156. 43. $x^4 + 2x^3 + 2x^2 + x$.
45. $8a + 4k$. 47. -2, 0, 5, 4. 49. -3, 0, -1, ab .
51. 16, -35, -99, $a^2 - 4b^2 + 20$. 53. -64, 8, 12, 5.
55. -1, -12, $-\pi, \pi^2$. 57. 3, 3, 17, 14. 59. $1, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{3}$.
61. $p = \sqrt{A}$. 63. $\frac{1}{2}x\sqrt{100-x^2}$. 65. $2\left(x + \frac{10}{x}\right)$.
67. $\frac{\sqrt{3}}{18}(40-x)^2 + \frac{x^2}{4\pi}$. 69. $d = 7l$.
71. $y = \sqrt{100-x^2}, A = x\sqrt{100-x^2}, P = 2(x + \sqrt{100-x^2})$.
73. $h = \frac{100}{\pi r^2}, S = 2\pi r^2 + \frac{200}{r}$.
75. $r = \sqrt{\frac{6h}{\sqrt[3]{\pi}}} - h^2, V = 2\pi^{\frac{1}{3}}h^2 - \frac{\pi h^3}{3}$.

Page 93. Sec. 3-5

1. $x = \frac{y}{3} + 6$. 3. $x = \pm\sqrt{y+9}$. 5. $x = \sqrt[3]{y}$.
7. $Z = \sqrt[3]{w-48}$. 9. $r = \frac{C}{2\pi}$. 11. $d = 2\sqrt{\frac{A}{\pi}}$.
13. $d = \sqrt[3]{\frac{6V}{\pi}}$. 15. $e = \sqrt[3]{V}$. 17. $I = \frac{E}{10}$.
19. $v = \frac{d}{3}$.

Page 94. Sec. 3-6

1. $3 < 5$. 3. $-5 < 2$. 5. $8 > -3$.
7. $8 > -1.2$. 9. $18 > -5$. 11. $-8 < \frac{2}{3} < \sqrt{2} < 4$.
13. $-5 < -3\frac{1}{2} < 5 < 6.7 < 7$. 15. $-8 < -5 < 4 < 6 < 7$.
17. $-5 < 1.6 < 4.7 < 12 < 18$.

Page 119. Sec. 3-10

55. 2.4, -3.4. 57. 0.6, 3.4. 59. -2.7, 0.7.
61. 3.7, -2.7. 63. 2.0, -5.0.

Pages 122-124. Sec. 3-11

- | | | |
|--|--|---------------|
| 1. 5. | 3. $g = 5.12 \times 10^8/d^2$; 14 ft. per sec. per sec. | |
| 5. 15 in. | 7. 12 in. | 9. 58,811 lb. |
| 11. $p = \frac{Av^2}{220}$; 1.5×10^3 lb. | 13. 2.2×10^{-3} coulomb. | 15. 6.3 in. |
| 17. 1.44 joules. | 19. 3622 ft. | 21. 38.7 ft. |
| 23. 0.09 ohm. | 25. 75%. | |

Page 126. Sec. 3-12

- | | | |
|------------------------|------------------------------------|--------------------------------------|
| 1. 5, 5. | 3. $-\frac{5}{4}, \frac{5}{2}$. | 5. $\frac{1}{3}, 1$. |
| 7. 2, 2. | 9. $\frac{39}{5}, \frac{39}{12}$. | 11. $\frac{15}{4}, 5$. |
| 13. -2, -4. | 15. $\frac{3}{4}, 6$. | 17. $-\frac{14}{3}, \frac{28}{15}$. |
| 19. $3, \frac{3}{2}$. | | |

Pages 128-129. Sec. 3-13

- | | | |
|--|---------------------------|---------------------------|
| 1. Intersect at (2, 3). | 3. Coincide. | 5. Parallel. |
| 7. Intersect at $(\frac{51}{8}, \frac{19}{8})$. | 9. Intersect at (-1, 2). | 11. Coincide. |
| 13. Coincide. | 15. Intersect at (-2, 5). | 17. Intersect at (-2, 4). |
| 19. Intersect at (-3, -2). | | |

Chapter 4

Pages 132-133. Sec. 4-2

- | | | |
|----------------------|----------------------------|----------------------------|
| 1. 21.20° . | 3. 173.70° . | 5. 174.42° . |
| 7. 0.37° . | 9. 13.196° . | 11. 127.039° . |
| 13. 0.020° . | 15. -52.271° . | 17. 21.22° . |
| 19. 32.00° . | 21. 142.290° . | 23. 41.968° . |
| 25. $13^\circ 6'$. | 27. $75^\circ 47'$. | 29. $465^\circ 16' 23''$. |
| 31. $16^\circ 30'$. | 33. $-15^\circ 33' 18''$. | 35. $42^\circ 6'$. |
| 37. $42^\circ 8'$. | 39. $-1'$. | |

Pages 136-139. Sec. 4-3

- | | | |
|--|--|--|
| 1. $\frac{\pi}{6}, \frac{\pi}{3}$. | 3. $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$. | 5. $\frac{5\pi}{6}, \frac{11\pi}{6}$. |
| 7. $\frac{5\pi}{18}, \frac{7\pi}{9}$. | 9. $\frac{8\pi}{3}$. | 11. $\frac{-2\pi}{3}$. |
| 13. $\frac{-17\pi}{18}$. | 15. $\frac{11\pi}{20}, \frac{-11\pi}{20}$. | 17. $45.00^\circ, 135.00^\circ$. |
| 19. $1080.00^\circ, 540.00^\circ$. | 21. $120.00^\circ, 240.00^\circ, 300.00^\circ$. | |
| 23. 10.00° . | 25. 577.50° . | 27. -210.00° . |
| 29. $-900.00^\circ, 900.00^\circ$. | 31. 1.08. | 33. 7.59. |
| 35. 0.31. | 37. 0.57. | 39. 0.06. |

41. 1.48. 43. 0.50. 45. 0.01.
 47. 362.7° . 49. 292.2° . 51. -0.6° .
 53. 72.1° . 55. 4.1° . 57. 0.5° .
 59. 5730.0° . 61. 1.548. 63. 1.54811.
 65. 0.8205. 67. 1.265. 69. 585.0° .
 71. 5850° . 73. 2370° . 75. 9.27772° .
 77. $5.60, 321^\circ$. 79. 3.0 in. 81. 2817.9 mi.
 83. 9607.2 mi.

85. 1038.04 mi. per hr., 1522.46 ft. per sec., 15.00° per hr., or 0.2618 rad. per hr.
 87. 8.6 in.

89. $R = \frac{17.6M}{\pi d}$, $A = \frac{35.2M}{d}$, 5.6 rev. per sec., 35.2 rad. per sec.

91. 19.6 sq. in., 16.8 sq. in., 35.3 sq. in., 9.53 sq. in. 93. 3.01 in.

95. 3 rad., one significant figure; 173° , three significant figures.

97. 4644 in. per sec., four significant figures.

Pages 146-147. Sec. 4-7

[For the answers to Exercises 1-55 the trigonometric functions are listed in this order: sine, cosine, tangent, cotangent, secant, and cosecant.]

1. $\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}$.
 3. $\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}, -2, -\frac{1}{2}, -\sqrt{5}, \frac{\sqrt{5}}{2}$.
 5. $\frac{8\sqrt{353}}{353}, \frac{17\sqrt{353}}{353}, \frac{8}{17}, \frac{17}{8}, \frac{\sqrt{353}}{17}, \frac{\sqrt{353}}{8}$.
 7. $\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}, \frac{1}{3}, 3, \frac{\sqrt{10}}{3}, \sqrt{10}$.
 9. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1, -1, \sqrt{2}, -\sqrt{2}$.
 11. 0, 1, 0, undefined, 1, undefined.
 13. -1, 0, undefined, 0, undefined, -1.
 15. $\frac{8}{17}, -\frac{15}{17}, -\frac{8}{15}, -\frac{15}{8}, -\frac{17}{15}, \frac{17}{8}$.
 17. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}, \frac{\sqrt{3}}{3}, -2, -\frac{2\sqrt{3}}{3}$.
 19. $\frac{5\sqrt{194}}{194}, \frac{13\sqrt{194}}{194}, \frac{5}{13}, \frac{13}{5}, \frac{\sqrt{194}}{13}, \frac{\sqrt{194}}{5}$.
 21. $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}$.
 23. $\frac{8}{15}, -\frac{\sqrt{161}}{15}, -\frac{8\sqrt{161}}{161}, -\frac{\sqrt{161}}{8}, -\frac{15\sqrt{161}}{161}, \frac{15}{8}$.
 25. $\frac{7}{113}, -\frac{4\sqrt{795}}{113}, -\frac{7\sqrt{795}}{3180}, -\frac{4\sqrt{795}}{7}, -\frac{113\sqrt{795}}{3180}, \frac{113}{7}$.
 27. $\frac{3}{4}, -\frac{\sqrt{7}}{4}, -\frac{3\sqrt{7}}{7}, -\frac{\sqrt{7}}{3}, -\frac{4\sqrt{7}}{7}, \frac{4}{3}$.

[For the answers to Exercises 29–55 the quadrant in which the angle terminates is given by Roman numerals I, II, III, IV.]

$$29. \text{I: } \frac{5}{13}, \frac{12}{13}, \frac{5}{13}, \frac{12}{13}, \frac{13}{5}; \text{II: } \frac{5}{13}, -\frac{12}{13}, -\frac{5}{13}, -\frac{12}{13}, \frac{13}{5}.$$

$$31. \text{I: } \frac{1}{3}, \frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, 2\sqrt{2}, \frac{3\sqrt{2}}{4}, 3; \text{II: } \frac{1}{3}, -\frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{4}, -2\sqrt{2}, -\frac{3\sqrt{2}}{4}, 3;$$

$$\text{III: } -\frac{1}{3}, -\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, 2\sqrt{2}, -\frac{3\sqrt{2}}{4}, -3; \text{IV: } -\frac{1}{3}, \frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{4}, -2\sqrt{2}, \frac{3\sqrt{2}}{4}, -3.$$

$$33. \text{I: } \frac{\sqrt{119}}{12}, \frac{5}{12}, \frac{\sqrt{119}}{5}, \frac{5\sqrt{119}}{119}, \frac{12}{5}, \frac{12\sqrt{119}}{119}; \text{IV: } -\frac{\sqrt{119}}{12}, \frac{5}{12}, -\frac{\sqrt{119}}{5}, -\frac{5\sqrt{119}}{119}, \frac{12}{5}, -\frac{12\sqrt{119}}{119}.$$

$$35. \text{I: } \frac{7\sqrt{149}}{149}, \frac{10\sqrt{149}}{149}, \frac{7}{10}, \frac{10}{7}, \frac{\sqrt{149}}{10}, \frac{\sqrt{149}}{7}; \text{II: } \frac{7\sqrt{149}}{149}, -\frac{10\sqrt{149}}{149}, -\frac{7}{10}, -\frac{10}{7}, -\frac{\sqrt{149}}{10}, -\frac{\sqrt{149}}{7}; \text{III: } -\frac{7\sqrt{149}}{149}, -\frac{10\sqrt{149}}{149}, \frac{7}{10}, \frac{10}{7}, -\frac{\sqrt{149}}{10}, -\frac{\sqrt{149}}{7}; \text{IV: } -\frac{7\sqrt{149}}{149}, \frac{10\sqrt{149}}{149}, -\frac{7}{10}, -\frac{10}{7}, \frac{\sqrt{149}}{10}, \frac{\sqrt{149}}{7}.$$

$$37. \text{III: } -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, -\frac{2\sqrt{3}}{3}, -2; \text{IV: } -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}, -\sqrt{3}, \frac{2\sqrt{3}}{3}, -2.$$

$$39. \text{I: } \frac{\sqrt{26}}{26}, \frac{5\sqrt{26}}{26}, \frac{1}{5}, 5, \frac{\sqrt{26}}{5}, \sqrt{26}; \text{III: } -\frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}, \frac{1}{5}, 5, -\frac{\sqrt{26}}{5}, -\sqrt{26}.$$

$$41. \text{II: } \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -1, -1, -\sqrt{2}, \sqrt{2}; \text{IV: } -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1, -1, \sqrt{2}, -\sqrt{2}.$$

$$43. -1, 0, \text{undefined}, 0, \text{undefined}, -1.$$

$$45. \text{I: } \frac{2}{3}, \frac{4}{3}, \frac{3}{4}, \frac{4}{3}, \frac{5}{4}, \frac{5}{3}; \text{II: } \frac{2}{3}, -\frac{4}{3}, -\frac{3}{4}, -\frac{4}{3}, -\frac{5}{4}, \frac{5}{3}.$$

$$47. (\text{Assume } m > 0.) \text{ I: } \frac{1}{\sqrt{1+m^2}}, \frac{m}{\sqrt{1+m^2}}, \frac{1}{m}, m, \frac{\sqrt{1+m^2}}{m}, \sqrt{1+m^2}; \text{III: } -\frac{1}{\sqrt{1+m^2}}, -\frac{m}{\sqrt{1+m^2}}, \frac{1}{m}, m, -\frac{\sqrt{1+m^2}}{m}, -\sqrt{1+m^2}.$$

$$49. (\text{Assume } a > 0.) \text{ II: } \frac{\sqrt{4-a^2}}{2}, -\frac{a}{2}, -\frac{\sqrt{4-a^2}}{a}, -\frac{a}{\sqrt{4-a^2}}, -\frac{2}{a}, \frac{2}{\sqrt{4-a^2}}; \text{III: } -\frac{\sqrt{4-a^2}}{2}, -\frac{a}{2}, \frac{\sqrt{4-a^2}}{a}, \frac{a}{\sqrt{4-a^2}}, -\frac{2}{a}, -\frac{2}{\sqrt{4-a^2}}.$$

$$51. \text{II: } \frac{\sqrt{5}}{3}, -\frac{2}{3}, -\frac{\sqrt{5}}{2}, -\frac{2\sqrt{5}}{5}, -\frac{3}{2}, \frac{3\sqrt{5}}{5}; \text{III: } -\frac{\sqrt{5}}{3}, -\frac{2}{3}, \frac{\sqrt{5}}{2}, \frac{2\sqrt{5}}{5}, -\frac{3}{2}, -\frac{3\sqrt{5}}{5}.$$

53. III: $-0.2354, -0.9719, 0.2422, 4.126, -1.029, -4.248$; IV: $-0.2354, 0.9719, -0.2422, -4.126, 1.029, -4.248$.
55. III: $-\frac{1}{5}, -\frac{2\sqrt{6}}{5}, \frac{\sqrt{6}}{12}, 2\sqrt{6}, -\frac{5\sqrt{6}}{12}, -5$; IV: $-\frac{1}{5}, \frac{2\sqrt{6}}{5}, -\frac{\sqrt{6}}{12}, -2\sqrt{6}, \frac{5\sqrt{6}}{12}, -5$.
57. $\frac{1}{3}$. 59. 0. 61. 0.583.
63. $2\sqrt{\frac{11}{13}}$. 65. $-\frac{3}{2}\frac{7}{5}$.
67. θ in III: $\frac{R_1(14\sqrt{13}-4)}{21}$; θ in IV: $-\frac{R_1(14\sqrt{13}+4)}{21}$.
69. $-40\sqrt{6}$. 71. E_1 .

Pages 150-151. Sec. 4-8

[For the answers to Exercises 1-31 the trigonometric functions are listed in this order: sine, cosine, tangent, cotangent, secant, and cosecant.]

1. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}, -\sqrt{3}, -\frac{2\sqrt{3}}{3}, 2$.
3. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{3}, -2, \frac{2\sqrt{3}}{3}$.
5. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}, \frac{\sqrt{3}}{3}, -2, -\frac{2\sqrt{3}}{3}$.
7. $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{3}, 2, -\frac{2\sqrt{3}}{3}$.
9. $-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1, 1, -\sqrt{2}, -\sqrt{2}$. 11. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1, -1, \sqrt{2}, -\sqrt{2}$.
13. 0, -1 , 0, undefined, -1 , undefined. 15. 0, 1, 0, undefined, 1, undefined.
17. 1, 0, undefined, 0, undefined, 1
19. $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}, -\sqrt{3}, \frac{2\sqrt{3}}{3}, -2$.
21. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}, \frac{\sqrt{3}}{3}, -2, -\frac{2\sqrt{3}}{2}$.
23. $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, 1, \sqrt{2}, \sqrt{2}$. 25. $-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1, 1, -\sqrt{2}, -\sqrt{2}$.
27. 0, -1 , 0, undefined, -1 , undefined. 29. 0, -1 , 0, undefined, -1 , undefined.
31. $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{3}, 2, -\frac{2\sqrt{3}}{3}$.
35. 1. 37. 1.
39. 5. 41. $\frac{40\sqrt{3}-9}{6}$.
43. -3 . 45. $Z_1\sqrt{3} + Z_2(\sqrt{3}-1)$.

Pages 153-154. Sec. 4-9

- | | | |
|--------------------------|--------------------------|----------------------------|
| 1. $\sin 25^\circ$. | 3. $\csc 6^\circ$. | 5. $\cot 88^\circ$. |
| 7. $\csc 17.65^\circ$. | 9. $\cos 75.57^\circ$. | 11. $\sec 89.987^\circ$. |
| 13. $\sin (-88^\circ)$. | 15. $\sec (-31^\circ)$. | 17. 40° . |
| 19. 18° . | 21. 25° . | 23. 12° . |
| 25. 0° . | 27. 5. | 29. 1. |
| 31. 7.5° . | 33. 7° . | 35. No θ satisfies. |

Page 156. Sec. 4-10

- | | | |
|--------------|--------------|-------------|
| 1. 0.15126. | 3. 0.01571. | 5. 0.5000. |
| 7. 0.7559. | 9. 0.3211. | 11. 0.5000. |
| 13. 1.524. | 15. 0.8171. | 17. 0.8232. |
| 19. 0.9407. | 21. 3.124. | 23. 0.7140. |
| 25. 0.5025. | 27. 0.4700. | 29. 1.990. |
| 31. 1.1419. | 33. 2.4454. | 35. 0.4252. |
| 37. 12.3729. | 39. 83.5453. | 41. 1362. |
| 43. 2.352. | 45. 57. | |

Page 157. Sec. 4-10

- | | | |
|-------------------------------------|-------------------------------------|------------------------------------|
| 1. 890° . | 3. 20.70° . | 5. 51.70° . |
| 7. 69.00° . | 9. 81.80° . | 11. 45.70° . |
| 13. 63.90° . | 15. 35.63° . | 17. 54.33° . |
| 19. 25.63° . | 21. 21.33° . | 23. 64.30° . |
| 25. 68.13° . | 27. 64.21° . | 29. 62.46° . |
| 31. No value of θ satisfies. | 33. No value of θ satisfies. | 35. No value of θ satisfies |

Pages 160-161. Sec. 4-11

- | | | |
|--------------------------------------|-----------------|-----------------|
| 1. 0.03490. | 3. -0.6494 . | 5. -0.8161 . |
| 7. 0.2456. | 9. -0.5821 . | 11. -0.8724 . |
| 13. -0.9658 . | 15. -0.3420 . | 17. -0.8796 . |
| 19. 0.7290. | 21. 0.2071. | 23. 0.7516. |
| 25. 0.15624. | 27. 0.5189. | 29. 0.5574. |
| 31. 0.9997 | 33. -0.500 . | 35. 16.7. |
| 37. $-0.2686 \cos x + 0.9633 \sin x$ | | 39. -43.41 . |

Page 163. Sec. 4-12

- | | | |
|--|---|-----------------------------------|
| 1. $24.70^\circ, 155.30^\circ$. | 3. $54.30^\circ, 234.30^\circ$. | 5. $136.40^\circ, 223.60^\circ$. |
| 7. $134.70^\circ, 314.70^\circ$. | 9. $45.00^\circ, 315.00^\circ$. | 11. $0.00^\circ, 180.00^\circ$. |
| 13. $46.90^\circ, 133.10^\circ, 226.90^\circ, 313.10^\circ$. | | 15. $26.24^\circ, 333.76^\circ$. |
| 17. $106.23^\circ, 286.23^\circ$. | 19. $25.56^\circ, 154.44^\circ, 205.56^\circ, 334.44^\circ$. | |
| 21. No value of θ satisfies. | 23. No value of θ satisfies. | |
| 25. $29.40^\circ, 150.60^\circ, 389.40^\circ, 510.60^\circ, -330.60^\circ, -209.40^\circ$, etc. | | |
| 27. $30.80^\circ, 210.80^\circ, 390.80^\circ, 570.80^\circ, -329.20^\circ, -149.20^\circ$, etc. | | |
| 29. $124.26^\circ, 304.26^\circ, 484.26^\circ, 664.26^\circ, -55.74^\circ, -235.74^\circ$, etc. | | |
| 33. $14.59^\circ, 194.59^\circ$ | 35. No value of θ satisfies. | 37. $65.07^\circ, 245.07^\circ$. |
| 39. $88.6^\circ, 271.4^\circ$. | | |

Page 165. Sec. 4-13

1. -0.6053 . 3. -0.7604 . 5. 1.016 .
 7. 0.108 . 9. 1.0 .
 11. Impossible to evaluate by method of this section.
 13. Impossible to evaluate by method of this section.
 15. 0 .

Page 167. Sec. 4-14

[For the answers to Exercises 1-13 the trigonometric functions are listed in this order: sine, cosine, tangent, cotangent, secant, cosecant.]

1. $\alpha: \frac{3}{5}, \frac{4}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{4}, \frac{5}{3}; \beta: \frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}$. 3. Same as Exercise 1.
 5. $\alpha: \frac{40}{11}, \frac{9}{11}, \frac{40}{9}, \frac{9}{40}, \frac{41}{10}, \frac{10}{41}; \beta: \frac{9}{41}, \frac{40}{41}, \frac{9}{40}, \frac{10}{9}, \frac{41}{40}, \frac{41}{9}$.
 7. $\alpha: \frac{6\sqrt{61}}{61}, \frac{5\sqrt{61}}{61}, \frac{6}{5}, \frac{5}{6}, \frac{\sqrt{61}}{5}, \frac{\sqrt{61}}{6}; \beta: \frac{5\sqrt{61}}{61}, \frac{6\sqrt{61}}{61}, \frac{5}{6}, \frac{6}{5}, \frac{\sqrt{61}}{6}, \frac{\sqrt{61}}{5}$.
 9. $\alpha: \frac{6}{13}, \frac{\sqrt{133}}{13}, \frac{6\sqrt{133}}{133}, \frac{\sqrt{133}}{6}, \frac{13\sqrt{133}}{133}, \frac{13}{6}; \beta: \frac{\sqrt{133}}{13}, \frac{6}{13}, \frac{\sqrt{133}}{6}, \frac{6\sqrt{133}}{133}, \frac{13\sqrt{133}}{133}, \frac{13}{6}$.
 11. α and $\beta: \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, 1, \sqrt{2}, \sqrt{2}$.
 13. $\alpha: 0, 1, 0$, not defined, 1 , not defined; $\beta: 1, 0$, not defined, 0 , not defined, 1 .
 15. $\alpha = 36.87^\circ, \beta = 53.13^\circ$. 17. $\alpha = \beta = 45.00^\circ$.
 19. $\alpha = 45.58^\circ, \beta = 44.42^\circ$. 21. $\alpha = 48.71^\circ, \beta = 41.29^\circ$.
 23. $\alpha = 51.32^\circ, \beta = 38.68^\circ$. 25. $\alpha = 53.13^\circ, \beta = 36.87^\circ$.
 27. $\alpha = 44.42^\circ, \beta = 45.58^\circ$. 29. $\alpha = 6.33^\circ, \beta = 83.67^\circ$.

Pages 171-173. Sec. 4-15

1. $\alpha = 30.34^\circ$. 3. $\alpha = 58.52^\circ, \beta = 31.48^\circ$.
 5. $a = 2.26, c = 2.27$. 7. $\alpha = 9.28^\circ, \beta = 80.72^\circ$.
 9. No triangle possible. 11. $\alpha = 28.11^\circ, \beta = 61.89^\circ, c = 4.03$.
 13. $\alpha = 52.57^\circ, \beta = 37.43^\circ, a = 8030$. 15. $\alpha = 78.74^\circ, a = 106.14, c = 108.25$.
 17. $\beta = 45.22^\circ, a = 0.9085, c = 1.290$. 19. $\alpha = 59.3^\circ, a = 6.39, b = 3.79$.
 21. 44.4 ft. 23. 55.2 ft.
 25. 39.6 ft. 27. 476.1 ft.
 29. $11,908$ ft. high, $21,630$ ft. from B , $33,850$ ft. from A .
 31. 887.4 ft. 33. The guy wires will hold the antenna tower safely.
 35. 1.3° . 37. 99.2° . 39. 349.0° .
 41. 4.7° . 43. 1.8 min. 45. 5720 ft., 3666 ft.
 47. 1620.7 ft., 439.3 ft. per min., 1 min. 41 sec. after the second observation.
 49. 1103.8 ft., 349.6 ft. above the building. 51. 778.4 ft., 420.5 ft.
 53. The apparent objective of the planes is the aircraft carrier; they will reach it 8.7 sec. after the second observation.

Chapter 5

Page 193. Sec. 5-9

1. 2.
7. $\frac{1}{2}$.
13. None.
19. None.

3. 3.
9. 6.
15. None.
21. None.

5. 5.
11. 3.
17. None.
23. None.

Page 197. Sec. 5-11

1. π , 1.
7. $\frac{2\pi}{3}$, 2.
13. $\frac{\pi}{2}$, none.
19. 2, none.
25. $\frac{\pi}{3}$, 5.

3. $\frac{\pi}{2}$, none.
9. $\frac{\pi}{2}$, 3.
15. 2, 1.
21. 1, 3.
27. $\frac{\pi}{2}$, none

5. $\frac{2\pi}{3}$, 2.
11. 4π , 3.
17. 1, none.
23. 3π , 10
29. $\frac{2\pi}{5}$, none.

Pages 202-203. Sec. 5-13

1. 2, π , π , $\frac{\pi}{2}$.
7. 3, $\frac{\pi}{2}$, $-\frac{\pi}{2}$, $-\frac{\pi}{8}$.
13. None, $\frac{\pi}{2}$, $-\frac{\pi}{2}$, $-\frac{\pi}{4}$.
19. None, π , $-\frac{\pi}{2}$, $-\frac{\pi}{4}$.
25. 10, 2, $\frac{\pi}{4}$, $\frac{1}{4}$.
31. None, $\frac{1}{5}$, $\frac{5\pi}{2}$, $\frac{1}{2}$.

3. 1, $\frac{\pi}{2}$, π , $\frac{\pi}{4}$.
9. 1, π , $-\frac{\pi}{2}$, $-\frac{\pi}{4}$.
15. None, $\frac{\pi}{4}$, $-\frac{\pi}{2}$, $-\frac{\pi}{8}$.
21. None, $\frac{2\pi}{3}$, $-\frac{\pi}{2}$, $-\frac{\pi}{6}$.
27. 2, 2, $-\frac{\pi}{3}$, $-\frac{1}{3}$.
33. None, 1, $-\frac{\pi}{4}$, $-\frac{1}{8}$.

5. 2, π , $-\pi$, $-\frac{\pi}{2}$.
11. None, π , $\frac{\pi}{3}$, $\frac{\pi}{3}$.
17. None, 2π , $\frac{\pi}{4}$, $\frac{\pi}{4}$.
23. 3, $\frac{8\pi}{3}$, $\frac{3\pi}{16}$, $\frac{\pi}{4}$.
29. 2, $\frac{1}{5}$, 2π , $\frac{1}{5}$.
35. 3, 2, $-\frac{\pi}{5}$, $-\frac{1}{5}$.

Pages 209-211. Sec. 5-17

1. $T = 1.4$ sec.; $f = 0.71$ cycles per sec.; $g = 684$ cm. per sec. per sec.
3. 24.8 cm.
7. 1.885×10^6 rad. per sec.
11. 7.96×10^7 cycles per sec.
15. 3 sec.; 2.09 rad. per sec.
17. 6×10^{-3} sec.; 1.05×10^3 rad. per sec.
19. 0.318 sec.
5. 377 rad. per sec.
9. 60 cycles per sec.
13. 0.005 sec.; 1256 rad. per sec.

*Chapter 6***Pages 221-222. Sec. 6-3**

- | | | |
|----------------|-----------------------------|--------------------|
| 1. 1.4, 15°. | 3. 2.2, 183°. | 5. 2.2, 313°. |
| 7. 1.75, 180°. | 9. 3.0, 120°. | 11. 1.0, 0°. |
| 13. 2.2, 183°. | 15. 1.1, 217°. | 17. 4.6, 325°. |
| 19. 1.6, 206°. | 21. 3.4, 105°. | 23. 0.8, 158°. |
| 25. 2.3, 202°. | 27. 102 m.p.h. bearing 11°. | 29. 50 lb. upward. |

31. 14 mi. from the starting point and bearing 15° from it.

33. 260 lb. 35. 137 mi

Pages 223-224. Sec. 6-4

29. 0.17 dyne, 0.47 dyne. 31. 125 lb., 34 lb. 33. 4.0 lb., 6.9 lb.

Pages 229-230. Sec. 6-5

- | | | |
|--------------------|---------------------|--------------------|
| 1. (4, 9). | 3. (-11, -8). | 5. (-6, -6). |
| 7. (4, -4). | 9. (1, -5). | 11. (3, 3). |
| 13. (-2, 2). | 15. (19, 5). | 17. (5, -7). |
| 19. (1, 2). | 21. 3.61, 56.3°. | 23. 6.00, 0.0°. |
| 25. 8.00, 90.0°. | 27. 5.39, 111.8°. | 29. 7.07, 135.0°. |
| 31. 8.25, 194.0°. | 33. 16.1, 209.7°. | 35. 5.83, 329.0°. |
| 37. 8.54, 339.4°. | 39. 5.39, 338.2°. | 41. (8.64, 5.03). |
| 43. (17.9, 47.9). | 45. (11.4, 10.1). | 47. (-115, 110). |
| 49. (-78.9, 34.6). | 51. (-35.4, -11.6). | 53. (3.75, -17.2). |
| 55. (29.9, -37.9). | 57. (50.6, -56.9). | 59. (1570, -243). |
| 61. 69.4, 62.9°. | 63. 1.04, 159.5°. | 65. 1220, 286.4°. |
| 67. 3.69, 199.8°. | 69. 5.06, 358.2°. | 71. 82.4, 128.1°. |
| 73. 7.12, 56.1°. | 75. 0.380, 76.3°. | |

Pages 238-240. Sec. 6-8

- | | | |
|------------|---------------|------------|
| 1. 634 lb. | 3. 115 lb. | 5. 207 lb. |
| 7. 172 lb. | 9. 18,750 lb. | |
11. 104 volts in the direction of the current vector.
13. 102 volts at an angle of 348° with the current vector.

*Chapter 7***Pages 242-243. Sec. 7-1**

- | | |
|---|------------------------|
| 1. $36K^2 - 25$. | 3. $9P^2 + 24P + 16$. |
| 5. $y^2 - 6y + 9$. | |
| 7. $4K^2 + 4N^2 + 4S^2 + 8KN - 8KS - 8NS$. | |

9. $4Cn + 2Ck + 2Ln + Lk$.
 11. $8Q^2 - 2$.
 13. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
 15. $8x^3 + 27$.
 17. $27a^3 - 64$.
 19. $343Q^{3n} + 294Q^{3n-1} + 84Q^{3n-2} + 8Q^{3n-3}$.
 21. $k^3 - 3h^2k + 3hk^2 - k^3$.
 23. $n^8 + 4n^6d + 6n^4d^2 + 4n^2d^3 + d^4$.
 25. $Z^{4n-4} - 4Z^{3n-3}I + 6Z^{2n-2}I^2 - 4Z^{n-1}I^3 + I^4$.
 27. $21p^2 - 40p - 21$.
 29. $4 + 2z + 2y + yz$.
 31. $-6a^4 + 13a^2 + 15$.
 33. $x^2 + y^2 + 2xy + 4x + 4y + 4$.
 35. $L^4 + 2L^3 - 5L^2 - 6L + 9$.
 37. $4X_1^2 + X_2^2 - 4X_1X_2 - 8X_1 + 4X_2 + 4$.
 39. $w^2 + 25x^2 + 9a^2 - 10wx + 6aw - 30ax$.
 41. $16x^4 + 9x^2y^4 + 25x^6 - 24x^3y^2 - 40x^5 + 30x^4y^2$.
 43. $625 - 500ei + 150e^2i^2 - 20e^3i^3 + e^4i^4$.
 45. $48a^{2n} - 75b^{2m}$.
 47. $b^{6n} - 1$.
 49. $32P^6 - 80P^4 + 80P^3 - 40P^2 + 10P - 1$.
 51. $r^6 - 18r^5s + 135r^4s^2 - 540r^3s^3 + 1215r^2s^4 - 1458rs^5 + 729s^6$.
 53. $c^9 - 343$.

Pages 245-246. Sec. 7-2

1. $3a(a + 2)$.
 3. $(x + 2)(3x + 1)$.
 5. $(x + 3y)^2$.
 7. $-x^2(x - a^2)^2$.
 9. $(x - 2y + z)^2$.
 11. $(2R_1 - R_2)^3$.
 13. $(q + p)^4$.
 15. $(I_1 - 2I_2)(I_1^2 + 2I_1I_2 + 4I_2^2)$.
 17. $(3m + 2n)(9m^2 - 6mn + 4n^2)$.
 19. $(x + 2)(5x - 6)$.
 21. $(3M - 4N)(3M + 4N)(9M^2 + 16N^2)$.
 23. $(8x^3 + 5)^2$.
 25. $(E - 1)(7E + 6)$.
 27. $(L - T)(L^2 + LT + T^2)$.
 29. $(X^2 + Y^2)(X^4 - X^2Y^2 + Y^4)$.
 31. $(c - 2d)(3c + d)$.
 33. $(a - b)(4a + 3b)$.
 35. $(3a - b)^3$.
 37. $(3x - 5)(7x + 2)$.
 39. $(x + y + z - 8)(x + y + z + 8)$.
 41. $(C - K - L)^2$.
 43. $2(Z + 2)(3Z - 1)$.
 45. $(p + q)(p^2 - pq + q^2)$.
 47. $(R_1S_1 + R_2S_2)(R_1^2S_1^2 - R_1R_2S_1S_2 + R_2^2S_2^2)$.
 49. $(a + b - d)(a + b + d)$.
 51. $(2X_1 - 3X_2)(5X_1 + 2X_2)$.
 53. $(2 - x - y - z)(4 + 2x + 2y + 2z + 2xy + 2xz + 2yz + x^2 + y^2 + z^2)$.
 55. $(a + 4)(a - 15)$.
 57. $2(M - 6)(4M + 1)$.
 59. $x^6(ax + w)(a^2x^2 - awx + w^2)$.
 61. $(a + b - c)(a^2 + b^2 + c^2 - ab + ac - 2bc)$.
 63. $(X^n - 2)(5X^n + 3)$.
 65. $(a - b^2 + c^2)(a^2 + ab^2 - ac^2 + b^4 - 2b^2c^2 + c^4)$.

67. $2x_0(x_0^2 + 3)$.

71. $2(3x^2 + 1)$.

75. $(a^3 + b)(a^5 - a^3b + b^2)$.

79. $(\mu + 2\beta - X_a + X_b^2)(\mu + 2\beta + X_a - X_b^2)$.

81. $(x - y - z)(x - y + z)$.

85. $(2E - 2E_1 - 3E_2)(2E + 2E_1 + 3E_2)$.

87. $(K - 1)(K + 1)(2K - 5)(2K + 5)$.

91. $(Q^2 + 2Q + 3)(Q^2 - 2Q + 3)$.

95. $(2\mu^2 - \mu - 7)(2\mu^2 + \mu - 7)$.

99. $(t - 2v)(t - 2v - 3x)$.

69. $(x - y - 1)(x^2 - 6)$.

73. $(2m + 3n)^3$.

77. $(1 + q)(1 - q + q^2)(1 - q^3 + q^6)$.

83. $(2x - 3y - z)^2$.

89. $(s^4 - 3)(s^4 - 4)$.

93. $(3a^2 - 5c)(ab - 2c)$.

97. $(S - r)(S^2 + Sr + r^2 - 1)$.

Pages 247-250. Sec. 7-3

1. $\frac{12 + 20p + 45q - 15pq}{80q^2}$.

5. $\frac{3(l^2 + c^2)}{(l - c)(l + c)}$.

9. $\frac{T + 6}{(T - 1)(T - 3)}$.

13. $\frac{3\mu + 2}{(\mu + 3)(\mu^2 - 3\mu + 9)}$.

17. $\frac{3(t^2 + 3)}{(t - 3)(t^2 + 3t + 9)}$.

21. $\frac{2(a^2 + T^2)}{(a + T)(a^2 - aT + T^2)}$.

25. $\frac{2s^2(s^3 - st^2 + t^2)}{s^6 - t^6}$.

29. 1.

33. $\frac{E_1^2 E_2^2}{E_1 - E_2}$.

37. $\frac{(E_g + E_p)^2}{E_g^2 + E_g E_p + E_p^2}$.

41. $\frac{P^2 - 3P + 9}{P^2 - 3}$.

45. $\frac{t(s - t)(s + t)^3}{s}$.

49. $(p + q)(p - q)$.

51. $\pi(4\pi^2 - 2\pi + 1)$.

55. $\frac{4s^3 + 21s^2t - 8st^2 + 7t^3}{(s - t)^2(s + t)^2}$.

3. $\frac{3x^2 + 2xy + 5y^2}{x^3y^3}$.

7. $\frac{3i^2 - 2ir - 6r^2}{(i - r)^3}$.

11. 1.

15. $\frac{c(3l + 7c)}{(l - 2c)(l^2 + 2lc + 4c^2)}$.

19. $\frac{x(2x - y)}{(x - y)(x^2 + xy + y^2)}$.

23. $\frac{x^2}{(x + y)(x^2 - xy + y^2)}$.

27. $\frac{(\pi - 1)(2\pi + 1)}{(\pi + 1)(2\pi - 1)}$.

31. $4P - 1$.

35. $-p$.

39. $\frac{a^2 + b^2}{(a + b)^2}$.

43. $\frac{K}{K + 1}$.

47. $\frac{I_1^2 + I_1 I_2 + I_2^2}{I_1 + I_2}$.

53. $\frac{2R(1 + 5R^2)}{(1 + R)^4}$.

57. $\frac{2(I_1 + I_2)}{(I_1 - I_2)(I_1^2 + I_1 I_2 + I_2^2)}$.

Pages 252-254. Sec. 7-4

1. $\frac{ET - IR}{ET + IR}$.
7. $\frac{5x - 3}{8x - 5}$.
13. 1.
19. $\frac{P^2 + 1}{P}$.
25. 1.
29. $\frac{2pq}{p^2 + q^2}$.
3. $\frac{2p - 3q}{6}$.
9. $\frac{ir + er + ei}{re^2 + ei^2 + ir^2}$.
15. $\frac{n(n^3 + n + 1)}{n^2 + 1}$.
21. $-\frac{P^2 + 1}{2P}$.
27. $\frac{(p + q)^4 + (p + q)^2 - 1}{(p + q)^3}$.
5. $\frac{x + 2y}{y}$.
11. $\frac{pq^2}{(p + q)(p - q)^2}$.
17. $\frac{x^2 + 2x + 4}{x^3 + 3x^2 + 8x + 6}$.
23. $\frac{1}{R_1^2 + R_1R_2 + R_2^2}$.

Pages 257-258. Sec. 7-5

1. 0; 5.
7. $\frac{3}{4}$.
13. 2; 5.
19. $\pm \frac{2}{3}$.
25. -15.
31. 0; $2\frac{1}{2}$.
37. $-\frac{2}{3}$.
3. -3; -17.
9. 2.
15. ± 6 .
21. $10\frac{1}{2}$.
27. $-2\frac{1}{2}$.
33. 1.
5. 16.
11. 4.
17. 0; $-\frac{2}{3}$.
23. -1.1.
29. $-\frac{2}{3}$.
35. $1\frac{1}{2}$.

Pages 260-261. Sec. 7-6

1. $1\frac{1}{5}$.
7. 82.
13. $6\frac{3}{5}$ mi. per hr.
19. 40 and 50 mi. per hr.
3. $\frac{9}{4}$.
9. 31.
15. 15 ft.
5. 44.
11. $\frac{3}{7}$.
17. 4.

Pages 262-264. Sec. 7-7

1. $I = \frac{E}{R}$; $R = \frac{E}{I}$.
5. $t = \frac{N\phi}{10^8 E}$.
9. $l = \frac{1.26N^2A\mu}{10^8 L_{av}}$.
13. $R = \frac{E_x - E_c}{I}$; $E_x = IR + E_c$.
17. $l = \frac{1.26N_1N_2A}{10^8 M}$; $N_1 = \frac{10^8 lM}{1.26N_2A}$; $N_2 = \frac{10^8 lM}{1.26N_1A}$.
19. $t = \frac{1}{E} \left(\frac{T^2}{R} - 1 \right)$; $R = \frac{T^2}{Et + 1}$.
21. $n_0 = \frac{W}{1 - \alpha QW}$; $Q = \frac{n_0 - W}{n_0 \alpha W}$.
3. $d^2 = \frac{K}{R}$.
7. $f = \frac{1}{2\pi C X_c}$; $C = \frac{1}{2\pi f X_c}$.
11. $R_x = \frac{R_2 R_3}{R_1}$.
15. $K = \frac{Cd}{0.0885A}$; $d = \frac{0.0885KA}{C}$.
23. $R_2 = \frac{(234.5 + t_2)}{(234.5 + t_1)} R_1$; $t_2 = \frac{R_2(234.5 + t_1) - 234.5R_1}{R_1}$.

25. $E_b = \frac{E(R + R_m)}{R_m}$; $E = \frac{E_b R_m}{R + R_m}$; $R_m = \frac{ER}{E_b - E}$.
27. $C_1 = \frac{C_2 C_3 C_T}{C_2 C_3 - C_2 C_T - C_3 C_T}$; $C_2 = \frac{C_1 C_3 C_T}{C_1 C_3 - C_1 C_T - C_3 C_T}$;
 $C_3 = \frac{C_1 C_2 C_T}{C_1 C_2 - C_1 C_T - C_2 C_T}$.
29. $R_1 = \frac{R_2 R_3 R_t}{R_2 R_3 - R_2 R_t - R_3 R_t}$; $R_2 = \frac{R_1 R_3 R_t}{R_1 R_3 - R_1 R_t - R_3 R_t}$;
 $R_3 = \frac{R_1 R_2 R_t}{R_1 R_2 - R_1 R_t - R_2 R_t}$.
31. $t_1 = \frac{t_2 i_{av} - C(e_2 - e_1)}{i_{av}}$; $t_2 = \frac{t_1 i_{av} + C(e_2 - e_1)}{i_{av}}$.
33. $r = \frac{n_s E - IR}{n_s I}$; $n_s = \frac{IR}{E - rI}$; $R = \frac{n_s(E - rI)}{I}$.
35. $n_s = \frac{n_p IR}{n_p E - rI}$; $n_p = \frac{rn_s I}{n_s E - IR}$; $r = \frac{n_p(n_s E - IR)}{n_s I}$; $R = \frac{n_s(n_s E - rI)}{n_p I}$.

Chapter 8

Pages 266-267. Sec. 8-1

- | | | |
|-------------------------------------|---|-----------------------------|
| 1. R^8 . | 3. a^8 | 5. $27s^3$. |
| 7. $s_1^6 s_2^3$. | 9. Y^4 . | 11. $\frac{Y^2}{X}$. |
| 13. $-27R^3 s^6$. | 15. $-\frac{1}{64}R^6$. | 17. $\frac{1}{rx^5}$. |
| 19. $\frac{9s_2^2}{64s_1^2}$. | 21. $\frac{1}{4x^{10}}$. | 23. $4m^3$. |
| 25. X^7 . | 27. $a^9 b^5$. | 29. 1. |
| 31. z^3 . | 33. $3XY$. | 35. $\frac{x^4}{25}$. |
| 37. $8r_1^6 r_2^9$. | 39. $25R_1^{10} R_2^2$. | 41. $\frac{12x^8}{y^7}$. |
| 43. $\frac{1}{C^4}$. | 45. $\frac{r_x^2}{r_y}$. | 47. $\frac{R_1^2}{R_2^2}$. |
| 49. $\frac{1}{X^{10} Y^{10}}$. | 51. $\frac{Z^{24}}{256X^{16} Y^{16}}$. | 53. R^2 . |
| 55. $-\frac{s^6}{r_x^6 r_y^{12}}$. | 57. $\frac{s_1 + s_2}{s_1 - s_2}$. | 59. $\tan^2 B$. |
| 61. $\frac{r^8 - 1}{r^4}$. | 63. s . | 65. $\frac{Y^8}{X^6}$. |
| 67. $\tan^4 \theta$. | | |

Pages 269-270. Sec. 8-2

- | | | |
|---------------------------------------|------------------------|-----------------------------|
| 1. 4. | 3. 9. | 5. 20. |
| 7. 2. | 9. 12. | 11. 13. |
| 13. 2. | 15. 5. | 17. $-a\sqrt{-a}$. |
| 19. $4a$. | 21. $2ab^2$. | 23. $\frac{5}{4}\sin^2 B$. |
| 25. $\frac{7}{12}$. | 27. $2\cos\theta$. | 29. 10. |
| 31. -2. | 33. $-\frac{3}{4}$. | 35. 20. |
| 37. $25\cos^2\theta$. | 39. $\frac{7}{8}E^2$. | 41. $-7\tan^2 B$. |
| 43. $-\frac{R^2\sin\theta}{S^2L^4}$. | 45. -3. | 47. 10. |
| 49. 9. | 51. $2\sqrt{11}$. | 53. $\frac{1}{7}$. |
| 55. 5. | 57. 15. | 59. 0.1. |
| 61. 17. | 63. $\frac{1}{4}$. | 65. 20. |
| 67. 4. | 69. 2. | 71. $\frac{1}{2}$. |
| 73. -4. | 75. $-\frac{2}{3}$. | 77. $\frac{3}{5}$. |
| 79. ± 9 . | 81. ± 25 . | 83. ± 2 . |
| 85. ± 12 . | | |

Pages 272-273. Sec. 8-4

- | | | |
|------------------------------------|-----------------------------------|------------------------------|
| 1. 3. | 3. $\frac{1}{18}$. | 5. $\frac{1}{2}$. |
| 7. -1. | | 9. $\frac{1}{2}$. |
| 11. 7. | 13. 1. | 15. 2. |
| 17. -7. | 19. 0.07. | 21. 0.2. |
| 23. $1\frac{1}{4}$. | 25. 4. | 27. 0.01. |
| 29. $\frac{1}{4}$. | 31. $\frac{r^2}{S^5}$. | 33. $\frac{m^2p^3}{n^4}$. |
| 35. $\frac{5}{\sin^2\theta}$. | 37. $\frac{5x^6}{y^5}$. | 39. $\frac{18z^2}{x^2y^3}$. |
| 41. $x^{-3}y^4$. | 43. N^{-3} . | 45. $3L^{-2}$. |
| 47. a^2x^{-4} . | 49. $x^{-2}y^{-2}z^{-2}$. | 51. $PS^{-2}r^{-1}$. |
| 53. $r_1^{-2}r_2^{-2}r_3^{-2}$. | 55. $\sqrt[3]{r}$. | 57. $\sqrt[7]{L^5}$. |
| 59. $5\sqrt{x}$. | 61. $7\sqrt[3]{S}$. | 63. $\sqrt[3]{2a}$. |
| 65. $\sqrt{3Sr}$. | 67. $\sqrt[3]{6\sin\theta}$. | 69. $\sqrt{2S_1 + 3S_2}$. |
| 71. $2\sqrt{\cos\theta}$. | 73. $S^{\frac{3}{2}}$. | 75. $N^{\frac{4}{3}}$. |
| 77. $x^{\frac{8}{5}}$. | 79. $P^{\frac{1}{3}}$. | 81. $x^{\frac{3}{2}}$. |
| 83. $(x+y)^{\frac{1}{3}}$. | 85. $(M^3 + N^3)^{\frac{1}{3}}$. | 87. $(a-b)^{\frac{2}{3}}$. |
| 89. $(\tan\theta)^{\frac{2}{3}}$. | | |

Page 276. Sec. 8-5

- | | | |
|-------------------------|--|--------------------------------------|
| 1. 1. | 3. $y^{\frac{7}{2}}$. | 5. $\frac{1}{4}$. |
| 7. $\frac{2}{L^2}$. | 9. m^{2x} . | 11. $\frac{6}{K^{\frac{1}{2}}}$. |
| 13. $E^{\frac{1}{2}}$. | 15. $2L^{\frac{3}{2}}$. | 17. r^3 . |
| 19. $\frac{2b^2}{L}$. | 21. $\frac{2K^{\frac{1}{2}}}{y^{\frac{1}{2}}}$. | 23. $\frac{(K_1 + K_2)^2}{K_1K_2}$. |

25. $\frac{c^{4x} - 1}{c^{2x}}$.

27. $\frac{5}{6}K$.

29. $\frac{c^{2x} - 1}{c^x}$.

31. 8.

33. 3.

35. 8.

37. 1.

39. 4.

41. $(3L^{\frac{2}{3}} + 2r^{\frac{1}{3}})(3L^{\frac{2}{3}} - 2r^{\frac{1}{3}})$.

43. $(r^4 - L^{\frac{1}{3}})^2$.

45. $(1 + 4a^{-\frac{5}{2}})^2$.

47. $(K^6 + 3r^{-\frac{7}{2}})^2$.

49. $(3 - y^{\frac{1}{3}})(1 - y^{\frac{1}{3}})$.

51. $(5 - x^{\frac{1}{4}})(25 + 5x^{\frac{1}{4}} + x^{\frac{1}{2}})$.

Pages 280-281. Sec. 8-6

1. $10\sqrt{2}$.

3. $2\sqrt[3]{2}$.

5. $2\sqrt[4]{2}$.

7. $-2\sqrt[3]{2}$.

9. $x^2\sqrt[3]{x}$.

11. $-3L^2\sqrt[3]{L}$.

13. $\cos A \sqrt{5 \cos A}$.

15. $3ax \sqrt{\cos \theta}$.

17. $yz \sqrt{6xz}$.

19. $0.2M \sqrt[3]{M^2}$.

21. $3\sqrt{1-x}$.

23. $(x+y)(x-y)\sqrt{(x+y)(x-y)}$.

25. $\sqrt{12}$.

27. $\sqrt{50}$.

29. $\sqrt{a^2bc}$.

31. $\sqrt[3]{125a}$.

33. $\sqrt[3]{8E^7K^4L}$.

35. $\sqrt{24ax^3}$.

37. $\sqrt[3]{56E^3}$.

39. $\sqrt{(1+y)^3}$.

41. $\sqrt{\sin \theta \cos \theta}$.

43. $\frac{\sqrt{6}}{3}$.

45. $\frac{\sqrt{3}}{3}$.

47. $\frac{\sqrt{6}}{4}$.

49. $\frac{\sqrt{15x}}{5}$.

51. $\frac{ab\sqrt{2ab}}{2}$.

53. $\frac{5\sqrt{3}}{3}$.

55. $\frac{\sqrt[3]{21}}{3}$.

57. $\frac{\sqrt[3]{2}}{2}$.

59. $\frac{\sqrt[3]{20xy}}{4y}$.

61. $\frac{\sqrt[3]{3m}}{3}$.

63. $\frac{\sqrt{a \sin A}}{\sin A}$.

65. $\frac{\sqrt{2b(5a+1)}}{4b^3}$.

67. $2\sqrt{2}$.

69. $\sqrt[3]{5}$.

71. $\sqrt[4]{6}$.

73. $x^2\sqrt{3}$.

75. $L\sqrt{6}$.

77. $y\sqrt{3xy}$.

79. $\sqrt{\sin \theta}$.

81. $\frac{\sqrt{3xy}}{2}$.

83. $\sqrt[3]{3 \tan A}$.

85. $\sqrt{3 \cot A}$.

87. $\frac{\sqrt{6xy}}{ab}$.

89. $\frac{a\sqrt{15}}{3}$.

91. $\frac{\sqrt{6x}}{2y}$.

93. $\frac{\sqrt{5E}}{5E^2}$.

95. $\frac{2\sqrt[5]{4L^4}}{L^2}$.

97. $\frac{\sqrt[3]{7ax^2}}{5x}$.

99. $\frac{E^2\sqrt{5}}{2}$.

101. $\frac{5x}{6y}$.

103. $\frac{2E}{3} \sqrt[3]{3ER}$.

105. $\frac{2y\sqrt[5]{3ay}}{3}$.

107. $\frac{(E+1)\sqrt{2E+1}}{(2E+1)}$.

109. $(x+y)^2\sqrt{x^2-xy+y^2}$.

111. $\frac{2a\sqrt[3]{18b^2cyz^2}}{3xyz}$.

Pages 282-283. Sec. 8-7

1. $10\sqrt[3]{2}$.
7. $\frac{21\sqrt[3]{6}}{2}$.
13. $(\sin A + \cos A)\sqrt{\sin A}$.
17. $7mn\sqrt{5m}$.
23. $\frac{7}{8}\sqrt[3]{36}$.
3. $(1 + 2x)\sqrt[3]{5a}$.
9. $(5a + 3b)\sqrt{\cos \theta}$.
15. $5\sqrt{\sin A}$.
21. $10a^2\sqrt[3]{4a^2}$.
25. $(5b - 4y^2)\sqrt{2b^2 - 3y}$.

Page 284. Sec. 8-8

1. $20\sqrt[3]{3}$.
7. 18.
13. 180.
19. $24ab^2c\sqrt[3]{bc^2}$.
25. $\frac{1}{8}\sqrt[12]{2^{11} \cdot 3^9}$.
31. $x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz}$.
35. $a\sqrt{a} + 3a\sqrt{b} + 3b\sqrt{a} + b\sqrt{b}$.
39. $mn\sqrt{m(m+n+1)}$.
45. $2\sqrt{\cot \theta}$.
3. $12\sqrt[3]{10}$.
9. $8xy^2$.
15. $2ab\sqrt[5]{ab}$.
21. $3E\sqrt[4]{3E}$.
27. $ab\sqrt[30]{a^9b^{13}}$.
33. $2 \cos A \sqrt[4]{2 \cos A}$.
37. $abc\sqrt[20]{a^3b^2c}$.
43. 8.
49. $RL^2\sqrt[3]{4a^2}$.
5. a^2 .
11. $2x^3\sqrt{3x}$.
17. $9Rx\sqrt{x}$.
23. $x\sqrt[12]{9x^7}$.
29. $2m\sqrt[4]{54mn}$.

Pages 286-287. Sec. 8-9

1. 2.
7. $2E$.
13. $3\sqrt{\tan \theta}$.
19. $\frac{1}{a}\sqrt[12]{2a^5x^3y^2}$.
23. $\frac{\sqrt[6]{(2R-3L)(2R+3L)^4}}{2R+3L}$.
27. $\frac{\sqrt{E}-\sqrt{R}}{E-R}$.
33. $\frac{x^2+2y^2+2y\sqrt{x^2+y^2}}{x^2}$.
37. $\frac{\cos A - 2 \sin A - 2\sqrt{\sin A(\sin A - \cos A)}}{\cos A}$.
3. 3.
9. $5\sqrt[3]{2} - 3\sqrt[3]{5}$.
15. $\sqrt[3]{6} + 2$.
21. $\frac{11 + 4\sqrt[3]{6}}{5}$.
25. $\frac{E - \sqrt{2E} - 4}{E - 8}$.
31. $\sqrt[3]{15}$.
35. $3\sqrt{2} + \sqrt[3]{13}$.
39. $\frac{4x\sqrt{1-4x^2} + 1}{8x^2 - 1}$.
5. $2\sqrt[3]{\sin \theta}$.
11. $\sqrt[6]{xy}$.
17. $30x\sqrt[12]{24x^2}$.

Page 290. Sec. 8-10

1. 51.
7. 3.
13. 5.
3. $12\frac{1}{4}$.
9. $\frac{2}{3}\frac{3}{8}$.
15. 49.
5. $\frac{1}{3}$.
11. 27.
17. No solution.

19. No solution.

21. -5.

23. 11.

25. 1.

27. 4.

29. $\frac{1}{2}$.

31. No solution.

33. 83.

35. $\frac{81}{49}$.37. $\frac{2}{3}$ and -5.

Pages 291-292. Sec. 8-11

1. 2.55.

3. 0.577.

5. 0.378.

7. 0.855.

9. 0.585.

11. 2.90.

13. -0.631.

15. 0.630.

17. 0.240.

19. 0.139.

21. 0.764.

23. 1.08.

Pages 292-294. Sec. 8-12

1. $r = \sqrt{\frac{A}{\pi}}$.

3. $r = \frac{1}{2} \sqrt{\frac{S}{\pi}}$.

5. $r = \sqrt{\frac{V}{\pi h}}$.

7. $W = I^2 R$, $R = \frac{W}{I^2}$.

9. $d = \sqrt{\frac{K \cdot l}{R}}$.

11. $D = \sqrt{\frac{M_1 M_2}{F}}$.

13. $r = \sqrt[3]{\frac{3V}{4\pi}}$.

15. $v = \sqrt{\frac{Fr}{m}}$.

17. $R = \sqrt{Z^2 - X^2}$; $X = \sqrt{Z^2 - R^2}$.

19. $l = 2d \sqrt{\frac{2}{W}}$.

21. $L = \frac{\lambda^2}{1884^2 C}$; $C = \frac{\lambda^2}{1884^2 L}$.

23. $a = \frac{bp}{\sqrt{b^2 - p^2}}$; $b = \frac{ap}{\sqrt{a^2 - p^2}}$.

25. $E_s = \frac{2\sqrt{r_p P_{\max}}}{\mu}$.

27. $r = \frac{1}{N} \sqrt{L(9r + 10l)}$.

29. $N = 10^4 \sqrt{\frac{lL_{av}}{1.26A\mu}}$.

31. $n = \frac{10^4}{2\pi r} \sqrt{\frac{10lL}{\mu}}$; $r = \frac{10^4}{2\pi n} \sqrt{\frac{10lL}{\mu}}$.

33. $I = \sqrt{\frac{H}{0.24RT}}$.

35. $R = \frac{XZ}{\sqrt{X^2 - Z^2}}$; $X = \frac{RZ}{\sqrt{R^2 - Z^2}}$.

37. $F = \left(\frac{10^3 d_1 d_2 r}{d_1 + d_2} \right)^2$.

39. $T = \sqrt{R(1 + El)}$.

41. $R_0 = \frac{1}{2I_L} \sqrt{2(E_0^2 - 2w^2 I_L^2 I_0^2)}$; $L_0 = \frac{1}{2wI_L} \sqrt{2(E_0^2 - 2I_L^2 R_0^2)}$.

43. $R = \frac{1}{I} \sqrt{E^2 - I^2 \left(2\pi FL - \frac{1}{2\pi fC} \right)^2}$.

45. $E_p + E_g = \frac{(a+b)[a+b(\mu+1)]^3 \cdot 10^{12}}{r_p^2 A_1^2}$.

Chapter 9

Page 305. Sec. 9-1

- | | | |
|-----------------------|-----------------------|----------------------------|
| 1. 1.4×10 . | 3. 3.8×10^3 | 5. 2.8×10^2 . |
| 7. 7.4×10 . | 9. 3.6. | 11. 1.2×10^{-1} . |
| 13. 9.8×10 . | 15. 4.6×10 . | |

Pages 307-308. Sec. 9-2

- | | | |
|------------------------------------|---|-----------------------------------|
| 1. $\log_2 4 = 2$. | 3. $\log_3 27 = 3$. | 5. $\log_{10} 0.01 = -2$. |
| 7. $\log_8 64 = 2$. | 9. $\log_{10} 0.001 = -3$. | 11. $\log_{11} 121 = 2$. |
| 13. $\log_{15} 225 = 2$. | 15. $\log_{10} (\frac{1}{10}) = -1$. | 17. $\log_2 (\frac{1}{8}) = -3$. |
| 19. $\log_3 (\frac{1}{27}) = -3$. | 21. $2^5 = 32$. | 23. $5^3 = 125$. |
| 25. $10^3 = 1000$. | 27. $2^{-2} = \frac{1}{4}$. | 29. $7^2 = 49$. |
| 31. $15^{-1} = \frac{1}{15}$. | 33. $10^{-3} = 0.001$. | 35. $1.2^{-2} = \frac{1}{1.44}$. |
| 37. $10^{-3} = 0.001$. | 39. $10^{\frac{1}{3}} = \sqrt[3]{10}$. | 41. -4 . |
| 43. 2. | 45. -3 . | 47. 3. |
| 49. 2. | 51. 0. | 53. 3. |
| 55. 2. | 57. $\frac{2}{3}$. | 59. 4. |
| 61. 8. | 63. $\frac{1}{5}$. | 65. 64. |
| 67. 125. | 69. 2.25. | 71. 1. |
| 73. 32. | 75. $\sqrt{2}$. | 77. 100. |
| 79. 1. | | |

Page 308. Sec. 9-3

- | | | |
|-------------------|-------------|---------------------|
| 1. -9 . | 3. -3 . | 5. $\frac{6}{5}$. |
| 7. -7 . | 9. 10. | 11. $\sin \theta$. |
| 13. 20. | 15. 1. | 17. $4x + 2$. |
| 19. 0. | 21. 10. | 23. 46. |
| 25. 49. | 27. x . | 29. 56. |
| 31. $a^2 + b^2$. | 33. A^2 . | 35. 10. |

Pages 312-313. Sec. 9-5

- | | | |
|--|--|---|
| 1. $\log_B P + \log_B Q + \log_B R$. | 3. $\log_B C - \log_B D - \log_B E$. | |
| 5. $3 \log_B C + 5 \log_B D$. | 7. $-n \log_{10} C$. | |
| 9. $-\frac{1}{q} \log_2 C$. | 11. $n \log_4 Q + \frac{1}{m} \log_4 R - 5 \log_4 S$. | |
| 13. $\frac{1}{3} (\log_B C + \log_B D - \log_B E)$. | 15. $\frac{1}{3} (2 \log_2 A + 5 \log_2 S - \log_2 T)$. | |
| 17. $\log_2 (\frac{3^0}{7})$. | 19. $\log_e (x^3 y^2)$. | 21. $\log_e xyz$. |
| 23. $\log_a \frac{\pi \sqrt[3]{3}}{x}$. | 25. $\log_{10} \sqrt[3]{15}$. | 27. $\log_B \sqrt[12]{\frac{x^6 y^4}{z^3}}$. |
| 29. $\log_B \frac{a^2 \sqrt[3]{b}}{C^5}$. | 31. 1.1461. | 33. 0.2386. |

35. -0.8451 .

41. 1.3316.

47. $\frac{3}{4}$.

53. 2.

59. $\frac{3}{2}$.

37. 0.1863.

43. 0.7417.

49. 6.4.

55. 6.

39. 1.1611.

45. -1.6565 .

51. $\frac{3}{4}$.

57. 0.

Pages 316-317. Sec. 9-9

1. 1.

7. 2.

13. 1.

19. -1 or $9 - 10$.

25. -3 or $7 - 10$.

31. 5689.

37. 568,900.

43. 5,689,000,000,000,000,000.

3. 1.

9. 4.

15. 1.

21. -3 or $7 - 10$.

27. -18 or $2 - 20$.

33. 568.9.

39. 0.005,689.

5. -2 or $8 - 10$.

11. -2 or $8 - 10$.

17. 2.

23. 5.

29. 15.

35. 0.5689.

41. 56,890,000,000.

Pages 318-319. Sec. 9-10

[In the answers to Exercises 1-19, the characteristic is given first and the mantissa second.]

1. 2, 0.9325.

7. $1 - 10$ or -9 , 0.5692.

13. 4, 0.3907.

19. $7 - 10$ or -3 , 0.6843.

25. 2.1271.

31. 4.7604.

37. 3.7604.

3. 0, 0.315796.

9. 5, 0.6392.

15. 5, 0.3268.

21. 0.9504.

27. 9.6618 $- 10$.

33. 7 7604 $- 10$.

39. 5.7604.

5. 5, 0.6194.

11. $6 - 10$ or -4 , 0.7706.

17. 3, 0.2159.

23. 3.9504.

29. 9.9504 $- 10$.

35. 8.9504.

Page 320. Sec. 9-11

1. 0.5105

7. 7.8274 $- 10$.

13. 5.6263.

19. 45,000.

25. 4.00.

3. 1.8932.

9. 9.9365 $- 10$.

15. 8.8082.

21. 0.0670.

27. 3.45.

5. 2.6946.

11. 1 8420.

17. 39.1.

23. 0.000877.

29. 790.

Page 323. Sec. 9-12

1. 0.8787.

7. 9.3751 $- 10$.

13. 2.1749.

19. 0.4972.

25. 2.7260.

31. 3.9595.

37. 716,300.

43. 0.0002014.

49. 10.35.

55. 0.0003985.

3. 2.0330.

9. 8.9710 $- 10$.

15. 2.5904.

21. 12.5529.

27. 5.8343 $- 10$.

33. 1.3318.

39. 3.002.

45. 0.0005005.

51. 19,960.

57. 0.6306.

5. 6.1723.

11. 9.0237 $- 10$.

17. 4.8063.

23. 7.9003 $- 10$.

29. 2.9282 $- 10$.

35. 6.554.

41. 0.04061.

47. 3.965.

53. 0.06957.

Pages 327-328. Sec. 9-13

- | | | |
|---------------|---------------|--------------|
| 1. 6843. | 3. 1.541. | 5. 118.5. |
| 7. 84.20. | 9. 21,870. | 11. 7.992. |
| 13. 0.002807. | 15. 0.006041. | 17. 0.1133. |
| 19. 0.001120. | 21. 0.2841. | 23. 0.03727. |
| 25. 1.383. | 27. 0.3245. | 29. 0.08818. |
| 31. 0.05309. | 33. 11.98. | 35. 3069. |
| 37. 0.003808. | 39. 24.01. | 41. 25.95. |
| 43. 0.09468. | 45. 0.9578. | 47. 3.471. |
| 49. 26.93. | 51. 28.39. | 53. 3.471. |
| 55. 1.279. | 57. 19.45. | 59. 6.306. |
| 61. 7.702. | 63. 0.9154. | 65. 17.29. |
| 67. 0.6802. | 69. 0.4725. | 71. 83.26. |
| 73. 0.05760. | 75. 2.985. | 77. 0.3536. |
| 79. 0.04088. | | |

Pages 329-330. Sec. 9-14

- | | | |
|---------------|-------------|------------------|
| 1. 524,800. | 3. 0.552. | 5. 0.00387. |
| 7. 47.52. | 9. 0.1079. | 11. 1.493. |
| 13. 5800. | 15. 0.0406. | 17. 3.99. |
| 19. 5.04. | 21. 0.816. | 23. 3100 meters. |
| 25. 26.5 sec. | | |

Pages 333-334. Sec. 9-15

- | | | |
|---------------------------|---------------------------|-----------------|
| 1. 0.02314. | 3. 0.007438. | 5. 2,796,000. |
| 7. 1.458. | 9. 1.484. | 11. 3.80. |
| 13. 0.026. | 15. 2.069. | 17. 0.06578. |
| 19. 1.12. | 21. -13,160. | 23. -8.824. |
| 25. -184,500,000. | 27. 2286. | 29. -1.569. |
| 31. -2.666. | 33. -5.96×10^8 . | 35. 6.610. |
| 37. 9.998×10^4 . | 39. 5.584×10^5 . | 41. 81.8 |
| 43. 0.560. | 45. 0.00577. | 47. 9.11. |
| 49. 0.005860. | 51. 0.8888. | 53. 74.95. |
| 55. 14.25. | 57. 45.6. | 59. -47.1. |
| 61. 1720. | 63. -463.7. | 65. Impossible. |

Pages 336-337. Sec. 9-16

- | | |
|--|---|
| 1. $\beta = 62.39^\circ$, $a = 76.56$, $b = 146.4$. | 3. $\beta = 74.24^\circ$, $b = 16,590$, $c = 17,240$ |
| 5. $\beta = 16.18^\circ$, $a = 0.1791$, $c = 0.1865$. | 7. $\alpha = 36.07^\circ$, $\beta = 53.93^\circ$, $c = 7.105$. |
| 9. $\alpha = 20.4^\circ$, $\beta = 69.6^\circ$, $a = 2.50$. | 11. $\beta = 57.9^\circ$, $a = 86.1$, $b = 137$. |
| 13. $\beta = 42^\circ$, $b = 16$, $c = 24$. | 15. $x_A = 68.30$, $y_A = 54.55$. |
| 17. $x_A = -0.3302$, $y_A = 0.1360$. | 19. $x_A = 384.4$, $y_A = -535.5$. |
| 21. $x_A = 4.51$, $y_A = 3.37$. | 23. $x_A = -0.0612$, $y_A = 0.00086$. |
| 25. $x_A = 1.7$, $y_A = -2.7$. | 27. $\alpha = 326.54^\circ$, $A = 5.833$. |
| 29. $\alpha = 70.57^\circ$, $A = 325.3$. | 31. $\alpha = 335.1$, $A = 7.62$. |
| 33. $\alpha = 52.0^\circ$, $A = 6.24$. | 35. $\gamma = 66.66^\circ$, $C = 92.10$. |
| 37. $\gamma = 12.23^\circ$, $C = 1345$. | 39. $\gamma = 1.2^\circ$, $C = 716$. |
| 41. $\delta = 329.7^\circ$, $D = 109$. | 43. $\delta = 149.82^\circ$, $D = 0.04606$. |
| 45. 196 2 ft. | 47. 1 min. 32 sec. |

Page 339. Sec. 9-17

- | | | |
|------------|------------|--------------|
| 1. 4.163. | 3. 1.860. | 5. -2.448. |
| 7. -7.792. | 9. 4.561. | 11. 9.98. |
| 13. 1.51. | 15. 0.100. | 17. 0.00361. |

Page 340. Sec. 9-18

- | | | |
|------------|-------------|-----------|
| 1. 1.3010. | 3. -0.2216. | 5. 1.369. |
| 7. 11.8. | 9. -43.8. | 11. 10.4. |
| 13. 5. | 15. 3.91. | |

Pages 345-346. Sec. 9-20

- | | | |
|--------------|--|-------------|
| 1. 1.761 db. | 3. 23.98 db. | 5. 1.76 db. |
| 7. 7.78 db. | 9. 5.12×10^3 maxwells per sq. cm. | |

Chapter 10

Pages 350-351. Sec. 10-1

$$13. \cos \theta = \sqrt{1 - \sin^2 \theta}, \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}, \cot \theta = \frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta},$$

$$\sec \theta = \frac{1}{\sqrt{1 - \sin^2 \theta}}, \csc \theta = \frac{1}{\sin \theta}.$$

$$15. \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}, \cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}}, \cot \theta = \frac{1}{\tan \theta},$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta}, \csc \theta = \frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}.$$

$$17. \sin \theta = \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}, \cos \theta = \frac{1}{\sec \theta}, \tan \theta = \sqrt{\sec^2 \theta - 1},$$

$$\cot \theta = \frac{1}{\sqrt{\sec^2 \theta - 1}}, \csc \theta = \frac{\sec \theta}{\sqrt{\sec^2 \theta - 1}}.$$

- | | | |
|-------------------------------------|--|--------------------|
| 19. 1. | 21. $\frac{1}{2}(E_1 + E_2) \cos \theta$. | 23. $I_m \cos A$. |
| 25. $2 \sec \theta - \tan \theta$. | 27. $\cot^2 \beta$. | 29. $\sin A$. |
| 31. $-\tan x$. | 33. $\tan \theta \sec \theta$. | 35. -1. |
| 37. $2 \sec \omega t - 1$. | 39. $\sec \theta$. | |

Pages 354-357. Sec. 10-2

$$13. \cot A = \csc 2A + \cot 2A. \quad 15. \csc A = \frac{\pm \sqrt{2(1 + \cos 2A)}}{\sin 2A}.$$

17. Values of A and B for which $\cot A + \cot B = 0$.

19. Either A or $B = 0^\circ, 90^\circ, 180^\circ, 270^\circ$, etc., and values of A and B such that $A \pm B = 90^\circ, 270^\circ$, etc.

21. $A = 45^\circ, 90^\circ, 135^\circ, 225^\circ, 270^\circ, 315^\circ$, etc.

23. $A = 0^\circ, 90^\circ, 180^\circ, 270^\circ$, etc.

25. $A = 90^\circ, 270^\circ$, etc.

27. $A = 90^\circ, 270^\circ$, etc.

29. $A = 90^\circ, 270^\circ$, etc.

31. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$.

33. $\frac{(\sqrt{6} + \sqrt{2})}{4}, \frac{(\sqrt{6} - \sqrt{2})}{4}, 2 + \sqrt{3}$.

35. 1, 0, not defined.

37. $\frac{(\sqrt{6} - \sqrt{2})}{4}, \frac{(\sqrt{6} + \sqrt{2})}{4}, 2 - \sqrt{3}$.

39. $\frac{(\sqrt{2} - \sqrt{6})}{4}, \frac{(\sqrt{6} + \sqrt{2})}{4}, \sqrt{3} - 2$.

41. $-\sin \beta$.

43. $\frac{1}{2}(\cos A - \sqrt{3} \sin A)$.

45. $\frac{1}{2}(\cos \alpha - \sqrt{3} \sin \alpha)$.

47. $\cos A$.

49. $-\tan A$.

51. $\frac{(2\sqrt{2} + \sqrt{3})}{6}, \frac{(1 - 2\sqrt{6})}{6}, \frac{(2\sqrt{2} - \sqrt{3})}{6}, \frac{(1 + 2\sqrt{6})}{6}$.

53. $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{10}, \frac{7\sqrt{2}}{10}$.

55. $-\frac{5}{8}, \frac{3}{8}, -\frac{1}{8}, \frac{9}{8}$.

57. $\frac{(6 + 4\sqrt{21})}{25}, \frac{(3\sqrt{21} - 8)}{25}, \frac{(6 - 4\sqrt{21})}{25}, \frac{(3\sqrt{21} + 8)}{25}$.

59. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}$.

63. $\sin 200^\circ = 2 \sin 100^\circ \cos 100^\circ, \cos 200^\circ = \cos^2 100^\circ - \sin^2 100^\circ,$

$$\sin 200^\circ = -\sqrt{\frac{1 - \cos 400^\circ}{2}}, \cos 200^\circ = -\sqrt{\frac{1 + \cos 400^\circ}{2}}.$$

65. $\sin 6x = 2 \sin 3x \cos 3x, \cos 6x = \cos^2 3x - \sin^2 3x, \sin 6x = \pm \sqrt{\frac{1 - \cos 12x}{2}},$

$$\cos 6x = \pm \sqrt{\frac{1 + \cos 12x}{2}}.$$

67. $\sin \pi = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2}, \cos \pi = \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2}, \sin \pi = \sqrt{\frac{1 - \cos 2\pi}{2}},$

$$\cos \pi = -\sqrt{\frac{1 + \cos 2\pi}{2}}.$$

69. $\sin 18\theta = 2 \sin 9\theta \cos 9\theta, \cos 18\theta = \cos^2 9\theta - \sin^2 9\theta,$

$$\sin 18\theta = \pm \sqrt{\frac{1 - \cos 36\theta}{2}}, \cos 18\theta = \pm \sqrt{\frac{1 + \cos 36\theta}{2}}.$$

71. 1, 0. 73. $\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$.
75. -1, 0. 77. $\frac{\sqrt{2-\sqrt{3}}}{2}$, $\frac{\sqrt{2+\sqrt{3}}}{2}$.
79. $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$. 81. 1, 0.
83. $-\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\frac{\sqrt{2+\sqrt{3}}}{2}$, $\frac{\sqrt{2-\sqrt{3}}}{2}$.
85. $-\frac{20}{29}$, $\frac{21}{29}$, $\sqrt{\frac{29+5\sqrt{29}}{58}}$, $\sqrt{\frac{29-5\sqrt{29}}{58}}$.
87. $\frac{4\sqrt{2}}{9}$, $-\frac{7}{9}$, $\frac{\sqrt{6}}{3}$, $-\frac{\sqrt{3}}{3}$.
89. 0, 1, 1, 0.

Pages 359-361. Sec. 10-3

9. $\frac{1}{2}(\cos 30^\circ - \cos 110^\circ)$. 11. $\frac{1}{2}(\sin 3x + \sin 9x)$.
13. $\frac{1}{2}\left(\cos \frac{\pi}{14} - \cos \frac{3\pi}{14}\right)$. 15. $\frac{1}{2}(\cos 2\alpha t + 1)$.
17. $\frac{1}{2}[\cos 2\alpha + \cos 2(\theta + \phi)]$. 19. $\frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right)$.
21. $-2\cos \frac{19\pi}{48} \sin \frac{13\pi}{48}$. 23. $-2\cos(\alpha + 45^\circ) \sin(3\alpha - 45^\circ)$.
25. $2\sin(45^\circ - 5\omega) \cos(25\omega - 45^\circ)$. 27. $2\cos \omega t \sin(2\omega t - \alpha)$.
29. $-2\sin\left(7\omega t - \frac{7\pi}{12}\right) \sin\left(\frac{7\pi}{12} - 3\omega t\right)$. 31. $-2\cos\left(y + \frac{9\pi}{4}\right) \sin\left(x - \frac{5\pi}{4}\right)$.

Pages 365-366. Sec. 10-4

1. $3 \sin x$. 3. $\sqrt{13} \sin(\omega t + 56.31^\circ)$.
5. $\sqrt{10 + 3\sqrt{3}} \sin(x - 52.63^\circ)$. 7. $3.3 \sin(x + 262^\circ)$.
9. $155 \sin(x + 268.9^\circ)$. 11. 0.
13. $18.4 \sin(x + 232.1^\circ)$.
15. Since the two sine functions do not have the same period, the theorem of this section does not apply.
17. $6.66 \sin(\omega t + 270.7^\circ)$.
19. $4.2 \sin(\omega t + 335^\circ)$.

Pages 369-370. Sec. 10-5

1. 27.00° , 153.00° . 3. 241.90° , 298.10° .
5. 160.60° , 199.40° . 7. 224.73° , 315.27° .
9. 56.14° , 123.86° . 11. 9.52° , 80.48° , 189.52° , 260.48° .
13. 143.91° . 15. 226.96° .
17. 36.90° , 53.10° , 216.90° , 233.10° . 19. 133.8° .
21. 30° , 150° , 210° , 330° . 23. 0° , 180° .

25. $30^\circ, 135^\circ, 225^\circ, 330^\circ$.
 29. $60^\circ, 180^\circ, 300^\circ$.
 33. $60^\circ, 120^\circ, 240^\circ, 300^\circ$.
 37. $0^\circ, 180^\circ$.
 41. $7.24^\circ, 45.00^\circ, 82.76^\circ, 135.00^\circ, 187.24^\circ, 225.00^\circ, 315.00^\circ$.
 43. All values of α .
 45. $29.61^\circ, 210.39^\circ, 269.61^\circ$.
 47. $0^\circ, 20^\circ, 100^\circ, 120^\circ, 140^\circ, 220^\circ, 240^\circ, 260^\circ, 340^\circ$.
 49. $0^\circ, 45^\circ, 60^\circ, 120^\circ, 135^\circ, 180^\circ, 225^\circ, 240^\circ, 300^\circ, 315^\circ$.
 51. $30.00^\circ, 54.58^\circ, 150.00^\circ, 174.58^\circ, 270.00^\circ, 294.58^\circ$.
 53. $0^\circ, 90^\circ, 180^\circ, 270^\circ$.
 55. $0^\circ, 105^\circ, 165^\circ, 180^\circ, 285^\circ, 345^\circ$.
 57. No value of x satisfies the equation.

27. $53.13^\circ, 56.31^\circ, 233.13^\circ, 236.31^\circ$.
 31. No values of α satisfy the equation.
 35. $61.04^\circ, 118.96^\circ, 199.47^\circ, 340.53^\circ$.
 39. $63.43^\circ, 66.42^\circ, 243.43^\circ, 293.58^\circ$.

Chapter 11

Page 376. Sec. 11-4

1. $b = 30, \gamma = 77^\circ, c = 30$.
 5. $\beta = 89.4^\circ, c = 0.934, b = 0.958$.
 9. $\alpha = 101.68^\circ, b = 940.4, c = 1314$.
 13. $c = 170$.
 17. $b = 3.40$.
 3. $\alpha = 128.3^\circ, c = 0.749, a = 2.89$.
 7. $\gamma = 165.4^\circ, a = 9.30, b = 12.6$.
 11. $\beta = 144^\circ$.
 15. $a = 17.4$.
 19. $c = 0.3073$.

Pages 378-379. Sec. 11-5

1. $a = 4.5, \beta = 53.6^\circ, \gamma = 105.0^\circ$.
 5. $b = 100, \alpha = 45^\circ, \gamma = 23^\circ$.
 9. $b = 0.5759, \gamma = 9.73^\circ, \alpha = 157.06^\circ$.
 13. $c = 238$.
 17. $\beta = 77.1^\circ$.
 3. $a = 15, \beta = 43.8^\circ, \gamma = 61.9^\circ$.
 7. $c = 0.813, \beta = 73.7^\circ, \alpha = 33.7^\circ$.
 11. $a = 22$.
 15. $c = 1280$.
 19. $\gamma = 155^\circ$.

Page 383. Sec. 11-6

1. $\beta = 32.3, \alpha = 115.4, c = 18.1$.
 5. $a_1 = 213, \beta_1 = 82.0^\circ, \alpha_1 = 19.2^\circ$,
 $a_2 = 36.1, \beta_2 = 98.0^\circ, \alpha_2 = 3.2^\circ$.
 9. No solution.
 13. No solution.
 17. $a = 0.1904, 0.04297$.
 3. $\alpha = 35^\circ, \gamma = 90^\circ, c = 0.160$.
 7. $\beta_1 = 43.9^\circ, \gamma_1 = 73.4^\circ, b_1 = 29.8$,
 $\beta_2 = 10.7^\circ, \gamma_2 = 106.6^\circ, b_2 = 7.98$.
 11. $\beta = 48.6^\circ$.
 15. $\alpha = 49.7^\circ$.
 19. No solution.

Page 385. Sec. 11-7

1. $\alpha = 16.8^\circ, \beta = 102.8^\circ, \gamma = 60.4^\circ$.
 5. $\alpha = 47.70^\circ, \beta = 96.82^\circ, \gamma = 35.53^\circ$.
 9. No solution.
 13. $\gamma = 53.8^\circ$.
 17. No solution.
 3. $\alpha = 27.9^\circ, \beta = 105.6^\circ, \gamma = 46.5^\circ$.
 7. $\alpha = 41.8^\circ, \beta = 44.7^\circ, \gamma = 93.5^\circ$.
 11. $\alpha = 32^\circ$.
 15. $\alpha = 81.4^\circ$.

Pages 388-392. Sec. 11-6

1. $C: C = 25; \gamma = 56.5^\circ$.
5. $C: C = 10.37; \gamma = 216.5^\circ$.
9. $C: C = 193.5; \gamma = 237.7^\circ$.
13. $C: C = 253; \gamma = 179.5^\circ$.
3. $C: C = 69; \gamma = 209^\circ$.
7. $C: C = 3060; \gamma = 120^\circ$.
11. $C: C = 5.51; \gamma = 242.8^\circ$.
15. $C: C = 406.1; \gamma = 175.8^\circ$.
17. $B: B = 574, \beta = 10^\circ; C: C = 393, \gamma = 112^\circ$.
 $B: B = 130, \beta = 330^\circ; C: C = 611, \gamma = 60^\circ$.
 $B: B = -260, \beta = 90^\circ; C: C = -836, \gamma = 240^\circ$.
19. $B: B = 544, \beta = 10^\circ; C: C = 1430, \gamma = 112^\circ$.
 $B: B = 1230, \beta = 60^\circ; C: C = -710, \gamma = 330^\circ$.
 $B: B = 1420, \beta = 90^\circ; C: C = 0, \gamma = 240^\circ$.
21. $B: B = -45.9, \beta = 112^\circ; C: C = 68.0, \gamma = 10^\circ$.
 $B: B = -23.9, \beta = 60^\circ; C: C = 41.40, \gamma = 330^\circ$.
 $B: B = -82.8, \beta = 90^\circ; C: C = -47.8, \gamma = 240^\circ$.
23. $B: B = 0.570, \beta = 10^\circ; C: C = 0.570, \gamma = 112^\circ$.
 $B: B = 0.718, \beta = 60^\circ; C: C = -0.0125, \gamma = 330^\circ$.
 $B: B = -0.696, \beta = 240^\circ; C: C = 0.0251, \gamma = 90^\circ$.
25. $B: B = -0.670, \beta = 112^\circ; C: C = 0.878, \gamma = 10^\circ$.
 $B: B = 1.22, \beta = 330^\circ; C: C = 0.135, \gamma = 60^\circ$.
 $B: B = -2.45, \beta = 90^\circ; C: C = -2.25, \gamma = 240^\circ$.
27. $B: B = 31.20, \beta = 112^\circ; C: C = -0.2789, \gamma = 10^\circ$.
 $B: B = 19.03, \beta = 60^\circ; C: C = -24.80, \gamma = 330^\circ$.
 $B: B = 49.60, \beta = 90^\circ; C: C = 23.93, \gamma = 240^\circ$.
29. $B: B = 1013, \beta = 10^\circ; C: C = 631.0, \gamma = 112^\circ$.
 $B: B = 278.5, \beta = 330^\circ; C: C = 1039, \gamma = 60^\circ$.
 $B: B = -557.0, \beta = 90^\circ; C: C = -1522, \gamma = 240^\circ$.
31. $B: B = -0.9424, \beta = 112^\circ; C: C = 18.65, \gamma = 10^\circ$.
 $B: B = 15.03, \beta = 330^\circ; C: C = 11.41, \gamma = 60^\circ$.
 $B: B = -30.06, \beta = 90^\circ; C: C = -37.44, \gamma = 240^\circ$.
33. $AC = 6127$ ft. $BC = 6145$ ft.
35. 14,470 ft.
37. Impossible.
39. 86.6 ft.
41. 25.4 mi.
43. 501,700 sq. ft.
45. 113 ft.
47. 13,240 ft.
49. 9300 ft.
51. Group I: 145° , 19 mi. per hr.; Group II: 6.7 mi. per hr.; Group III: 153° , 29 mi per hr.
53. 3157 ft.
55. 291 mi. south, 466 mi. east.
57. 157 mi. at 281.8° .
59. No solution.
61. 4.5 mi. per hr. at a bearing of 145° .
63. 6.1 lb., 25.1° , 81.2° .
65. 161.3 lb., 2.6° , 10.4° .
67. 336.7 lb.
69. 34.5° , 43.1° .
71. 74.5 lb.
73. 23.6° .
75. 2800 lb.
77. 2.87° ; 6.7 hr.
79. Bearing 227.7° of north.
81. 9.3° , 4.03 mi., 0.31 hr. or 19 min.

Pages 394-396. Sec. 11-9

1. $y = \arcsin x$. 3. $y = \operatorname{arccsc} x$. 5. $y = \frac{1}{2} \arccos 2x$.
 7. $y = \pi + \arcsin x$. 9. $y = \operatorname{arccsc} \left(\frac{x-1}{2} \right)$. 11. $y = \frac{1}{2} \arcsin \frac{t}{x}$.
 13. $y = \frac{1}{t} \operatorname{arccsc} 7t$ 15. $y = \arctan \frac{N}{M}$. 17. $y = 2 \arcsin \left(\frac{4x-3}{2} \right)$.
 19. $y = -1 + \arctan (2a + 4)$. 21. $x = \sin y$.
 23. $x = \cos \frac{\theta}{2}$. 25. $x = I_m \sin (\alpha + \theta)$.
 27. $x = a \sin \alpha$. 29. $x = \tan 8\theta$. 31. $x = \frac{N}{M} \tan \theta$.
 33. $x = 141.4 \sin (377t - 80^\circ)$. 35. $x = a \sin \frac{(2\theta - 2C)}{a^2}$.
 37. $x = -R \tan \alpha$. 39. $x = -\frac{5}{2} \cos \theta$. 41. $y = \frac{\pi}{6}$.
 43. $-\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$. 45. $\frac{\pi}{3}$.
 47. $-\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$. 49. $\frac{\pi}{6}$.
 51. 0. 53. $-5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi$.
 55. $-81^\circ, -45^\circ, -9^\circ, 27^\circ, 63^\circ, 99^\circ$. 57. $\frac{5\pi}{3}$.
 59. $\frac{5\pi}{8}$.
 61. $-\frac{9\pi}{8}, -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}$. 63. 3π .
 65. -54.0° .
 67. $-251.41^\circ, -108.59^\circ, -71.41^\circ, 71.41^\circ, 108.59^\circ, 251.41^\circ$.
 69. 1.3° . 71. -20.22° . 73. 3.00° .
 75. $-329.17^\circ, -210.83^\circ, 30.83^\circ, 149.17^\circ, 390.83^\circ, 509.17^\circ$.
 77. $-323.34^\circ, -216.66^\circ, 36.66^\circ, 143.34^\circ, 396.66^\circ, 503.34^\circ$.
 79. 55.98° . 81. 8.13° . 83. 90.63° .
 85. -0.4633 . 87. 1. 89. -233.60° .
 91. $\frac{2\sqrt{15} + \sqrt{5}}{10} = 0.9982$. 93. -0.937 . 95. 278.82° .

Chapter 12

Page 400. Sec. 12-1

- | | | |
|---------------------|----------------------|----------------|
| 1. -1 . | 3. j . | 5. $-j$. |
| 7. -1 . | 9. $-j$. | 11. j . |
| 13. $j7$. | 15. $j\frac{3}{5}$. | 17. $-j0.75$. |
| 19. $\frac{j}{7}$. | 21. $j3.46$. | 23. $j5.20$. |
| 25. $-j9.80$. | 27. $j0.433$. | 29. $j9$. |
| 31. $j13$. | 33. $j1.41$. | 35. $j2.65$. |
| 37. $j3.16$. | 39. $j6.71$. | 41. $-j2.45$. |
| 43. $j6.93$. | 45. $j6$. | 47. 2 . |
| 49. $j8$. | | |

Page 402. Sec. 12-3

- | | | |
|----------------------|---------------------------------|--|
| 1. $5 - j2$. | 3. $2a - jb$. | 5. $-7 + j$. |
| 7. 5 . | 9. -3 . | 11. $c + jd$. |
| 13. $x = 1, y = 2$. | 15. $x = 4, y = -\frac{3}{5}$. | 17. $x = -\frac{3}{2}\frac{3}{5}, y = -\frac{7}{2}\frac{3}{5}$. |

Pages 404-405. Sec. 12-4

- | | | |
|---|--|--|
| 1. $6 - j7$. | 3. $10 - j$. | 5. $2 - j10$. |
| 7. $7 - j2$. | 9. $0.2 - j0.7$. | 11. $0.3 + j0.8$. |
| 13. 13 . | 15. $31 - j5$. | 17. $\frac{1}{2}\frac{1}{2} - j1\frac{1}{2}$. |
| 19. $7 + j24$. | 21. $-\frac{3}{8}\frac{5}{6} + j\frac{1}{3}$. | 23. $-\frac{6}{1}\frac{1}{8} + j$. |
| 25. $-46 - j9$. | 27. $-5 - j1.41$. | 29. $-j8$. |
| 31. 1 . | 33. $\frac{3}{1}\frac{3}{3} - j1\frac{2}{3}$. | 35. $1 - j4$. |
| 37. $-\frac{6}{2}\frac{3}{5} + j\frac{1}{2}\frac{7}{3}$. | 39. $1 + j2$. | 41. $\frac{4}{2}\frac{3}{6} - j\frac{3}{2}\frac{9}{6}$. |

Page 412. Sec. 12-6

- | | | |
|---------------------------------------|--------------------------------------|---------------------------------------|
| 1. $1.41/\underline{45^\circ}$. | 3. $2/\underline{120^\circ}$. | 5. $8/\underline{180^\circ}$. |
| 7. $3/\underline{270^\circ}$. | 9. $8.54/\underline{290.56^\circ}$. | 11. $4.12/\underline{255.96^\circ}$. |
| 13. $9.22/\underline{282.53^\circ}$. | 15. $j7$. | 17. $-j3$. |
| 19. $3.194 + j2.407$. | 21. $-5.657 + j5.657$. | 23. $0.8880 + j1.209$. |
| 25. $-2.576 - j1.724$. | 27. $1.864 - j5.787$. | |

Page 414. Sec. 12-7

- | | | |
|-------------------------|---------------------------|---------------------------|
| 1. $2.828e^{j0.7854}$. | 3. $2e^{j1.0472}$. | 5. $7e^{j3.1416}$. |
| 7. $5e^{j4.7124}$. | 9. $9.434e^{j5.271}$. | 11. $7.816e^{j4.998}$. |
| 13. $3e^{j1.5708}$. | 15. $2e^{j0.7505}$. | 17. j . |
| 19. -1 . | 21. $-0.1288 + j0.9917$. | 23. $-0.6663 - j0.7457$. |

Pages 418-419. Sec. 12-10

- | | |
|--|---|
| 1. $6/\underline{58^\circ}, 3.18 + j5.09$. | 3. $18.4/\underline{45^\circ}, 13 + j13$. |
| 5. $42/\underline{110^\circ}, -14.4 + j39.5$. | 7. $117.4/\underline{81.18^\circ}, 18 + j116$. |
| 9. $2.69/\underline{22^\circ}, 2.49 + j1.01$. | 11. $2.61/\underline{32.5^\circ}, 2.2 + j1.4$. |

13. $58.9/143.9^\circ$, $-47.6 + j34.7$. 15. $0.0798/14.05^\circ$, $0.0774 + j0.01937$.
 17. $126.9/79.78^\circ$, $22.52 + j124.9$. 19. $-5 + j3$.
 21. $3/130^\circ$. 23. $-1 - j2$.
 25. $-1 - j5$. 27. $\pm(1.099 + j0.455)$.
 29. $\pm(0.707 - j0.707)$.
 31. $1.292 + j0.2013$, $-0.820 + j1.018$, $-0.4717 - j1.220$.

Page 420. Sec. 12-11

1. $26.5 + j20.2$, $33.3/37.3^\circ$. 3. $9.24 - j13.83$, $16.6/-56.2^\circ$.
 5. $16.6 - j1.73$, $16.7/-5.95^\circ$. 7. $174.5 + j6$, $174.6/1.97^\circ$.
 9. $0.0000310 + j0.000185$, $0.000188/80.5^\circ$.
 11. $-641.8 + j158.2$, $661/166.2^\circ$.

Page 425. Sec. 12-12

1. $Z = 16 + j42$, $I = 2.44$ amp. 3. $I = 1.19$ amp.
 5. $I = 2.74$ amp., $V = 110$ volts.

Chapter 13

Page 429. Sec. 13-1

1. $R_1 = 1$, $R_2 = 3$, $R_3 = 2$. 3. $x = -1$, $y = 4$, $z = 2$.
 5. $V_1 = 0$, $V_2 = 21$, $V_3 = 12$. 7. $I_1 = \frac{21}{5}$, $I_2 = \frac{3}{5}$, $I_3 = -\frac{36}{5}$.
 9. $x = \frac{2}{11}$, $y = -\frac{6}{11}$, $z = 1$. 11. $I_1 = \frac{79}{16}$, $I_2 = -\frac{115}{32}$, $I_3 = \frac{11}{4}$.
 13. $c_0 = 4$, $c_1 = -3$, $c_2 = 2$. 15. $x_1 = 9$, $x_2 = 10$, $x_3 = -2$.
 17. No solution. 19. $r = 15$, $s = 12$, $t = 10$.

Pages 434-435. Sec. 13-2

1. $x = -0.685$, $y = 1.686$. 3. $x = 13.69$, $y = 5.03$.
 5. $I_1 = -1.20$, $I_2 = 3.40$, $I_3 = -5.60$. 7. $a = 0.696$, $b = 0.794$, $c = 0.900$.

Page 442. Sec. 13-4

1. $x = \frac{4}{3}$, $y = \frac{1}{3}$. 3. $E_1 = 0$, $E_2 = 2$. 5. $v_1 = 1$, $v_2 = 1$.
 7. $z_1 = 1$, $z_2 = 1$. 9. $R = 5$, $x = 2$. 11. $A = 13.32$, $B = 6.30$.
 13. $r = \frac{23}{85}$, $s = \frac{177}{85}$. 15. $x = 0$, $y = 0$. 17. $m = \frac{64}{5}$, $n = \frac{133}{5}$.
 19. $u = 19$, $v = 3$.

Pages 445-446. Sec. 13-5

1. $x = 77$, $y = 107$. 3. Dependent. 5. Inconsistent.
 7. Dependent. 9. $p = 3$, $q = 2$. 11. $x = 4.5$, $y = 4.5$.
 13. $E_1 = 0.4$, $E_2 = 1.3$. 15. $k = -\frac{7}{2}$, dependent. 17. $k = \pm j$, dependent.
 19. $k = 1$, dependent; $k = 9$, inconsistent.
 21. $k = \frac{11 + \sqrt{5}}{2}$, $k = \frac{11 - \sqrt{5}}{2}$, inconsistent.
 23. $k = 16$, dependent.

Pages 449-450. Sec. 13-6

- | | | |
|-----------------------------|------------------------|--------------------------|
| 1. No. | 3. No. | 5. No. |
| 7. Yes. | 9. Yes. | 11. $k = \frac{23}{9}$. |
| 13. $k = \pm 0.9\sqrt{5}$. | 15. No value possible. | 17. $k = 13, 4$. |

Pages 459-460. Sec. 13-7

- | | | |
|--------------------------------|------------------------------|-----------------------------|
| 1. 42. | 3. 5. | 5. 88. |
| 7. 121. | 9. 38. | 11. -1. |
| 13. 24. | 15. 72. | 17. $a^2 + b^2 + c^2 + 1$. |
| 19. 0. | 21. 0. | 23. -12, 189. |
| 25. 0. | 27. 0. | 29. 0. |
| 31. 16. | 33. 30. | |
| 35. $-(A - B)(A - C)(B - C)$. | 37. $k^3 - 3k^2 + 8k - 16$. | |
| 39. 0. | | |

Pages 465-466. Sec. 13-8

- | | |
|---|---------------------------------------|
| 1. Independent. | 3. Independent. |
| 5. Independent. | 7. Independent. |
| 9. $x = 7t, y = 19t, z = 11t$. | 11. $x = 3t, y = -6t, z = -3t$. |
| 13. $x = 0, y = 0, z = 0$. | 15. $x = 11t, y = 4t, z = 7t$. |
| 17. $A_x = 4t, A_y = 29t, A_z = 43t$. | 19. $e_1 = 2t, e_2 = -5t, e_3 = 3t$. |
| 21. $A = 10t, B = 3t, C = -29t; A = 10, B = 3, C = -29$. | |
| 23. $A = (k - 1)t, B = (1 - k^2)t, C = k(k - 1)t; A = k - 1, B = 1 - k^2, C = k(k - 1)$. | |
| 25. $k = 2$. | 27. $k = 0, \frac{13}{31}$. |

Page 467. Sec. 13-9

1. $I_1 = -6.1$ amp., $I_2 = 9.9$ amp., $I_3 = 16.0$ amp.

Chapter 14

Pages 473-474. Sec. 14-3

- | | | |
|--------------|---------------|---------------|
| 5. 0.6, 1.6. | 7. -4.2, 1.2. | 9. -3.2, 4.2. |
|--------------|---------------|---------------|

Pages 475-477. Sec. 14-5

- | | | |
|--|-------------------|-----------------|
| 1. ± 2.67 . | 3. ± 1.32 . | 5. ± 2.23 . |
| 7. ± 1.90 . | 9. $\pm 0.606j$. | 11. 0, 0.187. |
| 13. 0, 0.186. | 15. 44.7 ohms. | |
| 17. (a) $B = 8490 \sqrt{\frac{F}{A}}$, (b) 39.1 lb. | 19. 36.2 volts. | |
| 21. 1.05 in. | 23. 8.66 in. | 25. 0.0224 in. |

Pages 483-485. Sec. 14-7

1. 2, 7. 3. $3 \pm j5$. 5. 3, 3.
 7. 6, -3. 9. $-5 \pm j\sqrt{5}$. 11. 3, -0.4.
 13. 1, -2.4. 15. $\frac{-3 \pm j\sqrt{11}}{2}$. 17. 0.750, -0.625.
 19. 3.5, -0.5. 21. 0.618, -1.62. 23. -1.5, 0.4.
 25. 1.84, -0.697. 27. 1.80, 0.618. 29. -1.78, 0.378.
 31. 26.4, -4.14. 33. 209, -85.9. 35. 2.44, 6.56.
 37. 2.18, 0.685. 39. 1.69, -1.59. 55. No such value.
 57. -4, 3. 59. 0, 8. 61. $(x + 3.45)(x - 1.45)$.
 63. $3.1(p - 1.01q)(p - 0.511q)$. 65. $2(s - 1.78)(s + 0.281)$.
 67. $35(L - 1.57)(L + 1.20)$. 69. $(2x - 3y)(4x + 5y)$.
 71. 3.56. 73. 18.2. 75. 5.
 77. 6. 79. No root. 81. 6.05.
 83. 5. 85. $\pm 18, \pm 19$. 87. 19.8 in.
 89. 25 mi. per hr. 91. 273 amp. or 37 amp.
 93. $e_z = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2^2}}{2c_2}$.

Page 487. Sec. 14-8

1. $\pm 0.792, \pm 2.52$. 3. $\pm 1.54, \pm j1.84$. 5. $\pm 3, \pm j$.
 7. $\pm 2.62, \pm 0.382$. 9. 2.77, -1.47. 11. 3, 2, -0.523, -11.48.
 13. $\pm 0.816, \pm j0.866$. 15. -6.85, -0.146, 0.209, 4.79.
 17. 2.28, -5.23, -1.5, $\pm j1.32$. 19. $71.0^\circ, 251.0^\circ, 150.2^\circ, 330.2^\circ$.
 21. $55.8^\circ, 235.8^\circ, 164.8^\circ, 344.8^\circ$. 23. $45.0^\circ, 225.0^\circ, 149.0^\circ, 329.0^\circ$.
 25. ± 2.09 . 27. $0^\circ, 90^\circ, 270^\circ$.
 29. $72.4^\circ, 252.4^\circ$. 31. No real roots.
 33. $x = \log_e(u + \sqrt{u^2 + 1})$.

Pages 491-492. Sec. 14-10

1. (1, 2), $(\frac{2}{3}, 1\frac{8}{15})$. 3. (5, 5), $(2\frac{2}{3}, -3\frac{1}{3})$. 5. (3.2, 2.6), (2.6, 3.2).
 7. $(2\sqrt{2}, \sqrt{2})$, $(2\sqrt{2}, -\sqrt{2})$, $(-2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$.
 9. (-6.02, 4.85), (2.81, 1.93).
 11. $(10 + 2\sqrt{15}, 10 - 2\sqrt{15})$, $(10 - 2\sqrt{15}, 10 + 2\sqrt{15})$.
 13. (-2, 1), $(3, -2\frac{1}{3})$. 15. (1, 2), (-1, -2), (2, 1), (-2, -1).
 17. (1, 2), (1, -2), (-1, 2), (-1, -2). 19. (6, 2), (6, -2), (-6, 2), (-6, -2).
 21. (2, 1), (2, -1), (-1.6, $j2.49$), (-1.6, $-j2.49$).
 23. ± 25 . 25. $\frac{1}{4}$.
 27. $x = \frac{-mb \pm \sqrt{r^2(1+m^2) - b^2}}{1+m^2}$, $y = \frac{b \pm m\sqrt{r^2(1+m^2) - b^2}}{1+m^2}$ (four answers).
 29. $x = \pm k\sqrt{\frac{1 \pm \sqrt{5}}{2}}$, $y = \pm k\sqrt{\frac{-1 \pm \sqrt{5}}{2}}$ (four answers).

Pages 493-494. Sec. 14-11

1. $v = \pm \sqrt{\frac{2dF}{M}}$.
3. $B = \pm \frac{10^4}{\sqrt{8.94}} \sqrt{\frac{F}{A}}$.
5. $r = \pm \sqrt{\frac{12I_z - h^2M}{3M}}$, $h = \pm \sqrt{\frac{12I_x - 3r^2M}{M}}$.
7. $\omega = \frac{-60 \pm \sqrt{3600 + 145.5S}}{145.5}$.
9. $r = \frac{-\pi + \sqrt{\pi^2 + 8\pi}}{4\pi}$ ft.
11. $t = \frac{10}{g} (-1 \pm \sqrt{1 - 0.1g(c-s)})$.
13. $t = \frac{1}{2c} (-b \pm \sqrt{b^2 - 4c(a-T)})$.
15. $t = \frac{1}{g} (kv_0 \pm \sqrt{k^2v_0^2 - 2gh})$.
17. $z_0 = \frac{X_l}{2R_c} (X_l \pm \sqrt{X_l^2 - 4R_c^2})$.
19. 4.53, 15.5.
21. $r^2 = \frac{1}{2\pi} (-\pi h^2 \pm \sqrt{\pi^2 h^4 + 4S^2})$.
23. 7.1, 16.9.

Page 498. Sec. 14-12

1. $3x^2 - 2x + 5$, 3.
3. $x^3 + 6x^2 + 18x + 55$, 170.
5. $2r - 5$, 0.
7. $x^2 + x + 1$, 0.
9. $x^2 - x + 1$, -2.
11. $x^3 + x^2 + x + 1$, 0.
13. $x^3 - x^2 + x - 1$, 0.
15. $x^3 - 2x^2 + 4x - 8$, 17.
17. $4x^2 + 6xy + 9y^2$, $54y^3$.
19. $x^2 + 4$, 0.
21. $2x^2 + x - 2$, 0.

Page 500. Sec. 14-13

1. $x^3 - x^2 + x - 1 + \frac{8}{x+1}$.
3. $x^2 + 8x + 33 + \frac{150}{x-5}$.
5. $2E^2 + 3E + 12 + \frac{29}{E-2}$.
7. $y^3 - 2y^2 + 2y - 1 + \frac{-3}{y+2}$.
9. $2z^3 - 10z^2 + 45z - 227 + \frac{1140}{z+5}$.
11. 29.
13. -7.
15. -157,957.
17. 2.25.
19. 4.
21. No.
23. No.
25. Yes.
27. No.
29. Yes.
31. Yes.
33. Yes.
35. No.
37. No.
39. $-2\frac{2}{3}$.

Page 503. Sec. 14-16

1. $-1.75 \pm j1.20$.
3. $-0.500 \pm j0.866$.
5. -2, 3.
7. -0.809, 0.309.
9. -2.414, 0.414.
11. 2, 3.
13. $\frac{1}{2}$, $-\frac{2}{3}$.
15. $\pm j2$.

Pages 506-507. Sec. 14-17

1. 2, $\frac{1}{2}$.
3. -2, 2, 2.
5. 1, 2, 3, 4.
7. 3.
9. 1, $-1\frac{2}{3}$, $\frac{1}{2}$.
11. 2, 2, -7.
13. 1, 3, 5.
15. 3, 4.
17. -2, -1, 1.
19. $-\frac{2}{3}$, $\frac{1}{2}(-1 \pm \sqrt{5})$.
21. 1, 2, -3 $\pm j$.
23. 2, $-1 \pm j\sqrt{3}$.
25. 3, $\frac{1}{2}(-5 \pm j\sqrt{15})$.
27. 3, $\frac{1}{2}(-3 \pm \sqrt{5})$.
29. $\frac{1}{2}$, $-1 \pm \sqrt{2}$.

Pages 510-511. Sec. 14-18

- | | | |
|-------------------|---------------------|---------------|
| 1. 5.3. | 3. -3.3, -1.8, 1.2. | 5. -2.5. |
| 7. No real roots. | 9. 3.0. | 11. -2, 1, 1. |
| 13. 2.6. | 15. 4.7. | 17. 0.6, 1.5. |
| 19. 0.74. | 21. 2.5. | 23. 1.9 rad. |

Pages 517-518. Sec. 14-19

- | | | |
|-------------|---------------------|------------------------|
| 1. 2.46. | 3. 1.15. | 5. 1.26, -1.76, -4.51. |
| 7. 2.96. | 9. -4, ± 1.41 . | 11. 2.23, -2.96. |
| 13. 3.68. | 15. ± 2.34 . | 17. 2.31. |
| 19. -0.965. | | |

Chapter 15

Pages 520-521. Sec. 15-1

- | | | |
|-------------------|-------------------|------------------------------|
| 1. $\sqrt{101}$. | 3. $3\sqrt{2}$. | 5. 11 |
| 7. $\sqrt{82}$. | 9. $\sqrt{145}$. | 27. (-2, -1), $2\sqrt{10}$. |
| 29. (2, 0). | | |

Page 522. Sec. 15-2

- | | | |
|--|---------------------------|--------------------------|
| 1. (3, 1). | 3. $(-4, -\frac{3}{2})$. | 5. $(-\frac{5}{2}, 6)$. |
| 7. $(3, 4)$, $(1, \frac{1}{2})$, $(-1, \frac{7}{2})$. | | 9. (10, 1). |

Pages 524-525. Sec. 15-3

- | | |
|---|------------------------|
| 1. -1, 135° . | 3. -3, 108.4° . |
| 5. $-\frac{8}{7} = -0.8571$, 139.4° . | 7. 1.5, 56.3° . |
| 17. 6. | 19. 6. |
| 21. -3. | 23. 3. |

Pages 526-527. Sec. 15-4

- | | | |
|---------------|---------------------------|---------------------------|
| 1. -5, 1, -1. | 3. -1, 1, $\frac{1}{7}$. | 5. -2, $\frac{1}{2}$, 3. |
|---------------|---------------------------|---------------------------|

Pages 529-530. Sec. 15-5

- | | | |
|--|---|---|
| 1. 8.1° . | 3. 18.4° . | 5. 52.7° . |
| 7. 53.1° . | 9. 90° . | 11. 0° , 0° , 180° . |
| 13. 63.4° , 63.4° , 53.2° . | 15. 26.6° , 11.3° , 142.1° . | 17. 20.8° , 54.0° , 105.2° . |
| 19. $\frac{1}{3}$. | 21. ∞ . | 23. $8 + 5\sqrt{3} = 16.66$. |

Page 532. Sec. 15-6

- | | | |
|-----------------------|-----------------------|---------|
| 1. 4. | 3. 20. | 5. 88. |
| 7. 3. | 9. 0. | 11. 6. |
| 13. 24. | 15. 34. | 17. 74. |
| 19. $30\frac{1}{2}$. | 21. $12\frac{1}{2}$. | 23. 24. |

Pages 536-537. Sec. 15-11

1. $x - 2y + 7 = 0$. 3. $x + 3y - 13 = 0$. 5. $3x + y + 5 = 0$.
 7. $x + 2y - 13 = 0$. 9. $y - 8 = 0$. 11. $y = \frac{1}{2}x + 3$.
 13. $y = -5x + 4$. 15. $y = \frac{2}{3}x + 2$. 17. $y = 4x$.
 19. $y = -3x - 2$. 21. $x - y = 0$. 23. $x + y + 1 = 0$.
 25. $x - \sqrt{3}y + 3 + 2\sqrt{3} = 0$. 27. $y = x + 5$.
 29. $y = \sqrt{3}x - 4$. 31. $y = -\sqrt{3}x$. 33. $-3, 6$.
 35. $-\frac{5}{2}, 5$. 37. $2, 4$. 39. $-\frac{2}{3}, 2$.
 41. $\frac{3}{8}, -4$. 43. $x - y + 1 = 0, x + y - 3 = 0$.
 45. $x - 2y + 8 = 0, 2x + y - 14 = 0$. 47. $3x - 5y = 0, 5x + 3y = 0$.
 49. $x - 5y + 34 = 0, 5x + y - 38 = 0$. 51. $x + 3y - 9 = 0$.
 53. $6x + 5y = 0$. 55. $2x + 5y - 1 = 0$. 57. $x = 0$.
 59. $x - 3 = 0$. 61. $y + 3 = 0$. 63. 161.6° .
 65. 92.7° . 67. 56.3° .

Page 538. Sec. 15-12

1. $5x - 4y = 0$. 3. $2x + 3y - 5 = 0$. 5. $4x + y = 0$.
 7. $3x - 4y + 1 = 0$. 9. $9x - 8y + 11 = 0$. 11. $x + 11y - 28 = 0$.
 13. $3x + y = 0$. 15. $x - 1 = 0$.
 17. $11x + 2y - 28 = 0, 7x + 10y - 12 = 0, x - 2y - 4 = 0$.
 19. $9x - 13y + 84 = 0, 3x - 7y + 44 = 0, 3x - 5y + 32 = 0$.
 21. $6x + y - 7 = 0$. 23. $x - 6y + 5 = 0$.

Pages 539-540. Sec. 15-13

1. $4x + 3y - 12 = 0$. 3. $3x - y + 6 = 0$. 5. $7x - 2y - 14 = 0$.
 7. $3x - 8y + 24 = 0$. 9. $x + 5y - 5 = 0$. 11. $-\frac{3}{2}, 1$.
 13. $3, -5$. 15. $7, -2$. 17. $\frac{3}{2}, -4$.
 19. $8, 4$. 21. $x + y - 4 = 0$ or $9x + y - 12 = 0$.
 23. $x + y - 4 = 0$. 25. $x + 3y - 12 = 0$ or $3x + y - 12 = 0$.

Pages 540-542. Sec. 15-14

1. $x - 2y - 8 = 0$. 3. $(0, 4), (5, 3)$. 5. $\frac{1}{2}\sqrt{370}$.
 7. $x + 3y - 16 = 0$. 9. $x - 7y + 2 = 0$.
 11. $y - 3 = 0, x - y = 0, x + y = 0$. 13. $67\frac{1}{2}$.
 17. $(1, 2)$. 19. $6x - y = 0$. 21. $A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.
 23. $x + y - 7 = 0$. 25. $(0, 0)$.
 27. (a) $(-3, -1), (2, 4), (6, -3)$; (b) $4x - 7y + 5 = 0, x + y - 3 = 0, 9x - 2y - 10 = 0$; (c) $7x + 4y + 25 = 0, x - y - 9 = 0, 2x + 9y - 40 = 0$; (d) $18x - 4y - 35 = 0, 8x - 14y - 25 = 0, x + y - 1 = 0$; (e) $27\frac{1}{2}$.
 29. 2. 31. 0.000342, 0.00057.

Chapter 16

Page 545. Sec. 16-1

- | | |
|---|---|
| 1. $x^2 + y^2 - 6x - 8y - 24 = 0$. | 3. $x^2 + y^2 - 6x + 2y - 6 = 0$. |
| 5. $x^2 + y^2 + 4x + 6y - 23 = 0$. | 7. $x^2 + y^2 - 2ax + 2ay - 2a^2 = 0$. |
| 9. $x^2 + y^2 - 12x = 0$. | 11. $x^2 + y^2 + 4x + 6y + 9 = 0$. |
| 13. $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. | 15. $x^2 + y^2 - 14x - 6y + 56 = 0$. |
| 17. $x^2 + y^2 - 4ax - 4ay = 0$. | 19. $x^2 + y^2 \pm 4y = 0$. |

Page 547. Sec. 16-2

- | | | |
|--------------------------------------|---------------------------------------|----------------|
| 1. (3, -5), 6. | 3. (3, -4), 5. | 5. (-2, 1), 0. |
| 7. (1, 3), 0. | 9. No circle. | 11. (2, 0), 4. |
| 13. $x^2 + y^2 - 2x - 4y - 20 = 0$. | 15. $x^2 + y^2 - 6x + 4y = 0$. | |
| 17. $x^2 + y^2 + 2x - 6y = 0$. | 19. $x^2 + y^2 + 6x \pm 6y + 9 = 0$. | |
| 21. $x^2 + y^2 - 14x + 32 = 0$. | 23. $x^2 + y^2 - 2y - 9 = 0$. | |
| 25. $x^2 + y^2 - 4x - 2y - 20 = 0$. | | |

Pages 548-549. Sec. 16-3

- | | |
|--|--------------------------------------|
| 1. $x^2 + y^2 - x + y = 0$. | 3. $x^2 + y^2 - 10y = 0$. |
| 5. $x^2 + y^2 + 4x - 4y - 17 = 0$. | 7. $x^2 + y^2 - 12x - 6y + 28 = 0$. |
| 9. $5x^2 + 5y^2 - 12x + 64y - 560 = 0$. | 11. $x^2 + y^2 - 3x - 2y = 0$. |
| 13. No. | 15. $x^2 + y^2 + 2x - 4y = 0$. |

Page 551. Sec. 16-4

- | | |
|---|--------------------------------------|
| 1. $x = 5$, straight line. | 3. $x + y - 10 = 0$, straight line. |
| 5. $x^2 - 3y^2 + 40y - 100 = 0$, a curve not previously studied. | |
| 7. $x + 2y - 10 = 0$, straight line. | |
| 9. $3x^2 + 4y^2 + 4x - 4 = 0$, a curve not previously studied. | |
| 11. $3x + 5y - 23 = 0$, $3x + 5y - 11 = 0$, straight lines. | |
| 13. $2(x_2 - x_1)x + 2(y_2 - y_1)y + x_1^2 - x_2^2 + y_1^2 - y_2^2 \pm k = 0$, where (x_1, y_1) and (x_2, y_2) are the fixed points; two straight lines. | |

Page 558. Sec. 16-7

- | | | |
|---|---|------------------------------------|
| 1. $x^2 + y^2 - 25 = 0$. | 3. $x - y = 0$. | 5. $7x - 11y = 0$. |
| 7. $y - 2 = 0$. | 9. $y - 2 = 0$. | 11. $x + 4 = 0$. |
| 13. $x^2 + y^2 - 2y = 0$. | 15. $x^4 + y^4 + 2x^2y^2 + 4x^3 + 4xy^2 - 4y^2 = 0$. | |
| 17. $\sqrt{3}x + y - 4 = 0$. | 19. $x^2 + y^2 - 2x + 2\sqrt{3}y = 0$. | |
| 21. $\sin \theta = 0$. | 23. $\tan \theta = \sqrt{3}$. | 25. $r \cos \theta + 1 = 0$. |
| 27. $r \sin \theta - 2 = 0$. | 29. $r \cos \theta - r \sin \theta = 2$. | 31. $r = 5$. |
| 33. $r = \tan \theta \sec \theta$. | 35. $r = 6$. | 37. $r(1 + a \cos^2 \theta) = a$. |
| 39. $r - \cos \theta + \sin \theta = 0$. | | |

Page 560. Sec. 16-8

- | | | |
|-----------------------|---------------------|-------------------|
| 1. $2x - y - 3 = 0$. | 3. $x = y$. | 5. $2x + y = 7$. |
| 7. $x^2 + y^2 = 25$. | 9. $4x^2 - y = 0$. | |

Chapter 17

Page 566. Sec. 17-2

1. $(0, 0), (1, 0), x + 1 = 0, 4.$
3. $(0, 0), (0, 1), y + 1 = 0, 4.$
5. $(0, 0), (0, 4), y + 4 = 0, 16.$
7. $(0, 0), (-4, 0), x - 4 = 0, 16.$
9. $(0, 0), (16, 0), x + 16 = 0, 64.$
11. $(0, 0), (0, \frac{9}{16}), 10y + 9 = 0, \frac{1}{8}.$
13. $(0, 0), (0, -\frac{5}{2}), 2y - 5 = 0, 10.$
15. $(0, 0), (-2, 0), x - 2 = 0, 8.$
17. $(0, 0), (\frac{1}{4}, 0), 4x + 1 = 0, 1.$
19. $(0, 0), (0, -\frac{3}{4}), 4y - 3 = 0, 3.$
21. $y^2 = 16x.$
23. $y^2 = -32x.$
25. $3x^2 = 32y.$
27. $y^2 = -24x.$
29. $y^2 = 16x.$
31. $x^2 = 40y.$
33. $y^2 = x.$
35. $x^2 = 4y.$
37. $x^2 = -16y.$
39. $x^2 + 24y + 144 = 0.$

Page 570-571. Sec. 17-3

1. $(\pm 3, 0), (\pm\sqrt{5}, 0), 3, 2.$
3. $(0, \pm 5), (0, \pm\sqrt{21}), 5, 2.$
5. $(\pm 4, 0), (\pm 2\sqrt{3}, 0), 4, 2.$
7. $(0, \pm 6), (0, \pm 3\sqrt{3}), 6, 3.$
9. $(\pm 5, 0), (\pm 4, 0), 5, 3.$
11. $(\pm 4, 0), (\pm\sqrt{7}, 0), 4, 3.$
13. $\left(0, \pm \frac{\sqrt{33}}{3}\right), \left(0, \pm \frac{\sqrt{33}}{6}\right), \frac{\sqrt{33}}{3}, \frac{\sqrt{11}}{2}.$
15. $\left(0, \pm \frac{\sqrt{115}}{5}\right), \left(0, \pm \frac{\sqrt{2415}}{30}\right), \frac{\sqrt{115}}{5}, \frac{\sqrt{69}}{6}.$
17. $(\pm 6, 0), (\pm\sqrt{11}, 0), 6, 5.$
19. $(\pm 10, 0), (\pm 8, 0), 10, 6.$
21. $(\pm \frac{2}{5}, 0), (\pm \frac{5}{2}\sqrt{21}, 0), \frac{2}{5}, 5.$
23. $(\pm 14, 0), (\pm 8\sqrt{3}, 0), 14, 2.$
25. $144x^2 + 169y^2 = 24,336.$
27. $225x^2 + 81y^2 = 18,225.$
29. $289x^2 + 64y^2 = 18,496.$
31. $25x^2 + 169y^2 = 4225.$
33. $289x^2 + 64y^2 = 18,496.$
35. $x^2 + 4y^2 = 16.$
37. $7x^2 + 4y^2 = 64.$
39. $2x^2 + 25y^2 = 200.$
41. $9x^2 + 25y^2 = 900.$
43. $16x^2 + 25y^2 + 32x - 384 = 0.$

Pages 575-576. Sec. 17-4

1. $(\pm 4, 0), (\pm 5, 0), 4, 3, y = \pm \frac{3}{4}x.$
3. $(\pm 4, 0), (\pm 4\sqrt{2}, 0), 4, 4, y = \pm x.$
5. $(0, \pm 8), (0, \pm 8\sqrt{2}), 8, 8, y = \pm x.$
7. $(0, \pm 2), (0, \pm 2\sqrt{2}), 2, 2, y = \pm x.$
9. $(\pm 12, 0), (\pm 13, 0), 12, 5, y = \pm \frac{5}{12}x.$
11. $(0, \pm 5), (0, \pm \sqrt{34}), 5, 3, y = \pm \frac{5}{3}x.$
13. $(0, \pm 6), (0, \pm 6\sqrt{2}), 6, 6, y = \pm x.$
15. $(\pm 2, 0), (\pm \sqrt{13}, 0), 2, 3, y = \pm \frac{3}{2}x.$
17. $(0, \pm 1), (0, \pm \sqrt{17}), 1, 4, y = \pm \frac{1}{4}x.$
19. $(0, \pm \frac{5}{2}), \left(0, \pm \frac{5\sqrt{13}}{6}\right), \frac{5}{2}, \frac{5}{3}, y = \pm \frac{3}{2}x.$

21. $(0, \pm 2), (0, \pm 2\sqrt{26}), 2, 10, y = \pm \frac{1}{5}x$.

23. $(\pm 2, 0), (\pm 2\sqrt{26}, 0), 2, 10, y = \pm 5x$.

25. $16x^2 - 9y^2 = 144$.

27. $144y^2 - 25x^2 = 3600$.

29. $225y^2 - 64x^2 = 14,400$.

31. $64x^2 - 225y^2 = 14,400$.

33. $3x^2 - 11y^2 = 64$.

35. $25x^2 - 144y^2 = 3600$.

Pages 578-579. Sec. 17-6

1. $(-2, -2), (-6, -2), (0, 1)$.

3. $(7, 1), (3, -1), (8, -8)$.

5. $(2, 8), (-4, 7), (-6, 1)$.

7. $x' - 3y' = 0$.

9. $x' - 2y' = 0$.

11. $4x' + 6y' + 31 = 0$.

13. $x'^2 + 4y'^2 - 16 = 0$.

15. $x'^2 + y'^2 - 16 = 0$.

17. $36x'^2 + 36y'^2 - 169 = 0$.

19. $x'^2 - 8y' = 0$.

21. $4x'^2 + 9y'^2 - 16 = 0$.

23. $x'^2 - y'^2 - 16 = 0$.

25. $x'^2 + y'^2 = r^2$.

Pages 581-582. Sec. 17-7

1. $(3, 2); (5, 2); (5, -2), (5, 6); 8$.

3. $(-4, -3); (0, -3); (0, -11), (0, 5); 16$.

5. $(-1, -3); (-1, 0); (5, 0), (-7, 0); 12$.

7. $(3, 0); (3, 1); (5, 1), (1, 1); 4$.

9. $(-2, 3); (-\frac{3}{2}, 3); (-\frac{3}{2}, 2), (-\frac{3}{2}, 4); 2$.

11. $(0, -6); (-\frac{1}{4}, -6); (-\frac{1}{4}, -5\frac{1}{2}), (-\frac{1}{4}, -6\frac{1}{2}); 1$.

13. $(2, -2); (2 \pm 2\sqrt{3}, -2); (6, -2), (-2, -2); (2, 0), (2, -4)$.

15. $(2, -2); (2, -2 \pm 2\sqrt{3}); (2, 2), (2, -6); (4, -2), (0, -2)$.

17. $(-1, 5); (-5, 5), (3, 5); (-6, 5), (4, 5); (-1, 2), (-1, 8)$.

19. $(1, -2); (1 \pm \sqrt{7}, -2); (5, -2), (-3, -2); (1, 1), (1, -5)$.

21. $(4, -2); (4, 2), (4, -6); (4, 3), (4, -7); (7, -2), (1, -2)$.

23. $(-5, 2); (-5 \pm 3\sqrt{3}, 2); (1, 2), (-11, 2); (-5, 5), (-5, -1)$.

25. $(1, -2); (5, -2), (-3, -2); (6, -2), (-4, -2); (1, 1), (1, -5); 3x - 4y - 11 = 0; 3x + 4y + 5 = 0$.

27. $(3, 1); (6, 1), (0, 1); (3 \pm \sqrt{13}, 1); (3, 3), (3, -1); 2x - 3y - 3 = 0, 2x + 3y - 9 = 0$.

29. $(-2, 3); (10, 3), (-14, 3); (11, 3), (-15, 3); (-2, 8), (-2, -2); 5x - 12y + 46 = 0, 5x + 12y - 26 = 0$.

31. $(0, -1); (0, 0), (0, -2); (0, -1 \pm \sqrt{17}); (4, -1), (-4, -1); x - 4y - 4 = 0, x + 4y + 4 = 0$.

33. $(-4, 0); (-4, 6), (-4, -6); (-4, \pm 3\sqrt{5}); (-7, 0), (-1, 0); 2x - y + 8 = 0, 2x + y + 8 = 0$.

35. $(3, -7); (3, -3), (3, -11); (3, -7 \pm 4\sqrt{2}); (7, -7), (-1, -7); x - y - 10 = 0, x + y + 4 = 0$.

37. $x^2 - 4x + 32y - 60 = 0$.

39. $16x^2 + 25y^2 + 100y - 300 = 0$.

41. $24x^2 + 25y^2 - 96x - 100y - 404 = 0$.

43. $169x^2 + 144y^2 + 1014x + 648y + 729 = 0$.

45. $9y^2 - 16x^2 - 72y - 32x - 16 = 0$.

47. $9x^2 - 16y^2 - 36x - 64y - 172 = 0$.

Page 583. Sec. 17-8

1. $(y-2)^2 = 4(x-3)$.

3. $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{7} = 1$.

5. $\frac{(x-6)^2}{16} - \frac{(y-3)^2}{9} = 1$.

7. $(y+4)^2 = -2(x-5)$.

9. $\frac{(x-4)^2}{8} + \frac{(y-1)^2}{16} = 1$.

11. $\frac{(y-\frac{5}{2})^2}{1} - \frac{(x+3)^2}{16} = 1$.

13. $(x+\frac{3}{2})^2 = 5(y-2)$.

15. $\frac{(x-1)^2}{\frac{1}{4}} + \frac{(y+1)^2}{\frac{1}{9}} = 1$.

17. $\frac{(x-\frac{1}{2})^2}{7} - \frac{(y+\frac{1}{2})^2}{2} = 1$.

19. $(x-6)^2 = -\frac{3}{2}(y+2)$.

21. $\frac{(x+1)^2}{\frac{4^5}{4}} + \frac{(y-3)^2}{10} = 1$.

23. $\frac{(y-3)^2}{8} - \frac{(x-1)^2}{9} = 1$.

25. $(y-2)^2 = -5(x+\frac{8}{15})$.

27. $(x-\frac{9}{2})^2 = -\frac{15}{2}(y-\frac{123}{36})$.

29. $\frac{(y-4)^2}{28} - \frac{(x+2)^2}{21} = 1$.

31. $(x-2)^2 = y$.

Chapter 18

Page 590. Sec. 18-3

1. $\sqrt{30}$.

3. $\sqrt{61}$.

5. $3\sqrt{2}$.

7. $2\sqrt{22}$.

9. $\sqrt{62}$.

Page 594. Sec. 18-5

1. $\cos \alpha = 0.4575, \cos \beta = 0.4575, \cos \gamma = 0.7625, \alpha = 62.8^\circ, \beta = 62.8^\circ, \gamma = 40.3^\circ$.

3. $\cos \alpha = 0.4472, \cos \beta = 0, \cos \gamma = 0.8944, \alpha = 63.4^\circ, \beta = 90^\circ, \gamma = 26.6^\circ$.

5. $\cos \alpha = 0.7553, \cos \beta = 0.4196, \cos \gamma = -0.5035, \alpha = 40.9^\circ, \beta = 65.2^\circ, \gamma = 120.2^\circ$.

7. $\cos \alpha = 0, \cos \beta = 0, \cos \gamma = 1, \alpha = 90^\circ, \beta = 90^\circ, \gamma = 0^\circ$.

9. $\cos \alpha = 0.3553, \cos \beta = -0.1421, \cos \gamma = 0.9239, \alpha = 69.2^\circ, \beta = 98.2^\circ, \gamma = 22.5^\circ$.

11. $\cos \alpha = 0.7071, \cos \beta = 0, \cos \gamma = -0.7071, \alpha = 45^\circ, \beta = 90^\circ, \gamma = 135^\circ$.

13. $\cos \alpha = 0, \cos \beta = 0, \cos \gamma = -1, \alpha = 90^\circ, \beta = 90^\circ, \gamma = 180^\circ$.

15. $\cos \alpha = 0, \cos \beta = 1, \cos \gamma = 0, \alpha = 90^\circ, \beta = 0^\circ, \gamma = 90^\circ$.

17. $\cos \gamma = \pm 0.8660, \gamma = 30^\circ, 150^\circ$.

19. $\cos \alpha = \pm 0.5000, \alpha = 60^\circ, 120^\circ$.

21. $\cos \alpha = 0, \alpha = 90^\circ$.

23. $\cos \gamma = 0, \gamma = 90^\circ$.

25. $\alpha = 0^\circ, \beta = 90^\circ, \gamma = 90^\circ, \cos \alpha = 1, \cos \beta = 0, \cos \gamma = 0$.

27. $\alpha = 90^\circ, \beta = 90^\circ, \gamma = 0^\circ, \cos \alpha = 0, \cos \beta = 0, \cos \gamma = 1$.

Page 598. Sec. 18-7

1. $\cos \alpha = 0.3333$, $\cos \beta = 0.6667$, $\cos \gamma = -0.6667$, $\alpha = 70.5^\circ$, $\beta = 48.2^\circ$, $\gamma = 131.8^\circ$.
3. $\cos \alpha = 0.3846$, $\cos \beta = 0$, $\cos \gamma = -0.9231$, $\alpha = 67.4^\circ$, $\beta = 90^\circ$, $\gamma = 157.4^\circ$.
5. $\cos \alpha = 1$, $\cos \beta = 0$, $\cos \gamma = 0$, $\alpha = 0$, $\beta = 90^\circ$, $\gamma = 90^\circ$.
7. $\cos \alpha = -0.3242$, $\cos \beta = -0.8115$, $\cos \gamma = 0.4870$, $\alpha = 108.9^\circ$, $\beta = 144.2^\circ$, $\gamma = 60.9^\circ$.
9. $\cos \alpha = -0.8889$, $\cos \beta = 0.1111$, $\cos \gamma = -0.4444$, $\alpha = 152.7^\circ$, $\beta = 83.6^\circ$, $\gamma = 116.4^\circ$.
11. 90° .
13. 90° .
15. 124.4° .
17. L_1 , L_3 , L_6 are parallel; L_2 , L_4 , L_5 are parallel.
19. L_2 , L_6 are parallel; L_3 , L_4 , L_5 are parallel.
21. L_1 and L_2 , L_1 and L_4 , L_1 and L_6 ; L_2 and L_3 , L_2 and L_4 , L_2 and L_5 ; L_3 and L_4 , L_3 and L_5 .
23. 15.1652 , 36.3° , 53.7° .

Pages 599-600. Sec. 18-8

1. $y^2 - z^2 = 0$.
3. $4x^2 - z^2 = 0$.
5. $z^2 = 25$.
7. $y^2 = 16$.
9. $x^2 + y^2 + z^2 = 16$.
11. $x^2 + y^2 = 25$.
13. $z = 4$.
15. $x^2 + y^2 + z^2 - 10z = 0$.
17. $3x - 9y - 4z = 0$.
19. The yz -plane.
21. The xy -plane.
23. The plane containing the x -axis and whose trace in the yz -plane is $y + z = 0$.
25. The plane perpendicular to the z -axis at $z = 6$.
27. The right circular cylinder whose axis is the z -axis and radius 5 units.
29. Two planes perpendicular to the y -axis at $y = -2$ and $y = +2$.

Pages 603-604. Sec. 18-11

1. $2x + y - 2z - 6 = 0$.
3. $4x - 2y + z - 10 = 0$.
5. $3x - 2y + 2z - 34 = 0$.
7. $y + z + 1 = 0$.
9. (a) $x = 6$, $y = 3$, $z = 2$; (b) 1, 2, 3.
11. (a) $x = 2$, $y = 5$, $z = 3\frac{1}{3}$; (b) 5, 2, 3.
13. (a) $x = 4$, $y = -6$, $z = -4\frac{1}{3}$; (b) 6, -4, -5.
15. (a) $x = -6\frac{1}{2}$, $y = 2\frac{3}{8}$, $z = -13$; (b) 2, -5, 1.
17. Parallel.
19. Perpendicular.
21. Parallel.
23. Neither.
25. $2x + y + z - 3 = 0$.
27. $17x - 4y - 16z = 0$.
29. $53x - 220y - 45z - 390 = 0$.
31. $z - 5 = 0$.
33. $x - 6y + 4z + 8 = 0$.
35. $x - 2y + 2z - 15 = 0$.
37. $x + 2y + 3z + 13 = 0$.
39. $x - y - z = 0$.
41. $5x + 7y + z - 22 = 0$.

Chapter 19

Pages 611–612. Sec. 19–1

1. 0.0872; 0.0458; 0.0096; 0.0048; 0.0010.
 3. 0.0669; 0.0347; 0.0071; 0.0036; 0.0007.
 5. 0.0326; 0.0164; 0.0082; 0.0017.
 7. 0.0363; 0.0181; 0.0090; 0.0018.
 9. 7.651; 2.871; 1.276; 0.234.
 11. At $x = 20$: 0.5279; 0.1104; 0.0112; at $x = 65$: 0.3043; 0.0618; 0.0062; at $x = 80$: 0.2753; 0.0557; 0.0056.
 13. -2 ; 2 ; 1 ; 0.2 .
 15. -0.0048 ; 0.0031 ; -0.0010 .
 17. -0.2591 ; -0.1328 ; -0.0272 .
 19. 1386.
 21. 7353.
 23. 3000; 2500; 100.

Pages 615–616. Sec. 19–2

1. (a) 0.0396; (b) 0.1250; (c) 0.143; (d) 0.15; (e) 0.00969; (f) 0.00144.
 3. (a) 16; (b) 160; (c) 112; (d) 160; (e) 160; (f) 112.
 5. (a) 3.2×10^{-6} ; (b) 6.4×10^{-5} ; (c) 5.8×10^{-5} .

Page 619. Sec. 19–3

1. (a) 0.215; (b) 0.06; (c) 1.45; (d) 0.105.
 3. (a) 0.878; (b) 0.697; (c) 0.070; (d) 1.000.
 5. (a) 0.795; (b) 0.422; (c) 0.031; (d) 0.169.
 7. (a) 0.040; (b) 0.092; (c) 0.010.
 9. (a) -4.26 ; (b) -5.00 ; (c) -6.34 .
 11. (a) 30,000; (b) 14,000; (c) 1685.
 13. (a) 280; (b) -75 ; (c) -135 .

Pages 622–623. Sec. 19–4

1. 4.
 3. 1.
 5. $\frac{3}{8}$.
 7. 1.000.
 9. 0.99997.
 11. 10.
 13. $\frac{E}{R}$.
 15. 0.
 17. 0.

Page 623. Sec. 19–6

1. (a) 90; (b) -54 ; (c) 0; (d) 27.
 3. (a) 4; (b) 4; (c) 1; (d) 5.32.
 5. $32t + 30$.
 7. $2 - 8t - 36t^2$.
 9. $-\frac{1}{3000C^2}$.
 11. $-\frac{10}{(2 + R)^2}$.

Pages 633–634. Sec. 19–9

1. $10x$.
 3. $3x^2 - 2$.
 5. $-\frac{28}{t^3} + \frac{3}{t^2}$.
 7. $\frac{3}{2\sqrt{x}} - \frac{5}{3\sqrt{x^3}}$.
 9. $\frac{12}{5\sqrt{x}} + \frac{16\sqrt[3]{x}}{3}$.
 11. $10t - 6 \cos t$.

13. $20(1+x)^4$. 15. $-\frac{1}{(3+\theta)^2}$. 17. $41,470 \cos 377t$.
19. $4 \sin \theta \cos \theta$. 21. $30 \sin 5t \cos 5t + 160 \sin 10t \cos 10t$.
23. $2t \cos 2t - 2t^2 \sin 2t$. 25. $\frac{(1+9x)(1+x)^3}{2\sqrt{x}}$.
27. $10(1+v)(31+35v)(5+7v)^2$. 29. $\frac{2x}{\sqrt{3+2x^2}}$.
31. $-\frac{1}{2(1+x)\sqrt{1+x}}$. 33. $-\frac{5}{2(2+5x)\sqrt{2+5x}}$.
35. $\frac{19}{(2x+3)^2}$. 37. $\frac{-4x}{(1+x^2)^2}$. 39. $\frac{2u(u^4+2u^2-1)}{(u^2+1)^2}$.
41. $-\frac{(1+x) \sin x + \cos x}{(1+x)^2}$. 43. $\frac{4 \sin \theta (\sin \theta + 2 \theta \cos \theta)}{\sqrt{1+8 \theta \sin^2 \theta}}$.
45. $-\frac{1}{3\omega^3 \sqrt{4+\frac{1}{3\omega^2}}}$. 47. $\frac{R^2 \sin t \cos t}{\sqrt{1+R^2 \sin^2 t}}$. 49. $A\omega \cos \omega t - B\omega \sin \omega t$.
51. $I_1\omega \cos(\omega t + \alpha_1) + 2I_2\omega \cos(2\omega t + \alpha_2) + 3I_3\omega \cos(3\omega t + \alpha_3)$.

Page 636. Sec. 19-10

1. (a) $2x - y - 3 = 0$; (b) $6x + 2y + 11 = 0$; (c) $y + 1 = 0$.
3. (a) $x - y = 0$; (b) $6\sqrt{3}x - 12y + 6 - \pi\sqrt{3} = 0$; (c) $y - 1 = 0$.
5. $15t^2$; $30t$.
7. $\frac{-2t}{(t^2+1)^2}$; $\frac{2(3t^2-1)}{(t^2+1)^3}$.
9. $\omega(A_1 \cos \omega t + 2A_2 \cos 2\omega t)$; $-\omega^2(A_1 \sin \omega t + 4A_2 \sin 2\omega t)$.
11. 200 volts.
13. $400 \sin(377t + 0.1) + 754 \cos(377t + 0.1)$ volts.

Pages 642-643. Sec. 19-11

1. Min.: $(\frac{3}{4}, 5\frac{7}{8})$. 3. Min. $(-0.39, -1.24)$.
5. Max.: $(-3, 333)$; Min.: $(2.5, 0.25)$. 7. Max.: $(-1, -2)$; Min.: $(1, 2)$.
9. Max.: $(0, 1)$.
11. Max.: $(k\pi, 2)$, $k = 0, \pm 2, \pm 4 \dots$. Max.: $(k\pi, 0)$, $k = \pm 1, \pm 3, \pm 5 \dots$. Min.: $(k\pi \pm 1.32, -1.12)$, $k = \pm 1, \pm 3, \pm 5 \dots$.
13. Side: 3. 15. Side: $\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$.
17. Diameter $= \frac{8}{\sqrt[3]{\pi}}$ in.; height $= \frac{4}{\sqrt[3]{\pi}}$ in.
19. Side of base: 8 yd.; height: 4 yd. 21. $\frac{E}{2R_i}$.
23. $M = \frac{\sqrt{R_1 R_2}}{\omega}$, $I = \frac{E}{2\sqrt{R_1 R_2}}$.

Page 646. Sec. 19-12

1. $dy = (21x^2 - 5) dx$. 3. $dy = -4x dx$. 5. $dy = \frac{(x-1)\sqrt{x}}{2x^2} dx$.
 7. $dy = 10 \cos x dx$. 9. $dy = 2 \cos 2x dx$. 11. 6.25 cu. ft.
 13. (a) 1.57 ohms; (b) 126 ohms; (c) 62.2 ohms. 15. 2.004.
 17. 0.868. 19. 1.01.

Chapter 20

Pages 652-653. Sec. 20-1

1. 9.50. 3. 14.6. 5. 0.987.
 7. 0.710. 9. 1.75. 11. 22.0.
 13. 1.95. 15. 2.01. 17. 12.0.
 19. 1.16.
 21. If $\Delta x = 1$: error = 0.83, 1.95%; if $\Delta x = \frac{1}{2}$: error, 0.20, 0.48%.
 23. 0.707.

Page 660. Sec. 20-2

13. $\frac{8}{3}$. 15. 35. 17. $\frac{8}{3}$.
 19. $\frac{3}{4}$. 21. $\frac{\pi}{2}$. 23. $\frac{2}{3}$.
 25. 1. 27. $\frac{2}{3}$. 29. 0.707.

Pages 663-664. Sec. 20-3

1. $\frac{x^6}{6} + C$. 3. $\frac{3}{4}u\sqrt[3]{u} + C$.
 5. $\frac{2\sqrt{10}}{3}S\sqrt{S} - \frac{2}{3}S^3 - \frac{1}{S^3} + C$. 7. $\frac{1}{3}(4x-3)\sqrt{4x-3} + C$.
 9. $\sqrt{x^2+3} + C$. 11. $\frac{1}{2}\sqrt{1+x^4} + C$.
 13. $-\frac{1}{3}\cos 5x + C$. 15. $-2\cos x + \frac{3}{2}\cos 2x - \frac{4}{3}\cos 3x + C$.
 17. $\frac{2}{3}\sin(5t+3) + \frac{1}{2}\sin(10t-7) + C$. 19. 4.
 21. $\frac{3}{4}a^4$. 23. $\frac{3^2}{9}$.
 25. 2. 27. 0.

Pages 669-670. Sec. 20-4

1. 6.25 ft.-lb. 3. $\frac{Q}{r_1} - \frac{Q}{r_2}$.
 5. (a) $x = \frac{F}{m\omega} \left(t - \frac{1}{\omega} \sin \omega t \right)$; (b) minimum of $\frac{F}{m\omega} (1 - \cos \omega t)$ is 0; maximum of $\frac{F}{m\omega} (1 - \cos \omega t)$ is $\frac{2F}{m\omega}$.

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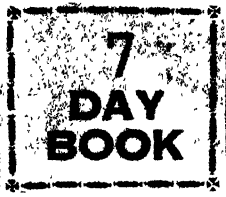
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